WEYL LAWS ON OPEN MANIFOLDS

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ABSTRACT. Under suitable invertibility hypothesis, the spectrum of the Dirac operator on certain open spin Riemannian manifolds is discrete, and obeys a growth law depending qualitatively on the (in)finiteness of the volume.

1. INTRODUCTION

Let M be a closed Riemannian manifold, $\mathcal{E} \to M$ a Hermitian vector bundle and $D : \mathcal{C}^\infty(M, \mathcal{E}) \to \mathcal{C}^\infty(M, \mathcal{E})$ an elliptic, symmetric, positive differential operator of order $k > 0$. Then the spectrum of D is discrete and the eigenvalues accumulate towards infinity obeying the Weyl law:

$$(1) \qquad \lim_{\lambda \to \infty} \frac{N(\lambda)}{\lambda^{\dim(M)/k}} = C$$

where $N(\lambda) := \#\{s \in \mathrm{Spec}(D); |s| < \lambda\}$ is the counting function and C is an explicit constant depending on $\dim(M)$ and on the principal symbol of D. This fact is classically proved using the heat trace expansion [26] when D is a Laplacian. Note that one can obtain better estimates of the remainder in (1), see [13].

In this paper we derive similar laws for Dirac operators on certain open spin Riemannian manifolds.

If the manifold M is not compact, not much can be said in general about the spectrum of D. However, if M is complete then the Laplacian on forms and the Dirac operator are essentially self-adjoint, and their essential spectra may be non-empty. Several results are also known for compact, incomplete manifolds with boundary. The original result of Weyl states that (1) holds for the Dirichlet Laplacian on a compact domain with smooth boundary in $\mathbb{R}^n$. Let now $\overline{X}$ be a compact manifold with boundary M, and X its interior. Then (1) holds [3] for the Dirac operator on X endowed with a product-type metric near M and a suitable non-local boundary condition. The results of [3] follow from

an explicit computation on the cylinder, which breaks down for perturbations of the product metric. A significant progress in this context was brought by Melrose [23], who computed the index of the Atiyah-Patodi-Singer non-local boundary value problem as an L^2 index, using his b-calculus on the complete manifold obtained from $\overline{X}$ by gluing infinite cylinders. Unlike the APS operator, the associated b-operator has non-empty essential spectrum.

The present work is motivated by a result of Bär [4], who showed that the essential spectrum of the Dirac operator on complete spin hyperbolic manifolds of finite volume is either empty or the whole real line. Thus, let $\overline{X}$ be a smooth n-dimensional compact manifold with closed boundary M, and x a boundary-defining function. This means

- $x \in \mathcal{C}^\infty(\overline{X}, [0, \infty))$;
- $M = \{x = 0\}$;
- dx never vanishes at $x = 0$.

Let g_0 be a *cusp metric* on $\overline{X}$, i.e., a (complete) Riemannian metric on $X := \overline{X} \setminus M$ which in local coordinates near the boundary takes the form

$$(2) \qquad g_0 = a_{00}(x, y)\frac{dx^2}{x^4} + \sum_{j=1}^{n-1} a_{0j}(x, y)\frac{dx}{x^2}dy_j + \sum_{i,j=1}^{n-1} a_{ij}(x, y)dy_i dy_j$$

such that the matrix $A = (a_{\alpha\beta})$ is smooth and non-degenerate down to $x = 0$. Let $h := \sum_{i,j=1}^{n-1} a_{ij}(0, y)dy_i dy_j$ denote the induced Riemannian metric on M. For simplicity, in this introduction we present our results under the additional assumption

$$(3) \qquad\qquad a_{00|M} \equiv 1, \quad a_{0j|M} \equiv 0.$$

Following Melrose [23], we call g_0 satisfying (3) an *exact cusp metric*. We are interested in the Riemannian metrics

$$(4) \qquad\qquad g_p := x^{2p}g_0$$

for $p \in \mathbb{R}$. Important particular cases are obtained when g_0 is a cylindrical metric on $(0, \epsilon) \times M \subset X$, $g_0 = x^{-4}dx^2 + h$. By the change of variable $y^q = x$, $q > 0$ (which changes the smooth structure of $\overline{X}$ but not that of X) and for varying $p \in \mathbb{R}$ we can get any metric of the type

$$x^a dx^2 + x^b h$$

with $a, b \in \mathbb{R}$ provided $b - a > 2$, in particular g_p cannot be conformally conical. Such metrics include all metric horns [18], and complete hyperbolic manifolds of finite volume. The above change of variable is rather artificial since it destroys smoothness of coefficients in the general case (2).

