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Dissertation paper

The Beilinson Spectral Sequence and Applications

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Introduction

The Beilinson spectral sequences were introduced as a powerful tool for characterizing some vector bundles on complex projective spaces in terms of their cohomology and the cohomology of their twists. Many results such as the Riemann-Roch Theorem, Serre duality, the long exact sequence in cohomology, the vanishing theorems etc show that we have a great deal of techniques for computing these invariants. Given a vector bundle M, the Beilinson spectral sequence associated to this bundle computes the successive quotients of a filtration on M. This mysterious filtration can be recuperated from the first few sheets of the spectral sequence if we impose enough restrictions on the cohomology of M. If we are lucky enough to obtain a filtration by only one sub-sheaf, then we have obtained our bundle M. If we were less fortunate and have obtained a filtration by two sub-sheaves, then we have M as an extension. If this extension is split, then we again recover M and so on.

By following Beilinson's proof from [OkScSp,88], a proof that we detail in Chapter 3, we see that the Koszul resolution of the diagonal inside the product $\mathbb{P}^n \times \mathbb{P}^n$ plays the main role in the proof. Therefore similar spectral sequences to Beilinson's can be obtained when we work over a variety X for which the diagonal Δ_X in $X \times X$ admits a Koszul resolution i.e. it is the scheme of zeros of a vector bundle of rank equal to the dimension of X over the product $X \times X$. The real power of these sequences becomes apparent if the vector bundle whose section resolves the diagonal is a product of bundles as we will explain later. Following [ApBr,06], we see that ruled surfaces have these properties and we explain the associated Beilinson type spectral sequence as well as the simpler form it has for rational ruled surfaces.

Chapter 4 gives an application of these techniques in proving two splitting criteria for rank 2 bundles on a rational ruled surface.

The first two chapters serve more as reference for the last two chapters than as information sources and should be treated accordingly.

Contents

Chapter 1

Spectral Sequences

A differential module is a an abelian group E equipped with some group endomorphism d that is a differential i.e. $d^2 \stackrel{def}{=} d \circ d = 0$. Associated to a differential module we have the usual notion of (co)homology $H(E) = \frac{\ker d}{Imd}$. The definition is consistent since $d^2 = 0 \Leftrightarrow \ker d \supseteq Imd$. With these definitions we can define our basic object of interest.

Definition 1.1. *A **spectral sequence** is a set of differential modules $(E_r, d_r)_{r\in\mathbb{N}}$ such that $E_{r+1} = H(E_r)$ for all r.*

While this is a very general definition, our fundamental example is that of the spectral sequence attached to a bicomplex which will appear as the spectral sequence attached to a filtered complex. In building these we follow [GrHa,78].

Let (C^*, d^*) be a complex indexed by $\mathbb{Z}$ with a given differential d that increases degrees i.e. $d^n : C^n \to C^{n+1}$. Define $Z_n = \ker d^n$ and $B_n = Imd^{n-1}$. We can see (C^*, d^*) as a differential module by defining $C = \bigoplus_{n\in\mathbb{Z}} C^n$ and taking the differential d as induced by d^*. C is a graded abelian group and d is an endomorphism of degree 1. We can do the same thing for Z^*, B^* and introduce the graded groups Z and B. Define $H^n(C^*, d^*) = \frac{Z_n}{B_n}$. Notice that $H(C, d) = \bigoplus_{n\in\mathbb{Z}} H^n(C^*, d^*)$.

By a filtration on C we mean a decreasing sequence of graded subgroups

$$C = F^0C \supseteq F^1C \supseteq \ldots \supseteq F^nC = 0$$

such that $dF^kC \subseteq F^kC$. Since the filtration is by graded subgroups, it induces a filtration on C^* and by restriction also on Z^* and B^*. Denote these filtration by F^*C^*, F^*Z^* and F^*B^* respectively. Since for all $m \in \mathbb{Z}$ and $k = \overline{0, n-1}$ we have $F^{k+1}Z^m \cap F^kB^m = F^{k+1}B^m$, we get natural inclusions $\frac{F^{k+1}Z}{F^{k+1}B} \hookrightarrow \frac{F^kZ}{F^kB}$ on. This sequence of inclusion is a filtration on the cohomology of C.

We define the associated graded complex to C^* corresponding to the filtration F^* by $Gr^k(C^*) = \frac{F^kC^*}{F^{k+1}C^*}$. We also have an associated graded complex $Gr^*H^*(C^*, d^*)$ on cohomology.

Theorem 1.2. *Let (C, d) be a filtered complex with filtration F^*. Then there exists a spectral sequence (E_r, d_r) such that:*

a. for all $r \geq 0$,

$$E_r = \bigoplus_{p,q\in\mathbb{Z}} E_r^{p,q}$$

and d_r is induced by

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

for all $p, q \in \mathbb{Z}$.

b.

$$\begin{aligned} E_0^{p,q} =& Gr^p C^{p+q} \\ E_1^{p,q} =& H^{p+q} Gr^p C^* \\ E_\infty^{p,q} =& Gr^p(H^{p+q}C^*). \end{aligned}$$

The meaning of E_∞ is that for fixed p, q, the sequence $E_r^{p,q}$ eventually stabilizes as r grows. The last assertion is usually denoted

$$E_\infty \Rightarrow H^*C^*.$$

We say that E_∞ approximates the cohomology of the complex C. What this means is that we have a filtration on $H^n(C^*, d^*)$ and $(E_\infty^{p,q})_{p+q=n}$ computes the successive quotients. $H(C)$ cannot always be recovered from the spectral sequence, but there are some special cases when it can.

For example if the filtration on $H(C)$ is by vector spaces we can express each $H^k(C)$ as a direct sum of members of $E_\infty^{*,*}$.

Another fortunate case is when all but at most one term in $(E_\infty^{p,q})_{p+q=n}$ are 0. In this case the nonzero term is $H^n(C^*, d^*)$. This is because of the simple observation that if $A = A_0 \supseteq A_1 \supseteq \ldots \supseteq A_n = 0$ is a filtration on the abelian group A, $k < n$ is an integer and $A_i/A_{i+1} = 0$ for all $i \neq k$, then $A_0 = \ldots = A_k$ and $A_{k+1} = \ldots = A_n = 0$.

Remark 1.3. *Even though we have chosen the filtration F^* on C to be finite, all the results above hold if we replace this finiteness condition with a weaker property; that is we require that the filtration is finite on every C^n, not necessarily on C. This will be the case in the examples and applications that follow.*

Proof. of 1.2 (sketch). Define

$$E_r^{p,q} = \frac{\{a \in F^pC^{p+q} |\ da \in F^{p+r}C^{p+q+1}\}}{dF^{p-r+1}C^{p+q-1} + F^{p+1}C^{p+q}}.$$

The convention is that when the denominator is not a subgroup of the numerator we take the intersection and then make the quotient. It is easy to see that the terms E_0 and E_1 defined in this manner are who they were supposed to be. Define on $E_r^{p,q}$

$$d_r a = [da] \in E_r^{p+r, q-r+1}.$$

For fixed n, large r and all $p + q = n$,

$$E_r^{p,q} = \frac{\{a \in F^pC^n |\ da = 0\}}{dC^{n-1} + F^{p+1}C^{p+q}} = \frac{F^pZ^n}{B^n + F^{p+1}C^n} = \frac{\frac{F^pZ^n}{F^pB^n}}{\frac{(B^n+F^{p+1}C^n)\cap F^pZ^n}{F^pB^n}}.$$

Since $(B^n+F^{p+1}C^n)\cap F^pZ^n = (B^n+F^{p+1}C^n)\cap Z^n \cap F^pC^n = (B^n\cap F^pC^n + F^{p+1}C^n\cap Z^n) = F^pB^n + F^{p+1}Z^n$,

$$E_r^{p,q} = \frac{\frac{F^pZ^n}{F^pB^n}}{\frac{F^pB^n+F^{p+1}Z^n}{F^pB^n}} = \frac{\frac{F^pZ^n}{F^pB^n}}{\frac{F^{p+1}Z^n}{F^pB^n\cap F^{p+1}Z^n}} = \frac{\frac{F^pZ^n}{F^pB^n}}{\frac{F^{p+1}Z^n}{F^{p+1}B^n}} = Gr^pH^n(C^*).$$

Notice that the technical difficulty with this proof is the verification of $H(E_r) = E_{r+1}$. □

The natural definition of the differentials d_r is what justifies the somehow strange direction of the arrows in E_r. We draw some pictures. At E_0 the differentials all go up:

$$\begin{tikzcd}
\cdots & E_0^{p,q+2} & E_0^{p+1,q+2} & \\
& E_0^{p,q+1} \arrow[u, "d_0^{p,q+1}"] & E_0^{p+1,q+1} \arrow[u, "d_0^{p+1,q+1}"] & \\
& E_0^{p,q} \arrow[u, "d_0^{p,q}"] & E_0^{p+1,q} \arrow[u, "d_0^{p+1,q}"] & \cdots
\end{tikzcd}$$

Next, on E_1 they all go one place to the right:

$$\begin{tikzcd}
\cdots \arrow[r] & E_1^{p,q+2} \arrow[r, "d_1^{p,q+2}"] & E_1^{p+1,q+2} & \\
& E_1^{p,q+1} \arrow[r, "d_1^{p,q+1}"] & E_1^{p+1,q+1} & \\
& E_1^{p,q} \arrow[r, "d_0^{p,q}"] & E_1^{p+1,q} \arrow[r, "d_0^{p+1,q}"] & \cdots
\end{tikzcd}$$

We now start building the spectral sequence associated to a bigraded complex.