We first study the self-adjointness of the Dirac operator of the metric g_p acting on spinors.

Theorem 1. *Let X have a spin structure such that the Dirac operator D^h for the induced spin structure on (M, h) is invertible. Assume that g_0 satisfies (3). Then the Dirac operator D_p with domain $\mathcal{C}_c^\infty(X, \Sigma)$ is essentially self-adjoint in $L^2(X, \Sigma, g_p)$.*

This result is standard for $p \leq 1$ even without the invertibility hypothesis, since then g_p is complete. For $p > 1$ and $g_0 = x^{-4}dx^2 + h$ we recover a result from [18] through the change of variable $y = x^{p-1}$. For us the most interesting phenomena will occur for $0 < p \leq 1/n$.

It is important to understand how restrictive the invertibility hypothesis really is. On one hand, it is conjectured that any spin manifold M admits metrics with harmonic spinors, provided $\dim(M) \geq 3$. On the other hand, it is also conjectured that (for M connected) for *generic* metrics the dimension of the space of harmonic spinors is the minimal dimension prescribed by index theory. Parts of this conjecture were proved by Maier [20] and Bär and Dahl [5]. In our case, by the cobordism invariance of the index, the minimal dimension is 0. Thus, for M connected, the invertibility hypothesis is automatically satisfied for generic metrics if $\dim(M) \leq 4$, or if $\pi_1(M)$ vanishes or is cyclic of odd order. An indirect way of ensuring invertibility of D^h is (following Lichnerowicz) asking that $\text{scal}_h \geq 0$, with strict inequality at least at one point.

Incidentally, under the hypothesis of Theorem 1, D_p^+ is in fact Fredholm, and for $\dim X$ even,

$$\text{index}(D_p^+) = \int_{\overline{X}} \hat{A}(g_p) - \eta(D^h)/2.$$

The $\hat{A}$ form is a conformal invariant, hence, like $\text{index}(D_p^+)$, it is independent of p. We will not pursue this fact further.

Next, we look for conditions on p to ensure the vanishing of the essential spectrum of D_p.

Theorem 2. *Under the hypothesis of Theorem 1, the operator D_p has pure point spectrum if and only if $p > 0$.*

This greatly generalizes the above cited result of Bär [4].

Our main result describes the asymptotics of the counting function for the eigenvalues of D_p.

Theorem 3. *Let $\overline{X}$ be a compact n-dimensional spin manifold with boundary M, g_p the metric on X given by (4) and D_p the associated Dirac operator on spinors. Assume that (3) holds and that the spin structure on $\overline{X}$ induces an invertible Dirac operator on (M, h). For*

$p > 0$ let $N_p(\lambda)$ denote the counting function for the eigenvalues of D_p (well-defined by Theorem 2). Then

$$
N_p(\lambda) \approx
\begin{cases}
\lambda^n \dfrac{\operatorname{Vol}(X,g_p)\operatorname{Vol}(S^{n-1})2^{[n/2]}}{n(2\pi)^n} & \text{for } 1/n < p < \infty, \\[2ex]
\lambda^n \log \lambda \dfrac{\operatorname{Vol}(M,h)\operatorname{Vol}(S^{n-1})2^{[n/2]}}{(2\pi)^n} & \text{for } p = 1/n, \\[2ex]
\lambda^{1/p} \dfrac{\Gamma\left(\frac{1-p}{2p}\right)\zeta\left(D^h,\frac{1}{p}-1\right)\epsilon_n}{2\sqrt{\pi}\Gamma\left(\frac{1}{2p}\right)} & \text{for } 0 < p < 1/n,
\end{cases}
$$

where ϵ_n equals 1 if n is odd, 2 if n is even.

Here Γ is the Gamma function and $\zeta(D^h, z) := \operatorname{Tr}\left((D^h)^2\right)^{-z/2}$ is the zeta function of the absolute value of D^h, which is holomorphic for $\Re(z) > n - 1 = \dim(M)$.

It is interesting to note that $\operatorname{Vol}(X, g_p))$ is finite exactly when it arises as coefficient in the asymptotic law. In that case the formula bears no difference from the case of closed manifolds. The critical conformal factor $p = 1/n$ marks the start of a remarkable change in the growth rate. As p approaches 0, the metric becomes close to the asymptotically cylindrical metric, for which the essential spectrum does not vanish. This is reflected by the growth rate becoming infinite. Remarkably, some sort of spectral asymptotics for the Laplacian survive for g_0, as shown by Christiansen and Zworski [8]. Let us note that a similar (though analytically quite different) result was obtained by Maniccia and Panarese [22] for ends isometric to the Euclidean space outside a ball.

Weyl asymptotics may be deduced by two related approaches – heat trace expansions, via the beautiful and elementary result of Karamata, and from meromorphic extensions of zeta functions via the Ikehara theorem. Complex powers are weaker but more conceptual than the heat kernel, in particular they can be constructed in wide generality [1] following the method of Guillemin [11] and Bucicovschi [7]. Thus our strategy of proof will be to construct the complex powers of the square of our Dirac operators inside a calculus of pseudodifferential operators, and examine the leading pole of the associated zeta function. Rather than constructing a different calculus for each p, which would be highly problematic, we rely on Melrose's *cusp calculus*, corresponding to $p = 0$ (note that geometrically it does not correspond to cuspidal but to cylindrical ends). For each p we find an isometry between $L^2(\Sigma_p)$ and $L^2(\Sigma_0)$ under which D_p transforms to a cusp operator of order $(1, p)$. The zeta function of this operator is a meromorphic function, with first pole at $\max\{n, 1/p\}$, double if $n = 1/p$ and simple otherwise. We use then the Ikehara theorem [14] and its generalization by Delange [9] to deduce Theorem 3. The invertibility of D^h means that the operator D_p is fully elliptic, hence determines the domain of its closure and yields

Theorem 1. As for Theorem 2, the condition $p > 0$ is equivalent to the parametrix of D_p inside the cusp calculus being compact.

It might be possible to retrieve our results by exhibiting small-time asymptotics of the heat kernel of D_p^2. Paul Loya (see e.g., [19]) has constructed such heat kernels and studied their asymptotics in a variety of non-standard situations. Such asymptotics would be stronger than the meromorphy of the zeta function; in exchange, they would yield immediately Theorem 3 via Karamata's theorem.

Let us mention a third possible approach, i.e., the asymptotics of the wave trace. Even the first step of identifying this trace with a tempered distribution depends mildly on some *a priori* knowledge of Weyl asymptotics. Nevertheless, this approach would likely give also an estimate of the error term in Theorem 3.

We finally mention without proof that our results seem to extend to manifolds with corners of codimension $d \geq 2$. For various conformal factors, one gets as possible growth rates $\lambda^n \log^k \lambda$, $k = 0, \ldots, d$, and also $\lambda^{1/p} \log^k \lambda$, $0 < p < 1/n$, $k = 0, \ldots, d - 1$.