A **bigraded complex** is a triplet $(C^{*,*}, d^{*,*}, \delta^{*,*})$ with $d^{p,q} : C^{p,q} \to C^{p+1,q}$ and $\delta^{p,q} : C^{p,q} \to C^{p,q+1}$ such that

$$\begin{cases} d^2 = 0 \\ \delta^2 = 0 \\ d\delta + \delta d = 0 \end{cases}$$

everywhere the compositions are possible. **The total complex** associated to $C^{*,*}$ is the complex $K^n = \bigoplus_{p+q=n} C^{p,q}$ with differential $D = d + \delta$. Since $D^2 = (d+\delta)^2 = d^2 + \delta^2 + (d\delta + \delta d) = 0$, we see that (K, D) is a true complex. It has two natural filtration given by

$$\begin{cases} F^p K^n = \bigoplus_{p'+q=n,\ p' \geq p} C^{p',q} \\ {}'F^q K^n = \bigoplus_{p+q'=n,\ q' \geq q} C^{p,q'} \end{cases}$$

These filtration are not finite but they satisfy the condition in the remark 1.3 of being finite on each K^n.

By 1.2 we get two spectral sequences E and $'E$ which both approximate the cohomology of K. Let's take a closer look at the first one. We have

$$E_0^{p,q} = \frac{F^p K^{p+q}}{F^{p+1} K^{p+q}} = \frac{C^{p,q} + C^{p+1,q-1} + \ldots}{C^{p+1,q-1} + \ldots} = C^{p,q}.$$

$d_0 : E_0^{p,q} \to E_0^{p,q+1}$ is induced from $D = d + \delta$ by passing to the quotient. We can identify this with δ because d is 0 on the quotient.

$E_1^{p,q}$ is by the definition of the spectral sequence the cohomology of $E_0^{*,*}$ at (p,q) and this is easily seen to be $H^q_\delta(C^{p,*}) \overset{def}{=} H^q(C^{p,*}, \delta)$. d_1 is also induced by D and similarly to the previous case we can see that it can be identified with d.

$E_2^{p,q} = H^{p,q}(E_1^{p,q}, d) = H^p_d(H^q_\delta(C^{*,*}))$ but this time we have no easy way of building d_2.

The computations for E can also be done for $'E$ and we have the two spectral sequences E, $'E$ approximating $H(K)$ such that

$$\begin{cases} E_2^{p,q} = H^{p,q}(E_1^{p,q}, d) = H_d^p(H_\delta^q(C^{*,*})) \\ 'E_2^{p,q} = H^{p,q}('E_1^{p,q}, \delta) = H_\delta^q(H_d^p(C^{*,*})) \end{cases}$$

Remark 1.4. *The condition $d\delta + \delta d = 0$ seems to go against the nature of our usual commutative diagram condition. We will see that any bicomplex $(C^{*,*}, d^{*,*}, \delta^{*,*})$ can de turned into a a triple $(C^{*,*},' d^{*,*}, \delta^{*,*})$ with*

$$\begin{cases} 'd^2 = 0 \\ \delta^2 = 0 \\ 'd \circ \delta = \delta \circ' d \end{cases}$$

with no affect on cohomology. We do this by setting

$$\begin{cases} 'd^{2*,*} = d^{2*,*} \\ 'd^{2*+1,*} = -d^{2*+1,*} \end{cases}.$$

The same change of signs could have been applied to δ instead of d for a similar result.

This remark obviously applies the other way around i.e. from a triple $(C^{*,*},' d^{*,*}, \delta^{*,*})$ with the commutativity condition we obtain a bicomplex. From now on I will use the term bicomplex indiscriminately for both types of triples.

An example of a typical construction of a spectral sequence that will be used in this paper, especially in the proof of the existence of the Beilinson spectral sequence in 3.1, is the **hyperdirect** image. Let $f : M \to N$ be a (continuous) map of topological spaces and take C^* be a complex of sheaves on M. Choose a Cartan-Eilenberg resolution $L^{*,*}$ of C^*. L is an injective resolution of C. By applying f_* to this complex we obtain the complex of sheaves $f_*L^{*,*}$ on N. The cohomology of the total complex associated to this bicomplex is called the hyperdirect image of f through C^* and is denoted by $\mathbb{R}^*f_*(C^*)$.

Chapter 2

Some Algebraic Geometry

2.1 Vector Bundles on Complex Projective Spaces

In this section we will deal with developing a basic vocabulary for this paper - we will introduce the basic definitions, establish our notation and state some useful results.

The n-dimensional **complex projective space** $\mathbb{P}^n(\mathbb{C})$ is the set of lines in $\mathbb{C}^{n+1}$. It can also be defined as $\mathbb{C}^{n+1} \setminus \{0\}$ modulo linear equivalence. $\mathbb{P}^n$ becomes a topological space with the quotient topology, but also a compact complex manifold. We denote the class of the element $(x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$ by $[x_0 : \ldots : x_n]$.

The same definition can be given for any $n+1$-dimensional complex vector space V. The associated n-dimensional projective space is denoted $\mathbb{P}(V)$.

The map $\mathbb{P}^n \to \{\text{hyperplanes in } \mathbb{P}^n\}$ given by

$$[a_0 : \ldots : a_n] \to \{\sum_{i=0}^{n} a_i X_i = 0\}$$

is a bijection and we denote the image by $(P^n)^*$. The same construction can be carried for any vector space V, but the bijection is not natural as it depends on the choice of a basis on V.

For a holomorphic vector bundle E of rank r over X, we have the associated sheaf of sections $\mathcal{O}_X(E)$ which is a locally free sheaf of rank r. We make the convention to identify E and $\mathcal{O}_X(E)$. Define E_x to be the stalk at x of $\mathcal{O}_X(E)$ and $E(x) = E_x/m_x E_x$, where m_x is the maximal ideal of the local ring $\mathcal{O}_x$. Notice that $E(x)$ is just the fibre of E over x, hence it is isomorphic to $\mathbb{C}^r$.

For a bundle morphism $f : E \to F$, we have associated maps $f_x : E_x \to F_x$ and $f(x) : E(x) \to F(x)$. Nakayama's lemma says that f_x is surjective if, and only if, $f(x)$ is. It can happen that f_x is injective even though $f(x)$ is not.

We shall now construct the holomorphic line bundles on $\mathbb{P}^n$. Define the **tautological line bundle** to be the bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ that has as fibre over every point of $\mathbb{P}^n$ the corresponding line in $\mathbb{C}^{n+1}$ i.e.

$$\mathcal{O}_{\mathbb{P}^n}(-1) = \{(l, v) \in \mathbb{P}^n \times \mathbb{C}^{n+1} |\ v \in l\}$$

Its dual is denoted $\mathcal{O}_{\mathbb{P}^n}(1)$ and we set $\mathcal{O}_{\mathbb{P}^n}(n) = \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes n}$ for all integers n.

Denote the set of isomorphism classes of line bundles on X by $Pic(X)$. This is a group with multiplication given by the tensor product and the inverse of each line bundle is its dual. Sending the isomorphism class of a line bundle L to the transition maps of the bundle which make a multiplicative Čech 1-cocycle proves that $Pic(X)$ and $H^1(X, \mathcal{O}_X^*)$ are isomorphic as groups, where $\mathcal{O}_X^*$ is the sheaf of non-vanishing holomorphic functions.

For $\mathbb{P}^n$ it can be proved by using the equivalence between divisors and line bundles that $Pic(\mathbb{P}^n) \simeq Z$ and that $\mathcal{O}_{\mathbb{P}^n}(1)$ is a generator.

By playing with the transition functions, we can prove that $H^0(\mathbb{P}^n, \mathcal{O}(n))$ is isomorphic to the linear space of complex homogeneous polynomials of degree n.

For an arbitrary sheaf $\mathcal{F}$ on $\mathbb{P}^n$, define the n-th twist of $\mathcal{F}$ by

$$\mathcal{F}(n) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}(n)$$

The Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{n+1} \to Q \to 0$$

on $\mathbb{P}^n$, where the first map is given by the geometric realization of the fibres of $\mathcal{O}(-1)$, defines a bundle Q that we call the twisted holomorphic tangent bundle and denote by

$$T_{\mathbb{P}^n}(-1).$$

Twisting by $\mathcal{O}(1)$ gives the tangent bundle of $\mathbb{P}^n$. The dual of $T_{\mathbb{P}^n}$ is the sheaf of rank 1 differential forms on $\mathbb{P}^n$ and is denoted by $\Omega_{\mathbb{P}^n}$. From the Euler sequence we obtain

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}^{n+1}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

For any complex manifold of dimension n, the sheaf of rank 1 differential forms on X, denoted by Ω_X is the dual bundle of the tangent bundle T_X. The n-th exterior power of Ω_X is denoted by ω_X and is called the canonical sheaf on X. Notice that it is the sheaf of differentials of maximal rank and it is a line bundle.