2. THE CUSP STRUCTURE

Consider a general cusp metric given by (2) without the condition (3), and the closely related metrics on X obtained by the conformal transformation of weight $x^p, p \in \mathbb{R}$. We call the metric $g_p = x^{2p} g_0$ a *p-metric*. Note that the volume of g_p is finite if and only if $p > 1/n$.
Example 4. A complete hyperbolic manifold X of finite volume is isometric (outside a compact set) to a warped product cylinder

$$((-\infty, 0] \times M, dt^2 + e^{-2t}h)$$

where M may be disconnected. Moreover, h is flat and independent of t. By the change of variables $x = e^{-t}$ near $-\infty$, the hyperbolic metric transforms to $x^2 h + \frac{dx^2}{x^2}$ near $x = 0$. Thus a hyperbolic metric of finite volume is a 1-metric on the compactification $\overline{X} := X \cup \{-\infty\} \times M$.

Let

$$^c\mathcal{V}(\overline{X}) := \{V \in \mathcal{C}^\infty(\overline{X}, T\overline{X}); dx(V) \in x^2 \mathcal{C}^\infty(\overline{X}, T\overline{X})\} \subset \mathcal{V}(\overline{X})$$

be the Lie sub-algebra of *cusp vector fields* on $\overline{X}$. In local coordinates (x, y_j) near $x = 0$, a vector field in $^c\mathcal{V}(\overline{X})$ can be uniquely written as a linear combination

$$a(x, y)x^2 \partial_x + \sum_{j=1}^{n-1} b_j(x, y)\partial_{y_j}$$

with smooth coefficients a, b_j. Thus ${}^c\mathcal{V}(\overline{X})$ is a locally free $\mathcal{C}^\infty(\overline{X})$-module. By the Serre-Swan theorem there exists a vector bundle ${}^c T\overline{X} \to \overline{X}$ such that ${}^c\mathcal{V}(\overline{X}) = \mathcal{C}^\infty(\overline{X}, {}^c T\overline{X})$. Moreover, the inclusion ${}^c\mathcal{V}(\overline{X}) \hookrightarrow \mathcal{C}^\infty(\overline{X}, T\overline{X})$ gives rise to a bundle map

$$c : {}^c T\overline{X} \to T\overline{X}$$

which is an isomorphism outside $x = 0$. A curious feature of the cusp tangent bundle is that it has a canonical *normal* subbundle to the boundary, rather than conormal as in the standard case. Namely, the vector field $x^2 \partial_x$ is well-defined (and non-vanishing) regardless of changes in the other local coordinates $y_1, \ldots, y_{n-1}$.

Lemma 5. *A metric g_0 on X extends to a Riemannian metric on the bundle ${}^c T\overline{X} \to \overline{X}$ if and only if it has the form* (2).

Proof. Clearly, cusp metrics have been defined so that this lemma holds. $\square$

In particular, g_0 takes the same form for different local coordinates y_j, which might not have been *a priori* clear. Notice that near M we can write

$$(5) \qquad g_0 = a_{00} \frac{dx^2}{x^4} + \frac{dx}{x} \otimes \theta^X + \theta^X \otimes \frac{dx}{x} + h^X$$

where θ^X and h^X are a smooth 1-form, respectively a symmetric 2-tensor which restricts to be non-degenerate on the level sets $\{x = \text{constant}\}$. There is an ambiguity in the definition of θ^X, h^X which can be removed by choosing a smooth extension Z_0 of the canonical vector field $x^2 \partial_x$ to $\overline{X}$ such that $dx(Z_0) = x^2$, and asking that θ^X, h^X vanish on Z_0. This vector field corresponds to a product decomposition of a neighborhood of M.

Let now $x' = ax + Bx^2$, with $a \in \mathcal{C}^\infty(M, \mathbb{R}_+^*)$, $B \in \mathcal{C}^\infty(\overline{X}, \mathbb{R})$, be the Taylor limited development of another boundary-defining function x'. Then x' and x define the same Lie algebra of cusp vector fields if and only if a is a constant. The set of such functions is called a cusp structure, which is henceforth fixed. However, it proves useful not to fix the function x inconsiderately inside the cusp structure since by Lemma 5, the metric g_0 is also a cusp metric with respect to x'. The canonical vector fields $x^2 \partial_x$ and $x'^2 \partial_{x'}$ are related by $x^2 \partial_x = a^{-1} x'^2 \partial_{x'}$, thus the cusp normal bundle is canonically trivialized up to a constant.

Let $\mathcal{I}$ be the Lie ideal $x \cdot {}^c\mathcal{V}(\overline{X})$ inside ${}^c\mathcal{V}(\overline{X})$. We also denote by $\mathcal{I}$ the space $x\mathcal{C}^\infty(\overline{X}, {}^c T^*\overline{X})$, and more generally any ideal of the form $x\mathcal{C}^\infty(\overline{X}, E)$ where E is a smooth vector bundle over $\overline{X}$.

Lemma 6. *Let g_0 be a cusp metric written in the form* (5). *Let $h := h^X_{|M}$, $\theta := \theta^X_{|M}$ and $q := a_{00}|_M \in \mathcal{C}^\infty(M)$. Define a 1-form α and a*

metric g^M on M by

$$\alpha := \frac{\theta}{q}, \quad g^M := \frac{h}{q} - \alpha \otimes \alpha.$$

Then q, g^M and the residue class $\alpha + d\Lambda^0(M)$ are independent, up to a multiplicative constant, of the boundary-defining function x inside the fixed cusp structure.

Proof. There is an ambiguity in the choice of θ^X and h^X but only up to $\mathcal{I}^2$, so θ and h are well-defined. Clearly, $0 < \|x^2 \partial_x - \theta^\#\|_{g_0}^2 = q - \|\theta\|_h^2$ so g^M is non-degenerate. We write

$$g_0 = a_{00}\left(\left(\frac{dx}{x^2} + \frac{\theta^X}{a_{00}}\right)^2 + \frac{h^X}{a_{00}} - \frac{\theta^X}{a_{00}} \otimes \frac{\theta^X}{a_{00}}\right).$$

Let x' be another boundary-defining function in the cusp structure of x, i.e., $x = ax' + Bx'^2$ with $a > 0$ constant, $B \in \mathcal{C}^\infty(\overline{X})$. Then

$$\frac{dx}{x^2} = \frac{1}{a}\frac{dx'}{x'^2} + \frac{dB}{a^2} + \mathcal{I}$$

so $q' = \frac{q}{a^2}$, $\alpha' = a\alpha + \frac{db}{a}$, where $b = B_{|M}$, and the lemma follows. $\square$

Melrose [23] calls cusp metrics with property (3) *exact*. In light of the above lemma, a cusp metric with $a_{00|M} \equiv 1$ and θ exact can be put in the form (3). By the Hodge decomposition, we can always modify x so that α is coexact with respect to g^M. It makes sense therefore to distinguish *closed* cusp metrics as being those with θ/q closed; in that case, a change of boundary-defining function can make θ/q harmonic with respect to g^M.

The invariant $\alpha + d\Lambda^0(M)$ has no equivalent in the case of finite-length cylinders, see [6].

3. p- versus 0-Dirac operators

Assume that (X, g_0) has a spin structure $\pi : P_{\mathrm{Spin}}(X) \to P_{SO}(X, g_0)$. Let $T_p : (TX, g_p) \to (TX, g_0)$ be the isometry defined by $V \mapsto x^p V$. It induces a $SO(n)$-isomorphism

$$T : P_{SO}(X, g_p) \to P_{SO}(X, g_0)$$

between the orthonormal frame bundles of (X, g_p) and (X, g_0). We define a spin structure on (X, g_p) by $T^{-1}\pi : P_{\mathrm{Spin}}(X) \to P_{SO}(X, g_p)$. Thus the cusp- and p-spinors have the same underlying vector bundle Σ, with the same metrics but different Clifford module structures. The Clifford action c^{g_p} with respect to g_p obeys the rule

$$c^{g_p}(V)\phi = c^{g_0}(T(V))\phi.$$

Proposition 7. *The unbounded operator D_p in $L^2(X, \Sigma, g_p)$ with domain $\mathcal{C}_c^\infty(X, \Sigma)$ is isometric to*

$$A_p = x^{-p} \left(D_0 - \tfrac{p}{2} x c^0 \left(\frac{dx}{x^2} \right) \right)$$

acting in $L^2(X, \Sigma, g_0)$ with domain $\mathcal{C}_c^\infty(X, \Sigma)$.