For $X = \mathbb{P}^n$, we have by raising Ω's Euler sequence to the top power that

$$\bigwedge^{n} \Omega_{\mathbb{P}^n} \otimes \bigwedge^{1} \mathcal{O}_{\mathbb{P}^n} \simeq \bigwedge^{n+1} (\mathcal{O}_{\mathbb{P}^n}(-1)) \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1),$$

which proves

$$\omega_{\mathbb{P}^n} \simeq \mathcal{O}(-n-1).$$

Theorem 2.1 (Hirzebruch)**.** *Let E be a holomorphic line bundle of rank r over the complex manifold X. Then, for all q,*

$$\bigwedge^{q} E^* \simeq \bigwedge^{r} E^* \otimes \bigwedge^{r-q} E,$$

with the convention $\bigwedge^{<0} E = 0$.

Theorem 2.2. *Let X be a compact complex manifold. Then for any analytic sheaf $\mathcal{F}$ on X, we have*

$$h^p(X, \mathcal{F}) \stackrel{def}{=} dim_{\mathbb{C}} H^p(X, \mathcal{F}) < \infty$$

Theorem 2.3 (Serre duality). *Let X be a compact projective complex manifold and let ω_X be its canonical line bundle. For any holomorphic vector bundle E on X, we then have*

$$H^i(X,E)^* \simeq H^{n-q}(X, E^* \otimes \omega_X)$$

Let $f : X \to Y$ be a proper holomorphic map of complex spaces, F a coherent analytic sheaf on X and E a holomorphic vector bundle on Y.

Theorem 2.4 (Coherence). *The i-th direct image sheaf $R^i f_* F$ is a coherent analytic sheaf over Y for all $i \geq 0$.*

Proposition 2.5. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathcal{F}$ be a sheaf of abelian groups, and assume that $R^i f_*(\mathcal{F}) = 0$ for all $i > 0$. Then there are natural isomorphisms, for each $i \geq 0$,*

$$H^i(X, \mathcal{F}) \simeq H^i(Y, f_*\mathcal{F})$$

Theorem 2.6 (Semicontinuity). *If f is projective and E is a flat coherent sheaf over Y, then for all $i, s \geq 0$,*

$$\{y \in Y |\ h^i(f^{-1}(y), E|_{f^{-1}(y)}) \geq s\}$$

is a closed analytic subset of Y, where the fibres have the induced complex structure $\mathcal{O}_{f^{-1}(y)} = \mathcal{O}_X / m_y \mathcal{O}_X$.

Theorem 2.7 (Grauert). *If $f : X \to Y$ is a projective morphism of noetherian schemes, Y is integral, $\mathcal{F}$ is a coherent analytic sheaf on X, flat over Y, and if for some i the function $h^i(y, \mathcal{F})$ is constant on Y, then $R^i f_*(\mathcal{F})$ is locally free on Y, and for every y the natural map $R^i f_*(\mathcal{F}) \otimes k(y) \to H^i(X_y, \mathcal{F}_y)$ is an isomorphism.*

Theorem 2.8 (Base-change). *Assume that f is flat, that Y is reduced and that there exists some $i \geq 0$ such that $s(y) = h^i(f^{-1}(y), E|_{f^{-1}(y)})$ is independent of y. Denote the common value by s. If*

$$\begin{tikzcd} X' \arrow[r, "\psi"] \arrow[d, "g"] & X \arrow[d, "f"] \\ Y' \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

is some arbitrary base change, then the canonical $\mathcal{O}_Y$-modules homomorphism

$$\varphi^* R^i f_*(E) \to R^i g_*(\psi^* E)$$

is an isomorphism. In particular, for any $y \in Y$

$$(R^i f_*(E))(y) \simeq H^i(f^{-1}(y), E|_{f^{-1}(y)}).$$

$R^i f_*(E)$ is a rank s complex bundle over Y.

Theorem 2.9 (Projection formula). *Let E' be a vector bundle over Y. Then for all $i \geq 0$,*

$$R^i f_*(f^* E' \otimes F) \simeq E' \otimes R^i f_*(F).$$

Proposition 2.10. *Let Y be a noetherian scheme, and let $\mathcal{E}$ be a locally free $\mathcal{O}_Y$-module or rank $n+1$, $n \geq 1$. Let $X = \mathbb{P}(\mathcal{E})$, with the associated invertible sheaf $\mathcal{O}_X(1)$ and the projection morphism $\pi : X \to Y$.*

a. Then $\pi_*(\mathcal{O}(l)) \simeq S^l(\mathcal{E})$ *for* $l \geq 0$, $\pi_*(\mathcal{O}(l)) = 0$ *for* $l < 0$; $R^i\pi_*(\mathcal{O}(l)) = 0$ *for* $0 < i < n$ *and all* $l \in \mathbb{Z}l$ *and* $R^n\pi_*(\mathcal{O}(l)) = 0$ *for* $l > -n-1$.

b. There is a natural exact sequence

$$0 \to \Omega_{X/Y} \to (\pi^*\mathcal{E})(-1) \to \mathcal{O} \to 0,$$

so the relative canonical sheaf $\omega_{X/Y} = \bigwedge^n \Omega_{X/Y}$ *is isomorphic to the invertible sheaf* $(\pi^* \bigwedge^{n+1} \mathcal{E})(-n-1)$. *Furthermore, there is a natural isomorphism* $R^n\pi_*(\omega_{X/Y}) \simeq \mathcal{O}_Y$.

c. For any $l \in \mathbb{Z}$,

$$R^n\pi_*(\mathcal{O}(l)) \simeq \pi_*(\mathcal{O}(-l-n-1))^* \otimes (\bigwedge^{n+1} \mathcal{E})^*.$$

d. $p_a(X) = (-1)^n p_a(Y)$ *and* $p_g(X) = 0$.

2.1.1 Koszul Complexes

Computing the cohomology of sheaves is a very important problem in Algebraic Geometry because it makes it easy to distinguish objects. One method used for computing cohomology is by taking injective resolutions of sheaves. These exist and are natural, which allows us to compute some other derived functors such as *Ext*. They however present a dramatic computational disadvantage. Čech cohomology computed with respect to an acyclic covering has the advantage that it can be computed easier, but since it is not even a sheaf resolution, it does not behave well with respect to functors. The Koszul complex has both advantages of the injective resolutions and of the Čech complex, but it does not always exist. You can think of the Koszul complex as of a finite, locally free resolution of some special class of sheaves of ideals $\mathcal{I}$ on a variety X. Without further ado, we unwrap the result:

Theorem 2.11 (Koszul complex)**.** *Let* X *be a complex variety of dimension* n *and* E *a vector bundle on* X *of rank* r. *For a section* s *of* E, *let* S *be the corresponding scheme of zeros and* $\mathcal{I}_s$ *be the corresponding sheaf of ideals such that* $\mathcal{O}_S \simeq \mathcal{O}_X/\mathcal{I}_s$. *Then, if the codimension of* S *in* X *is* r, *we have a locally free resolution:*

$$0 \to \det E^* \to \bigwedge^{r-1} E^* \to \ldots \to \bigwedge^{2} E^* \to E^* \to \mathcal{O}_X \to \mathcal{O}_S,$$

where the map $E^* \to \mathcal{O}_X$ *is the dual map of the natural map* $\mathcal{O}_X \to E$ *induced by* s.

Since by definition $\mathcal{I}_s$ is the image of the map $E^* \to \mathcal{O}_X$, we see that we obtain the complex above by splicing a locally free resolution of $\mathcal{I}_s$ with $\mathcal{O}_X$.

Notice that if D is an effective divisor, then its associated invertible sheaf (or line bundle) $\mathcal{O}(D)$ has a section whose divisor of zeros is D. Effective divisors are actually the only subschemes of X whose ideal sheaves are the sheaves of sections of vector bundles on X i.e. locally free sheaves.

2.2 Ruled Surfaces

This section contains the basic results about ruled surfaces and especially rationally ruled surfaces that will be referred to later, but lets start with some general facts about surfaces. The bibliography for this section is mainly [Ha,77].

Let S be a smooth complex projective surface. If C and D are nonsingular curves of S meeting transversally, then we define the intersection number $C.D$ as the number of intersection points. This definition extends by linearity to a bilinear pairing $Div(S) \times Div(S) \to \mathbb{Z}$ such that for any divisor D and irreducible nonsingular curve C, $C.D = deg_C(\mathcal{O}(D) \otimes \mathcal{O}_C)$.

Theorem 2.12 (Adjunction Formula). *If C is nonsingular on S of genus $g(C)$ and if $K(S)$ denotes the canonical divisor of S, then*

$$2g(C) - 2 = C.(C + K).$$

Theorem 2.13 (Riemann-Roch for divisors on surfaces). *If $D \in Div(S)$, then*

$$\chi(\mathcal{O}(D)) = \frac{1}{2}D.(D - K) + 1 + p_a,$$

where p_a denotes the arithmetic genus of S.

Theorem 2.14 (Riemann-Roch for vector bundles of rank 2 on a surface). *If M is a vector bundle of rank 2 on a surface S, then*

$$\chi(M) = \frac{1}{2}c_1(M)(c_1(M) - K(S)) - c_2(M) + 2\chi(\mathcal{O}_S)$$

Theorem 2.15 (Nakai-Moishezon Criterion). *$D \in Div(S)$ is very ample if, and only if, $D^2 > 0$ and $D.C > 0$ for all irreducible curves C in S.*

Let C be a smooth complex projective curve (a Riemann surface). There are several ways to think of a **ruled surface** X over C. We may see it as a locally trivial $\mathbb{P}^1$ bundle over C that is itself a projective variety. An equivalent way to see it is as the projective line bundle associated to the dual of a rank 2 vector bundle $\mathcal{E}$ over C i.e $X = \mathbb{P}(\mathcal{E}^*) = Proj(Sym(\mathcal{E}))$. Since a morphism from an open subset of a projective curve to a projective space extends on the curve, we see that any ruled surface has a section. Any of these sections and the pullback image of $Pic(C)$ in $Pic(X)$ generate $Pic(X)$. More precisely, we have:

Proposition 2.16. *Let X be a ruled surface over C and let $C_0 \subseteq X$ be a section. Then $Pic(X) \simeq \mathbb{Z} \cdot C_0 \oplus \pi^* Pic(C)$. We have the intersection numbers $C_0 \cdot f = 1$ and $f^2 = 0$ for any fibre f of $\pi : X \to C$.*

Lemma 2.17. *Let D be a divisor on the ruled surface X and assume that $D \cdot f \geq 0$. Then $\pi_*\mathcal{O}(D)$ is locally free of rank $D \cdot f + 1$ on C. In particular $\pi_*\mathcal{O}_X = \mathcal{O}_C$. Also $R^i\pi_*\mathcal{O}(D) = 0$ for $i > 0$, and for all i, $H^i(X, \mathcal{O}(D)) \simeq H^i(C, \pi_*\mathcal{O}(D))$.*

As a consequence of this lemma we prove that if C is a curve of genus g, then the arithmetic genus of X is $-g$, the geometric genus of X is 0 and $h^1(X, \mathcal{O}_X) = g$.

By the correspondence between sections of π and invertible sheaves $\mathcal{L}$ on C with a surjective mapping $\mathcal{E} \to \mathcal{L}$, there exists a section C_0 such that $\mathcal{O}(C_0) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. If we denote $e = -deg(\bigwedge^2 \mathcal{E})$, then it can be proved that

$$C_0^2 = -e.$$

Proposition 2.18. *The canonical divisor K on X is given by*

$$K(X) \sim -2C_0 + \pi^*(K(C) + \mathcal{O}(\bigwedge^2 \mathcal{E})),$$

where $K(C)$ is the canonical divisor on C.

2.2.1 Rational Ruled Surfaces

A rational ruled surface is a ruled surface over $\mathbb{P}^1$. It can be proved that any such surface is $\mathbb{P}(\mathcal{E}^*)$ with $\mathcal{E} = \mathcal{O} \oplus \mathcal{O}(-e)$ for some $e \geq 0$. For $e \geq 0$, we denote by X_e the rational ruled surface defined by $\mathcal{O} \oplus \mathcal{O}(-e)$. Denote any divisor $aC_0 + \pi_*(\mathcal{O}(b))$ on X_e, by $aC_0 + bf$. Then $Pic(X_e)$ is freely generated by C_0 and by any of the fibres f. For the basis $\{C_0, f\}$ of $Pic(X_e)$, that we denote by $(1,0)$ and $(0,1)$ respectively, we have the following intersection numbers:

$$(1,0).(1,0) = -e \quad (1,0).(0,1) = 1 \quad (0,1).(0,1) = 0 \ .$$

With this notation, the canonical divisor is

$$K(X_e) = (-2, -e-2).$$

Theorem 2.19 (Riemann-Roch for rank 2 vector bundles on X_{-n})**.** *Let M be a rank 2 vector bundle on X_{-n} for some $n \leq 0$. Then*

$$\chi(M) = \frac{1}{2}c_1(c_1 - (-2, n-2)) - c_2 + 2$$

Proof. This only differs from 2.14 in that it says $\chi(\mathcal{O}) = 1$. This is because $\chi(\mathcal{O}) = \chi(\pi_*\mathcal{O}) = \chi(\mathcal{O}_{\mathbb{P}^1}) = 1$. $\square$

Chapter 3

The Beilinson Spectral Sequence

Theorem 3.1 (Beilinson)**.** *Let E be a holomorphic vector bundle of rank r on $\mathbb{P}^n$. Then there exists a spectral sequence $E_r^{p,q}$ with E_1 term*

$$E_1^{p,q} = H^q(\mathbb{P}^n, E(p)) \otimes \Omega^{-p}(-p)$$

which converges to

$$E^i = \begin{cases} E, & \textit{for i=0} \\ 0, & \textit{otherwise} \end{cases}$$

i.e. $E_\infty^{p,q} = 0$ for $p+q \neq 0$ and $\bigoplus_{p=0}^n E_\infty^{-p,p}$ is the associated graded sheaf of a filtration of E.

The point of this theorem is that a holomorphic vector bundle on $\mathbb{P}^n$ can be sometimes recuperated and at least approximated by twists of the sheaf of differentials and by the cohomology groups of twists of E.

Proof. of 3.1. A Koszul complex that resolves the diagonal of $\mathbb{P}^n \times \mathbb{P}^n$ gives by tensoring with E a locally free resolution of E. The solution to our problem is a spectral sequence that gives the hyperdirect image of a map through this complex. In short and with a strong abuse of terminology, this is the idea behind the proof of Beilinson's theorem.

We will first explain the construction of the Koszul complex. Consider the diagram

$$\begin{tikzcd} \mathbb{P}^n \times \mathbb{P}^n \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathbb{P}^n \\ \mathbb{P}^n & \end{tikzcd}$$

Make the convention $F \boxtimes G = p_1^* F \otimes p_2^* G$. We will construct a bundle on $\mathbb{P}^n \times \mathbb{P}^n$ and prove that it has a section whose scheme of zeros is $\Delta_{\mathbb{P}^n}$.

Let Q be the twisted tangent sheaf $T_{\mathbb{P}^n}(-1)$. It can be computed from the Euler sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n}^{\oplus n+1} \to Q \to 0$$

Consider the vector bundle on $\mathbb{P}^n \times \mathbb{P}^n$

$$F = \mathcal{O}_{\mathbb{P}^n}(1) \boxtimes Q \simeq Hom(p_1^* \mathcal{O}_{\mathbb{P}^n}(-1), p_2^* Q)$$

Since $H^1(\mathcal{O}_{\mathbb{P}^n}(-1)) = 0$, from the Euler sequence we have that $H^0(Q) = H^0(\mathcal{O}_{\mathbb{P}^n}^{\oplus n+1})/H^0(\mathcal{O}_{\mathbb{P}^n}(-1)) \simeq \mathbb{C}^{n+1}$, so

$$H^0(F) = H^0(\mathbb{P}^n \times \mathbb{P}^n, p_1^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes p_2^*Q) \simeq H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \otimes H^0(\mathbb{P}^n, Q) \simeq$$

$$\simeq (\mathbb{C}^{n+1})^* \otimes \mathbb{C}^{n+1} \simeq End_{\mathbb{C}}(\mathbb{C}^{n+1})$$

Let $s \in H^0(F)$ be the section that corresponds to the identity by the above isomorphism. For $x, y \in \mathbb{P}^n$,

$$F_{(x,y)} = (\mathcal{O}_{\mathbb{P}^n}(1) \boxtimes Q)_{(x,y)} = (p_1^*\mathcal{O}_{\mathbb{P}^n}(1) \otimes p_2^*Q)_{(x,y)} =$$

$$= (p_1^*\mathcal{O}_{\mathbb{P}^n}(1))_{(x,y)} \otimes (p_2^*Q)_{(x,y)} = \mathcal{O}_{\mathbb{P}^n}(1)_x \otimes Q_y = (\mathcal{O}_{\mathbb{P}^n}(-1))_x^* \otimes Q_y \simeq$$

$$\simeq Hom((\mathcal{O}_{\mathbb{P}^n}(-1))_x, Q_y)$$

The above isomorphisms are all natural as isomorphisms of germs of sheaves and moreover induce natural isomorphism on the fibres. Since $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the tautological line bundle of $\mathbb{P}^n$, the fibre at x is canonically identified to the lined spanned by x in $\mathbb{C}^{n+1}$. The same identification and the Euler sequence show that $Q(y)$ is canonically identified to the quotient $\mathbb{C}^{n+1}/\mathbb{C}y$ of $\mathbb{C}^{n+1}$ by the one dimensional subspace spanned by y. So we have a canonical isomorphism

$$F(x,y) \simeq Hom_{\mathbb{C}}(\mathbb{C}x, \mathbb{C}^{n+1}/\mathbb{C}y).$$

Let v, w pe fixed nonzero elements of the lines x, y in $\mathbb{C}^{n+1}$. The naturality of all our isomorphisms shows us that $s(x,y)(av) = av$ for all $a \in \mathbb{C}$. In particular, s vanishes at (x,y) if, and only if, $v = 0$ in $\mathbb{C}^{n+1}/\mathbb{C}y$ which is equivalent to saying that v lies on the line spanned by y i.e. $x = y$. Therefore the zero locus of s is supported as a scheme on the diagonal Δ of $\mathbb{P}^n \times \mathbb{P}^n$.