Proof. Note that dx/x^2 is a smooth section in ${}^cT^*\overline{X}$ over $\overline{X}$, thus $c^0(dx/x^2)$ is actually non-singular down to $x = 0$.

For every conformal transformation $\tilde{g} = f^2 g$, the Dirac operators $D_{\tilde{g}}$ and D_g are related by [12, Prop. 1.3]:

(6) $$D_{\tilde{g}} = f^{-\frac{n+1}{2}} D_g f^{\frac{n-1}{2}}.$$

Note also the relationship between the two volume densities:

$$dg_p = x^{np} dg_0.$$

Together with the fact that the metric on Σ is the same for all p, we see that the map

$$\mathcal{C}_c^\infty(X, \Sigma, g_0) \to \mathcal{C}_c^\infty(X, \Sigma, g_p), \quad \phi \mapsto x^{-\frac{np}{2}} \phi$$

is an isometry with respect to the L^2 inner product. Let $A_p := x^{\frac{np}{2}} D_p x^{-\frac{np}{2}}$ be the conjugate of D_p under this isometry. By using (6) with $f = x^p$, we get $A_p = x^{-\frac{p}{2}} D_p x^{-\frac{p}{2}}$. The commutation formula $[D_0, \mu] = c(d\mu)$ for $\mu \in \mathcal{C}^\infty(X)$ ends the proof. $\square$

In particular, the spectra of the closure of D_p and A_p are the same. It is therefore enough to study A_p, which is a cusp operator in the sense of the next section.

4. REVIEW OF CUSP OPERATORS

Definition 8. The algebra $\mathcal{D}_c(\overline{X})$ of cusp differential operators is the universal enveloping algebra of the Lie algebra ${}^c\mathcal{V}(\overline{X})$ of cusp vector fields on $\overline{X}$.

The definition extends easily to cusp operators acting on sections of smooth vector bundles over $\overline{X}$.

The spectral properties of cusp differential operators acting as unbounded operators in $L^2(X, g_0)$ are well-understood, thanks to the calculus of cusp pseudo-differential operators of Melrose, a cousin of the celebrated b-algebra. We review below several results about cusp operators, and then use them to derive the spectral properties of a class of differential operators on $L^2(X, g_p)$ which includes the p-Dirac operator in the spin case. The cusp algebra is described e.g., in [25, Appendix A]. It is a particular case of the fibered cusp algebra [21]

when the boundary fibers over a point, and of the cusp algebra on manifolds with corners [15] when the corners are of codimension 1.

4.1. The principal symbol.

Let A be a cusp differential operator. Then the principal symbol of A, which lives *a priori* on the cotangent bundle to $\overline{X}$, lifts in fact to ${}^c T^*\overline{X}$ through the (dual) canonical map $T^*\overline{X} \to {}^c T^*\overline{X}$. We call this lifted symbol the *cusp principal symbol*, or simply the principal symbol when no confusion can occur.

4.2. The normal operator.

This is a "boundary symbol" map, associating to any cusp operator $A \in \mathcal{D}_c(\overline{X}, \mathcal{E}, \mathcal{F})$ a family of differential operators on M with one real polynomial parameter ξ as follows:

$$\mathcal{N}(A)(\xi) = \left(e^{\frac{i\xi}{x}} A e^{-\frac{i\xi}{x}} \right) |_M$$

where restriction to M is justified by the mapping properties

$$A : \mathcal{C}^\infty(\overline{X}, \mathcal{E}) \to \mathcal{C}^\infty(\overline{X}, \mathcal{F})$$
$$A : x\mathcal{C}^\infty(\overline{X}, \mathcal{E}) \to x\mathcal{C}^\infty(\overline{X}, \mathcal{F})$$