What we want to prove is that this zero locus is the diagonal as a scheme also. In any case, if we denote this scheme by S, we have a canonic map $\mathcal{O}_S \to \mathcal{O}_\Delta$ and the Koszul locally free resolution:

$$0 \to \wedge^n F^* \to \wedge^{n-1} F^* \to \ldots \to F^* \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_S \to 0$$

which is the same as

$$0 \to \wedge^n(\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes Q^*) \to \wedge^{n-1}(\mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes Q^*) \to \ldots \to$$

$$\to \mathcal{O}_{\mathbb{P}^n}(-1) \boxtimes Q^* \to \mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n} \to \mathcal{O}_S \to 0 \tag{3.1}$$

From the Koszul complex it is now immediate that $\mathcal{O}_S$ is zero exactly outside Δ and a vector bundle when restricted to Δ. These are sufficient to prove that the map $\mathcal{O}_S \to \mathcal{O}_\Delta$ is an isomorphism and $S = \Delta$. Now that we have built the Koszul complex, we make the next step in the proof.

Since $\wedge^n(L \otimes F) = L^{\otimes n} \otimes \wedge^n F$ for any line bundle L, bundle F and any $n \in \mathbb{N}$, and $Q^* = T_{\mathbb{P}^n}(-1)^* = \Omega_{\mathbb{P}^n}(1)$, we can tensor the locally free resolution 3.1 by p_1^*E to obtain the complex of sheaves:

$$0 \to E(-n) \boxtimes \Omega^n(n) \to E(-n+1) \boxtimes \Omega^{n-1}(n-1) \to \ldots \to E(-1) \boxtimes \Omega^1(1) \to$$

$$\to E \boxtimes \mathcal{O}_\Delta \to 0$$

Denote by C^* the complex $C^{-k} = E(-k) \boxtimes \Omega^k_{\mathbb{P}^n}(k)$.

Let $\mathbb{R}^i p_{2*}(C^*)$ be the i^{th} hyperdirect image defined as the limit of any of the two spectral sequences $E_*^{*,*}$ or $'E_*^{*,*}$ attached to the double complex obtained from applying p_{2*} to a resolution $L^{*,*}$ of C^*. These two spectral sequences have second terms

$$E_2^{p,q} = H^p(R^q p_{2*}(C^*))$$

$$'E_2^{p,q} = R^p p_{2*}(H^q(C^*)).$$

Since C^* is a locally free resolution of $p_1^* E|_\Delta$, it follows that

$$H^q(C^*) = \begin{cases} p_1^* E|_\Delta & \text{,if } q = 0 \\ 0 & \text{,otherwise} \end{cases} \Rightarrow$$

$$'E_2^{p,q} = R^p p_{2*}(H^q(C^*)) \simeq \begin{cases} E & \text{,if } p = q = 0 \\ 0 & \text{,otherwise} \end{cases}$$

The last follows from the infamous formula $p_{2*}(p_1^* E|_\Delta) \simeq E$ which may be seen as a fancy way of saying that if we take E, pull it back through p_1, restrict to the diagonal and push through p_2 we get E; or may be proved by the Base Change Theorem.

Since we know the limit of $'E_*^{*,*}$, we conclude that

$$\mathbb{R}^i p_{2*}(C^*) \simeq \begin{cases} E & \text{,if } i = 0 \\ 0 & \text{,otherwise} \end{cases}$$

The solution to the problem is not $'E_*$, but E_*. We only have to compute E_1.

$$E_1^{p,q} = R^q p_{2*}(C^p) = R^q p_{2*}(E(p) \boxtimes \Omega^{-p}(-p)) =$$

$$= R^q p_{2*}(p_1^* E(p)) \otimes \Omega^{-p}(-p) = H^q(E(p)) \otimes \Omega^{-p}(-p),$$

by the Projection Formula and Base Change. □

A similar result could have been obtained by using the complex

$$D^{-k} = \mathcal{O}(-k) \boxtimes (E \otimes \Omega^k(k)).$$

Theorem 3.2 (Beilinson II). *Let E be a holomorphic vector bundle of rank r over $\mathbb{P}^n$. Then there exists a spectral sequence E_* with E_1 term*

$$E_1^{p,q} = H^q(\mathbb{P}^n, E \otimes \Omega^{-p}(-p)) \otimes \mathcal{O}_{\mathbb{P}^n}(p),$$

with

$$E_* \Rightarrow E^i = \begin{cases} E & \text{,if } i = 0 \\ 0 & \text{,otherwise} \end{cases}$$

i.e. $E_\infty^{p,q} = 0$ for $p + q \neq 0$ and $\bigoplus_{p=0}^n E_\infty^{-p,p}$ is the associated graded sheaf of a filtration of E.

Now this has been the classical Beilinson spectral sequence. If we look at it closely, we see that the main ingredients are the Koszul complex and the resolution of the diagonal, so the main recipe in generalizing this result would be to find for a given variety X of dimension n, a bundle E of rank n on $X \times X$ and a section $s \in H^0(E)$ such that its scheme of zeros is the diagonal Δ. Not all varieties X meet these criteria. Examples and counter-examples of such varieties ar given in [PrSrPa,07].

We will concern ourselves with proving that a Beilinson type spectral sequence exists for ruled surfaces. The reason why we consider this case is that in these conditions we can choose the bundle whose section solves Δ to be of type $A \boxtimes B$ which will appease the pain of the cumbersome formulas we would have to work with otherwise. The inspiration for this is [ApBr,06].

Let C be a Riemann surface of genus g and let X be a ruled surface associated to a vector bundle $\mathcal{E}$ of rank 2 on C. Consider the natural projections $\pi : X \to C$ and $p : X \times X \to C \times C$. The diagonal is a divisor in $C \times C$, so the associated line bundle $\mathcal{O}(\Delta_C)$ on $C \times C$ has a section (because Δ_C is effective) whose scheme of zeros is Δ_C. Denote the line bundle by L and the section by s. It is clear that $Y = p^*(\Delta_C)$ is $X \times_C X$, the scheme theoretic product of X with itself as a scheme over C by π. Y is the scheme of zeros of the section $p^*(s)$ of the line bundle p^*L.

Consider the line bundle on $X \times X$

$$F = T_{X/C}(-1) \boxtimes \mathcal{O}_X(1) \stackrel{def}{=} p_1^*(T_{X/C}(-1)) \otimes p_2^*(\mathcal{O}_X(1)),$$

where $\mathcal{O}(1)$ is naturally associated to $\mathbb{P}(\mathcal{E}^*) = X$ with the property $\pi_*(\mathcal{O}(1)) = \mathcal{E}$ and $T_{X/C}$ is the relative tangent bundle given by the generalized Euler sequence

$$0 \to \mathcal{O}_X(-1) \to \pi^*(\mathcal{E}^*) \to T_{X/C}(-1) \to 0.$$

There exists a section σ of $F|_Y$ whose scheme of zeros is Δ_X. The associated Koszul complex yields a resolution of $\mathcal{O}_{\Delta_X}$ over $\mathcal{O}_Y$. A resolution of the same structural sheaf, but over $\mathcal{O}_{X \times X}$ can be constructed by use of the following extension lemma (cf. [ApBr,06]):

Lemma 3.3. *Let Z be a smooth projective, irreducible variety, Y an effective divisor on Z, F a vector bundle on Z, and $\sigma \in H^0(Y, F|_Y)$. If G denotes the vector bundle on Z given by the extension*

$$0 \to F \to G \to \mathcal{O}_Z(Y) \to 0,$$

corresponding to the image of σ by the canonical morphism $H^0(Y, F|_Y) \xrightarrow{\delta} H^1(Z, F(-Y))$, then there exists $u \in H^0(Z, G)$ such that $u|_Y = \sigma$. In particular, zero(u) = zero$(\sigma) \subset Y$.

Proof. The proof of this lemma is a diagram chase at its very best. First of all, an extension of vector bundles is always a vector bundle; being a vector bundle is a local property and an extension of free modules is always split. We have the large diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F(-Y) \arrow[r] \arrow[d] & G(-Y) \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & \mathcal{O}(Y) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F|_Y \arrow[r] \arrow[d] & G|_Y \arrow[r] \arrow[d] & \mathcal{O}_Y(Y) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

In this diagram, all the rows and columns are short exact sequences and we will also see that it is commutative. The bottom row is the restriction to Y of the second row which is the given extension of $\mathcal{O}(Y)$ through F, so it is exact. It should be easy to see that restricting the picture to the last two rows we have a commutative diagram. That the entire diagram is commutative follows from a Snake-Lemma-type argument.

We check that the image of σ through the natural map $H^0(Y, F|_Y) \to H^0(Y, G|_Y)$ belongs to

$$Ker(H^0(Y, G|_Y) \to H^1(Z, G(-Y))) = Im(H^0(Z, G) \to H^0(Y, G|_Y)).$$

For this, notice that we have another commutative diagram coming from the long exact sequences associated to the first two exact columns of the diagram:

$$\begin{tikzcd}
H^0(Z,F) \arrow[r] \arrow[d] & H^0(Z,G) \arrow[d] \\
H^0(Y,F|_Y) \arrow[r] \arrow[d] & H^0(Y,G|_Y) \arrow[d] \\
H^1(Z,F(-Y)) \arrow[r] & H^1(Z,G(-Y))
\end{tikzcd} \quad . \tag{3.2}$$

Since Y is an effective divisor on Z, $\mathcal{O}(Y)$ is a line bundle. The extension $0 \to F \to G \to \mathcal{O}(Y) \to 0$ corresponds to the element of $Ext^1(\mathcal{O}(Y), F)$ given by the image of the identity endomorphism via the map $Hom(\mathcal{O}(Y), \mathcal{O}(Y)) \xrightarrow{\delta} Ext^1(\mathcal{O}(Y), F)$ induced from the long exact sequence of the derived functors of $Hom(\mathcal{O}(Y), \cdot)$ applied to our extension. $Hom(\mathcal{O}(Y), \mathcal{O}(Y)) = H^0(Z, \mathcal{O})$ since $\mathcal{O}(Y)$ is a line bundle. Also $Ext^1(\mathcal{O}(Y), F) = Ext^1(\mathcal{O}, F(-Y)) = H^1(Z, F(-Y))$ for the same reason of $\mathcal{O}(Y)$ being a line bundle. By these identifications, the extension corresponds to the image of 1 in $H^1(Z, F(-Y))$ by the map $H^0(Z, \mathcal{O}) \to H^1(Z, F(-Y))$. The last map fits into an exact sequence induced from the first row of the large diagram:

$$H^0(Z, \mathcal{O}) \to H^1(Z, F(-Y)) \to H^1(Z, G(-Y)). \tag{3.3}$$

By all that we have said, it should be clear that σ and 1 map to the same element of $H^1(Z, F(-Y))$. This element must map further to 0 in $H^1(Z, G(-Y))$ because 3.3 is a complex. By using the commutativity of 3.2, we have checked that the image χ of σ through the natural map $H^0(Y, G|_Y) \to H^0(Y, G|_Y)$ lies where we said it would, so we can extend χ to a section $u \in H^0(Z, G)$.

We now prove that $zero(u) = zero(\sigma)$. One inclusion is obvious because u extends σ, so any zero of σ is a zero of u. Let ρ be the image of u in $H^0(Z, \mathcal{O}(Y))$. Because we have the complex $0 \to H^0(Y, F|_Y) \to H^0(Y, G|_Y) \to H^0(Y, \mathcal{O}(Y)|_Y)$, the restriction of ρ to Y vanishes. But then $zero(\rho) \geq Y$. Since $zero(\rho)$ is an effective divisor of Z linearly equivalent to Y, we must have $zero(\rho) = Y$. This means that u cannot have zeros outside Y, and since u and σ coincide in Y, we are done. □

We apply the lemma for the above σ and the extension

$$0 \to F \to G \to p^*(L) \to 0 \tag{3.4}$$

to obtain an extension u in G of σ to $X \times X$ such that $zero(u) = \Delta_X$. We can therefore repeat all the arguments in the construction of the Beilinson spectral sequence to obtain (cf. [ApBr,06]):

Theorem 3.4. *With the notation above, for any complex vector bundle M of rank 2 on X, there exists a spectral sequence $E_r^{p,q}$ abutting to M if $p+q=0$ and to 0 otherwise, such that*

$$E_1^{p,q} = R^q p_{1*}(\bigwedge^{-p} G^\vee \otimes p_2^* M)$$

and this first sheet of the spectral sequence can be computed from the long exact sequence:

$$\ldots \to R^q p_{1*}(p^*(L^\vee) \otimes p_2^* M(p+1)) \otimes \Omega_{X/C}^{-p-1}(-p-1) \to$$

$$\to E_1^{p,q} \to H^q(X, M(p)) \otimes \Omega_{X/C}^{-p}(-p) \to$$

$$\to R^{q+1} p_{1*}(p^* L^\vee \otimes p_2^* M(p+1)) \otimes \Omega_{X/C}^{-p-1}(-p-1) \to E_1^{p,q+1} \to \ldots$$

Moreover, for all q:

$$E_1^{0,q} \simeq H^q(X, M) \otimes \mathcal{O}_X$$

$$E_1^{-2,q} \simeq R^q p_{1*}(p^* L^\vee \otimes p_2^* M(-1)) \otimes \mathcal{O}_X(-1) \otimes \pi^*(\bigwedge^2(\mathcal{E})).$$

Proof. The spectral sequence is, as we said, constructed as in 3.1. What needs explaining is the long sequence. For this, by taking exterior powers in the dual of 3.4, we obtain the short exact sequences for all negative p:

$$0 \to \bigwedge^{-p-1} F^\vee \otimes p^* L^\vee \to \bigwedge^{-p} G^\vee \to \bigwedge^{-p} F^\vee \to 0.$$

The long exact sequence is obtained by applying p_{1*} to this sequence after tensoring by $p_2^* M$ and then using the identifications

$$R^q p_{1*}(p^* L^\vee \otimes \bigwedge^{-p-1} F^\vee \otimes p_2^* M) \simeq R^q p_{1*}(p^* L^\vee \otimes p_2^* M(p+1, 0)) \otimes \Omega_{X/C}^{-p-1}(-p-1, 0)$$

$$R^q p_{1*}(\bigwedge^{-p} F^\vee \otimes p_2^* M) \simeq H^q(X, M(p, 0)) \otimes \Omega_{X/C}^{-p}(-p, 0)$$

in the long exact sequence for $R^* p_{1*}$ that follows. □

Remark 3.5. *The previous result can be generalized to the case when M is any complex vector bundle over any curve. For the proof of the general result, see [ApBr,06].*

This theorem has a simpler form if $C = \mathbb{P}^1$. In this case,

$$p^*(L) = p_1^*(0,1) \otimes p_2^*(0,1)$$

with the notation $(0,1) = \mathcal{O}(f)$ which does not depend on the choice of the fibre f. Then

$$R^q p_{1*}(p^*(L^\vee) \otimes p_2^*(M(p+1,0))) = H^q(X, M(p+1,-1)) \otimes (0,-1),$$

and as a consequence of 3.4, we have (cf. [Buch,87]):

Theorem 3.6. *Let $\mathcal{E} \to \mathbb{P}^1$ be a rank 2 vector bundle with deg$\mathcal{E} = n$ and let $X = \mathbb{P}(\mathcal{E}^*)$ be the associated rational ruled surface. Then, for any vector bundle M on X, there exists a spectral sequence $E_r^{p,q}$ abutting to M if $p+q=0$ and to 0 otherwise, and whose first sheet can be computed from the long exact sequence*

$$\ldots \to H^q(X, M(p+1,-1)) \otimes \Omega^{-p-1}_{X/\mathbb{P}^1}(-p-1,-1) \to E_1^{p,q} \to$$

$$\to H^q(X, M(p,0)) \otimes \Omega^{-p}_{X/\mathbb{P}^1}(-p,0) \to$$

$$\to H^{q+1}(X, M(p+1,-1)) \otimes \Omega^{-p-1}_{X/\mathbb{P}^1}(-p-1,-1) \to \ldots$$

Moreover, for all q, we have

$$E_1^{0,q} \simeq H^q(X, M) \otimes \mathcal{O}_X$$

and

$$E_1^{-2,q} \simeq H^q(X, M(-1,-1)) \otimes \mathcal{O}(-1, n-1).$$

Caution: Notice the sign change. In the definition of the ruled surface X_e, we had the associated rank two vector bundle $\mathcal{E}$, but it had degree $-e$ whereas in the above theorem we chose a vector bundle of degree n. The conclusion is that we have no problem as long as we remember to replace n by $-n$ in any of the previous formulas we had about ruled surfaces.

In his paper [Buch,87], Buchdahl actually uses this generalization of the Beilinson spectral sequence arising from the second version of the spectral sequence that in the case of a projective space was used to prove 3.2.