and by the isomorphism $\mathcal{C}^\infty(M) = \mathcal{C}^\infty(\overline{X})/x\mathcal{C}^\infty(\overline{X})$. For example, $\mathcal{N}(ix^2\partial_x)(\xi) = -\xi$. The normal operator depends on x, nevertheless if we change x inside its cusp structure, $x' = ax + Bx^2$ with $a > 0$ constant and $B \in \mathcal{C}^\infty(\overline{X}, \mathbb{R})$, then the normal operator changes by a conjugation:

$$\mathcal{N}'(A)(\xi) = e^{-ib\xi/a^2} \mathcal{N}(A)(\xi/a) e^{ib\xi/a^2}$$

where $b := B_{|M}$. In particular, the invertibility of the normal operator for all values of $\xi \in \mathbb{R}$ is independent on the particular boundary-defining function x.

4.3. Cusp Sobolev spaces.

For $k \in \mathbb{N}$, the cusp Sobolev space $H_c^k(\overline{X}, S)$ is defined as the space of those sections $\phi \in L^2(\overline{X}, S)$ such that $P(\phi) \in L^2(\overline{X}, S)$ for all $P \in \mathcal{D}_c^k(\overline{X}, S)$. It is a Hilbertable space in which $\mathcal{C}_c^\infty(\overline{X}, S)$ sits as a dense subspace; moreover, for all $a \in \mathbb{R}$ an operator $A \in \mathcal{D}_c^k(\overline{X}, \mathcal{E}, \mathcal{F})$ has the mapping property

$$A : x^a H_c^k(\overline{X}, \mathcal{E}) \to x^a L_c^2(\overline{X}, \mathcal{F})$$

which justifies the definition

$$\mathcal{D}_c^{k,a} := x^{-a} \mathcal{D}_c^k, \quad H_c^{k,a} := x^a H^k.$$

4.4. **The cusp calculus.** As part of his larger program of quantizing singular structures, Melrose constructed a calculus $\Psi_c^{z,s}(X)$ of pseudo-differential operators, $z, s \in \mathbb{C}$, in which $\mathcal{D}_c$ sits as the differential (i.e., local) operators. For the construction we refer the reader to [25]. Operators in $\Psi_c^{z,s}(X)$ extend by duality to distributions and map $H_c^{k,a}(X)$ into $H_c^{k-\Re(z),a-\Re(s)}(X)$. The cusp-principal symbol and the normal operator extend from $\mathcal{D}_c(X)$ to this larger calculus. The normal operator takes as values certain families of pseudodifferential operators on M, called *suspended operators* [24].

Definition 9. An operator $A \in \Psi_c^{z,s}(X)$ is called *fully elliptic* if the cusp principal symbol $\sigma_z(x^s A)$ is invertible outside the zero section in $\mathcal{T}^*\overline{X}$, and if moreover $\mathcal{N}(x^s A)(\xi)$ is invertible as an operator from $H^{\Re(z)}(M)$ to $L^2(M)$ for all values of the parameter ξ.

Fully elliptic operators admit parametrices with respect to the two symbol structures. By a standard argument we get

Lemma 10. *Let $A \in \Psi_c^{z,s}(X)$ be fully elliptic. Then the domain of the closure of A inside $L_c^2(X)$ is $H_c^{\max\{\Re(z),0\},\max\{\Re(s),0\}}(X)$. If moreover A is symmetric then it is essentially self-adjoint.*

The following lemma is the basic tool for the analysis of the spectrum of cusp operators.

Lemma 11 (Cusp Rellich lemma)**.** *Let $\overline{X}$ be a compact manifold with boundary, x a boundary-defining function and g_0 a cusp metric on X. Then for $p, k \in \mathbb{R}$ the inclusion $x^p H_c^k(\overline{X}) \hookrightarrow L_c^2(\overline{X})$ is compact if and only if $p > 0$ and $k > 0$.*

Proof. Easy, using a partition of unity and the classical Rellich lemma on a compact exhaustion of a neighborhood of infinity in X. $\qquad\square$

Corollary 12 (Melrose)**.** *Let $z, s \in \mathbb{C}$, $a, b \in \mathbb{R}$ and $A \in \Psi_c^{z,s}(\overline{X}, \mathcal{E}, \mathcal{F})$. Then A is continuous as an operator*

$$A : H_c^{a+\Re(z),b+\Re(s)}(\overline{X}, \mathcal{E}) \to H_c^{a,b}(\overline{X}, \mathcal{F}),$$

and it is Fredholm if and only if it is fully elliptic.