Theorem 3.7. *Under the same hypotheses as the previous theorem, there exists a spectral sequence $E_*^{*,*}$ with the following properties:*

a. $E_\infty^{p,q} \Rightarrow M$ *if* $p+q=0$ *and* $E_\infty^{p,q} \Rightarrow 0$ *otherwise.*

b. $E_1^{p,q} = 0$ *if* $|p+1| > 1$ *or* $|q-1| > 1$. $E_1^{0,q} = H^q(M) \otimes \mathcal{O}$ *and* $E_1^{-2,q} = H^q(M(-1,n-1)) \otimes (-1,-1)$.

c. There exists a long exact sequence

$$\ldots \to H^q(M(0,-1)) \otimes (0,-1) \to E_1^{-1,q} \to H^q(M(-1,n)) \otimes (-1,0) \to \ldots$$

Chapter 4

Two Splitting Criteria

In this chapter we apply the Beilinson type spectral sequence in order to prove some splitting criteria for rank 2 bundles on rational ruled surfaces. Throughout this chapter, Σ_n denotes the rational ruled surface corresponding to the rank 2 vector bundle of degree n, $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ on $\mathbb{P}^1$. Recall that this Σ_n corresponds to X_{-n} defined in the section on ruled surfaces and we must have $n \leq 0$. With this notation:

$$K(\Sigma_n) = (-2, n-2)$$

$$\Omega_{\Sigma_n/\mathbb{P}^1} = (-2, n)$$

by 2.10.b.

Denote $(1,0) = \mathcal{O}_{\Sigma_n}(1)$ and $(0,1) = \pi^*\mathcal{O}_{\mathbb{P}^1}(1)$. Recall the intersection formulas

$$(1,0).(1,0) = n$$

$$(1,0).(0,1) = 1$$

$$(0,1).(0,1) = 0.$$

The first result we are looking for is:

Theorem 4.1. *Let M be a holomorphic complex vector bundle of rank 2 on Σ_n. Then M is isomorphic to the direct sum*

$$M \simeq (0,-1) \oplus (-1, n-1) \tag{4.1}$$

if, and only if,

$$\begin{cases} h^0(M) = 0 & , h^1(M(-1,-1)) = 0 \\ h^2(M(0,-1)) = 0 & , h^2(M(-1,0)) = 0 \\ c_1(M) = (-1, n-2) & , c_2(M) = 1 \end{cases}.$$

In these conditions, the two direct summands of M can be recovered from the first Beilinson spectral sequence.

A step towards proving this theorem is:

Proposition 4.2. *Let M be a holomorphic complex vector bundle of rank 2 on Σ_n that is given by an extension*

$$0 \to (0,-1) \to M \to (-1, n-1) \to 0.$$

Then the extension is ***split****, $E_1^{-1,1} = E_\infty^{-1,1} = (0,-1)$ and $E_1^{-2,2} = E_\infty^{-2,2} = \mathcal{O}(-1, n-1)$, where E is the Beilinson spectral sequence of M given in 3.1.*

Proof. We first check that the extension is split. The extensions of this type are parameterized by

$$Ext^1((-1, n-1), (0, -1)) = H^1(\Sigma_n, (1, -n)).$$

Since $(1, -n).f = 1 \geq 0$, we have $h^1(1, -n) = h^1(\mathbb{P}^1, \pi_*(\mathcal{O}_{\Sigma_n}(1) \otimes \pi^*\mathcal{O}_{\mathbb{P}^1}(-n))) = h^1(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \otimes \mathcal{O}_{\mathbb{P}^1}(-n)) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2) \otimes (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(n-2)) = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-2)) = 0$, because n is negative. We have used the projection formula and Serre duality for $\mathbb{P}^1$. Therefore all the extensions of type 4.1 are split.

From 3.6 we have

$$E_1^{0,q} = H^q(\Sigma_n, M) \otimes \mathcal{O}_{\Sigma_n}$$

and

$$E_1^{-2,q} \simeq H^q(\Sigma_n, M(-1,-1)) \otimes \mathcal{O}(-1, n-1).$$

The long sequence that computes E_1 is:

$$\ldots \to H^q(M(p+1,-1)) \otimes (-2,n)^{-p-1}(-p-1,-1) \to E_1^{p,q} \to$$

$$\to H^q(M(p,0)) \otimes (-2,n)^{-p}(-p,0) \to$$

$$\to H^{q+1}(M(p+1,-1)) \otimes (-2,n)^{-p-1}(-p-1,-1) \to \ldots$$

We see from the sequence and from Grothendieck's vanishing theorem that $E_1^{p,q}$ is concentrated in $p \in \{-2,-1,0\}$ and $q \in \{0,1,2\}$.

We will first prove that all $E_1^{0,q}$ are zero by proving that $H^q(M) = 0$ for all q. From the splitting 4.1 we have for all i:

$$h^i(M) = h^i(0,-1) + h^i(-1, n-1).$$

$H^0(0,-1) = H^0(-1, n-1) = 0$ because $(0,-1)$ and $(-1, n-1)$ are not effective divisors, so $H^0(M) = 0$.

Because $(0,-1).f = 0 \geq 0$, $H^1(0,-1) = H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$ and $H^2(0,-1) = H^2(\mathbb{P}^1, \mathcal{O}(-1)) = 0$.

The line bundle $(-1, n-1)$ is not as easy to work with. It has negative degree -1 along the fibres, so the cohomology along the fibres vanishes in all dimensions. By Grauert 2.7, it follows that $R^i\pi_*(-1, n-1) = 0$ for all i and so $H^i(-1, n-1) = H^i(\pi_*(-1, n-1))$. But $\pi_*(-1, n-1) = \pi_*((-1,0) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(n-1))) = \pi_*(-1,0) \otimes \mathcal{O}_{\mathbb{P}^1}(n-1) = 0$ by the projection formula and 2.10. So all the cohomology of $(-1, n-1)$ vanishes. Summing up, gives $H^*(M) = 0$, so $E_1^{0,q} = 0$ for all q.

We now move our attention to $E_1^{-2,q}$. We know

$$E_1^{-2,q} = H^q(M(-1,-1)) \otimes \mathcal{O}(-1, n-1).$$

If we tensor the splitting 4.1 by $(-1, 1))$ and use additivity for cohomology on direct sums, we get for all i:

$$h^i(M(-1,-1)) = h^i(-1,-2) + h^i(-2, n-2).$$

The same arguments used for $(-1, n-1)$ can be used now to prove that all the cohomology of $(-1,-2)$ vanishes. $(-2, n-2)$ is the canonical sheaf, so by Serre Duality, $h^i(-2, n-2) = h^{2-i}(\Sigma_n, \mathcal{O}_{\Sigma_n}) = h^{2-i}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})$. So the cohomology of the canonical sheaf is zero in dimension 0 or 1, but is 1-dimensional as a complex vector space in dimension 2. If we combine these arguments with the splitting, we see that the canonical sheaf and $M(-1,-1)$ have the same cohomology. This proves that $E_1^{-2,2} \simeq (-1, n-1)$ and $E_1^{-2,q} = 0$ for $q \neq 2$.

To compute $E_q^{-1,q}$, we use the long exact sequence in 3.6 which is

$$\ldots \to H^q(M(0,-1)) \otimes (0,-1) \to E_1^{-1,q} \to$$

$$\to H^q(M(-1,0)) \otimes (-1,n) \to H^{q+1}(M(0,-1)) \otimes (0,-1) \to \ldots,$$

and the part that counts is

$$0 \to H^0(M(0,-1)) \otimes (0,-1) \to E_1^{-1,0} \to H^0(M(-1,0)) \otimes (-1,n) \to$$

$$\to H^1(M(0,-1)) \otimes (0,-1) \to E_1^{-1,1} \to H^1(M(-1,0)) \otimes (-1,n) \to$$

$$\to H^2(M(0,-1)) \otimes (0,-1) \to E_1^{-1,2} \to H^2(M(-1,0)) \otimes (-1,n) \to 0.$$

From 4.1 follow the obvious formulas:

$$h^i(M(0,-1)) = h^i(0,-2) + h^i(-1,n-2)$$

$$h^i(M(-1,0)) = h^i(-1,-1) + h^i(-2,n-1).$$

We have $(0,-2).f = 0$, so $h^i(0,-2) = h^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = h^{1-i}(\mathcal{O}_{\mathbb{P}^1})$ by the same techniques we have used before. So the cohomology of $(0,-2)$ is isomorphic to $\mathbb{C}$ in dimension 1 and vanishes elsewhere. Just like for $(-1,-2)$, we prove that the cohomology of $(-1,n-2)$ vanishes in all dimensions. In conclusion, $H^1(M(0,-1)) \simeq \mathbb{C}$ and $M(0,-1)$ has no other nonzero cohomology groups.

Again, $(-1,-1)$ has no cohomology. By Serre duality, $h^i(-2,n-1) = h^{2-i}(0,-1) = h^{2-i}(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)) = 0$ for all i. So all the cohomology groups of $M(-1,0)$ vanish.

We conclude that $E_1^{-1,1} = (0,-1)$ and $E_1^{-1,q} = 0$ for $q \neq 1$.

The first sheet of the spectral sequence E is:

$(-1,n-1)$	0	0
0	$(0,-1)$	0
0	0	0

By the construction of the spectral sequence and the way the arrows go in higher sheets, we see that $E_1 = E_\infty$. □

We now prove 4.1.

Proof. From the previous proof, we get that if M is given by the splitting 4.1, then all the required vanishing condition are fulfilled and $(0,-1)$ and $(-1,n-1)$ are recovered from the long sequence associated to the Beilinson spectral sequence. Computing the Chern classes is a very easy exercise. We are more interested in the converse.

If M has no nonzero sections, then the same must hold for $M(0,-1)$, $M(-1,0)$ and $M(-1,-1)$, because of the general result: If M is a vector bundle on a smooth projective variety such that M has no nonzero sections and if L is an invertible sheaf corresponding to an effective divisor, then $M \otimes L^\vee$ has no nonzero section. For a proof, notice that a nonzero section of $M \otimes L^\vee$ is a nontrivial (injective) sheaf morphism $\mathcal{O} \to M \otimes L^\vee$ which is equivalent to a morphism $L \to M$. Since L has sections i.e. there exists a nonzero morphism $\mathcal{O} \to L$, the composition $\mathcal{O} \to L \to M$ gives a nonzero section of M.

If we have the given vanishing conditions, then it immediately follows from the long sequence in 3.6:

$$0 \to H^0(M(0,-1)) \otimes (0,-1) \to E_1^{-1,0} \to H^0(M(-1,0)) \otimes (-1,n) \to$$
$$\to H^1(M(0,-1)) \otimes (0,-1) \to E_1^{-1,1} \to H^1(M(-1,0)) \otimes (-1,n) \to$$
$$\to H^2(M(0,-1)) \otimes (0,-1) \to E_1^{-1,2} \to H^2(M(-1,0)) \otimes (-1,n) \to 0,$$

that $E_1^{-1,0} = E_1^{-1,2} = 0$. All the sheets of the spectral sequence E are concentrated in $\overline{-2,0} \times \overline{0,2}$ and by what we know so far, we have for E_1:

$E_1^{-2,2}$	0	$E_1^{0,2}$
0	$E_1^{-1,1}$	$E_1^{0,1}$
0	0	0

.

We have used the formulas:

$$E_1^{0,q} = H^q(\Sigma_n, M) \otimes \mathcal{O}_{\Sigma_n}$$
$$E_1^{-2,q} \simeq H^q(\Sigma_n, M(-1,-1)) \otimes \mathcal{O}(-1, n-1).$$

$E_\infty^{p,q} = 0$ for $p+q \neq 0$ from 3.6. Keeping in mind the direction of the arrows in different sheets of E, we see that $E_2^{p,q} = E_\infty^{p,q}$ for

$$(p,q) \in \{(-2,0),\ (-1,2),\ (-1,1),\ (-1,0),\ (0,2)\}.$$

Because $E_1^{-1,2} = 0$, $E_1^{0,2} = E_2^{0,2} = E_\infty^{0,2} = 0$. From $E_1^{-2,1} = 0$, it follows that $E_1^{-1,1} = E_2^{-1,1} = E_\infty^{-1,1}$.

We compute $\chi(M)$. Riemann-Roch (2.19) tells us:

$$\chi(M) = \frac{1}{2}(-1, n-2).(1,0) - 1 + 2 = \frac{1}{2}(-n+n-2) - 1 + 2 = 0.$$

For any bundle L on Σ_n, a formal splitting of M as a sum of line bundles can be used to prove the following relations for the Chern classes of twists of M:

$$c_1(M \otimes L) = c_1(M) + 2c_1(L)$$
$$c_2(M \otimes L) = c_2(M) + c_1(M)c_1(L) + c_1^2(L).$$

With these relations and the Riemann-Roch formula, we can compute $\chi(M(-1,0)) = 0$, $\chi(M(0,-1)) = -1$ and $\chi(M(-1,-1)) = 1$. From these, from the formulas for $E_1^{0,q}$ and from the long sequence in 3.6, we draw the following picture:

$E_1^{-2,2}$	0	0
0	$(0,-1)$	0
0	0	0

,

which proves also $E_1^{-2,2} = E_\infty^{-2,2}$. But $E_1^{-2,2} = H^2(M(-1,-1)) \otimes (-1, n-1)$ and from Riemann-Roch again $h^2(M(-1,-1)) = 1$, so $E_1^{-2,2} = (-1, n-1)$.

From the general theory of spectral sequences, we see that we have a filtration of M by two terms $M = F^{-2} \supseteq F^{-1} \supseteq F^0 = 0$ such that $E_\infty^{-1,1} = F^{-1}$ and $E_\infty^{-2,2} = F^{-2}/F^{-1}$. This means that we have the extension $0 \to (0,-1) \to M \to (-1, n-1) \to 0$ that arises from the Beilinson spectral sequence. The proof ends here since we have already proved that any such extension is split. $\square$

Since we have a plural in the title, we should present another application of the Beilinson spectral sequence. Not stretching too far from the previous criterion, we can prove:

Theorem 4.3. *The rank 2 vector bundle M on Σ_n is isomorphic to*

$$\mathcal{O}_{\Sigma_n} \oplus (-1, n-1)$$

if, and only if, $c_1(M) = (-1, n-1)$, $c_2(M) = 0$, and $h^0(M(-1,0)) = h^1(M) = h^1(M(-1,-1)) = h^2(M(0,-1)) = 0$. In this case, the direct summands of M can be recovered from the Beilinson spectral sequence associated to M.

Even more, if $n = 0$ i.e. $\Sigma_0 = \mathbb{P}^1 \times \mathbb{P}^1$, then we can drop the assumptions on $h^0(M(-1,0))$ and $h^2(M(0,-1))$.

Proof. Since computing the Chern classes and the required cohomology spaces presents no difficulty and can be done in similar fashion to the previous case, we skip to the converse, but not before mentioning that just like before we can prove that any extension

$$0 \to (0,0) \to M \to (-1, n-1) \to 0$$

is actually split.

We will use (repetitively) this generalization of a result in the proof of the first splitting criterion:

Over any projective variety, if M is a vector bundle and L is a line bundle such that $h^0(L) > h^0(M)$, then $h^0(M \otimes L^\vee) = 0$.

To prove this, assume for a contradiction that $M \otimes L^\vee$ has nonzero sections. Any such section corresponds to an injective bundle map $\mathcal{O} \to M \otimes L^\vee$ which produces an injective map $L \to M$. Taking global sections yields an injective linear map $H^0(L) \to H^0(M)$ which cannot exist by a dimension count.

Serre duality and the previous result use $h^2(M(0,-1)) = 0$ to prove that $h^2(M) = 0$. This, $h^1(M) = 0$ and Riemann-Roch prove that $h^0(M) = 1$. Since $h^0(0,1) = 2$, the previous result shows $h^0(M(0,-1)) = 0$. Again our ubiquitous tool can be used to prove $h^0(M(-1,-1)) = 0$. Together with $h^1(M(-1,-1)) = 0$ and Riemann-Roch, $h^2(M(-1,-1)) = 1$. From $h^0(0,1) = 2$, $h^2(M(-1,-1)) = 1$, from Serre duality and our result, we obtain $h^2(M(-1,0) = 0$. Riemann-Roch can be used to prove that $h^1(M(0,-1)) = h^1(M(-1,0)) = 0$. By plugging all the data that we have collected so far in the long exact sequence computing $E_1^{-1,*}$, we see that all these sheaves are 0. From the spectral sequence we can extract an extension $0 \to (0,0) \to M \to (-1, n-1) \to 0$ that we know is split, so we are done.

Note that we could not have used our proposition as shown to prove that $h^2(M(-1,-1)) = 1$ implies $h^2(M(0,-1)) = 0$ or that $h^0(M) = 1$ implies $h^0(M(-1,0)) = 0$ if $n < 0$, because $h^0(1,0) = h^0(\mathbb{P}^1, \mathcal{O} \oplus \mathcal{O}(n)) = 1$. But if $n = 0$, then $h^0(1,0) = 2$ and the previous arguments work because in any case $h^0(M) \leq 1$ and $h^2(M(-1,-1)) \leq 1$ from $h^1(M) = h^1(M(-1,-1)) = 0$ and Riemann-Roch. So if $h^1(M) = 0$, then $h^0(M(-1,0)) = h^0(M(0,-1)) = h^0(M(-1,-1)) = 0$ and if $h^1(M(-1,-1)) = 0$, then the following vanishing results are proved: $h^2(M(-1,0)) = h^2(M(0,-1)) = h^2(M) = 0$. These, and again Riemann-Roch, imply $h^0(M) = h^2(M(-1,-1)) = 1$ and $h^1(M(-1,0)) = h^1(M(0,-1)) = 0$. Just like in the case $n < 0$, the conclusion follows. ☐

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