Corollary 13. *The spectrum of a symmetric, fully elliptic operator $A \in \Psi_c^{z,s}(X, S)$ is discrete and accumulates towards infinity if and only if $\Re(z) > 0$ and $\Re(s) > 0$.*

5. Complex powers and eigenvalue growth

Analytic families of cusp operators have been introduced in [25] to define trace functionals in the spirit of Wodzicki's residue and to derive

an index formula. This idea has been exploited in [15] for cusp manifolds with corners, and in [17] for fibered-cusp metrics. We need here a variant of Melrose' and Nistor's results.

Proposition 14. *Let $p \in \mathbb{R}_+^*$ and $\mathbb{C} \supset \Omega \ni z \mapsto A(z) \in \Psi_c^{z,pz}(X, S)$ an analytic family of cusp operators indexed by a connected open subset $\Omega \subset \mathbb{C}$. Then the trace map*

$$\{z \in \Omega; \Re(z) < -n, \Re(pz) < -1\} \ni z \mapsto \mathrm{Tr}(A(z))$$

is holomorphic, and extends meromorphically to Ω with (at most double) possible poles occurring at the superposition of the sets $(\mathbb{N} - n)$ and $(\mathbb{N} - 1)/p$.

The first occurring pole is

- *simple at $z = -n$, if $n > 1/p$, with residue*

$$-\frac{1}{(2\pi)^n} \int_{cS^*\overline{X}} \mathrm{tr}\, \sigma_{-n} A(-n);$$

- *double at $z = -n$ if $n = 1/p$, with leading coefficient*

$$\frac{n}{(2\pi)^n} \int_{cS^*\overline{X}_{|M}} \mathrm{tr}\, \sigma_{-n}(\mathcal{N}(x^{-1}A(-n)));$$

- *simple at $z = -1/p$, if $n < 1/p$, with residue*

$$-\frac{n}{2\pi} \int_{\mathbb{R}} \mathrm{Tr}\, \mathcal{N}(x^{-1}A(-1/p))(\xi)d\xi.$$

Proof. (sketch) By an analytic family of operators of varying order we mean that A is holomorphic at $z \in \Omega$ inside the Banach space of bounded operators $B(H_c^{k,b}, L_c^2)$ for $k > \Re(z)$, $b > p\Re(z)$.

For a trace-class cusp operator A we write its trace as the integral of the Schwartz kernel of A on the diagonal in the cusp double-space [25]. By the Fourier inversion formula, this becomes the integral on $cT^*\overline{X}$ of the pointwise trace of the full symbol of A, times the (singular) canonical volume form given by the cusp symplectic form. Then we try to extend this integral meromorphically beyond the critical z. This extension follows by a standard elementary argument from the polyhomogeneity of the symbol at the boundary of the unit ball bundle, see [25] or [15]. Along the way we collect the information about the leading coefficient in the first pole (we could give all asymptotic coefficients in the same way, however only the first one matters in the rest of the paper) . We have omitted the canonical volume forms on $cS^*\overline{X}$, resp. $cS^*\overline{X}_{|M}$, which are obtained from the symplectic volume form by contraction with the canonical radial, respectively cusp-normal vector field to M. $\square$

Proposition 15. *Let $A \in \Psi_c^{a,b}(X, \mathcal{E})$, $a, b > 0$, be fully-elliptic, symmetric (hence self-adjoint) and positive. Then the complex powers A^z form an analytic family as in Proposition 14.*

Proof. (sketch) This is a particular case of [16, Thm. 7.1]. Here is the idea of the proof. Recall that for classical pseudodifferential operators on a closed manifold this was shown by Seeley [28]. Guillemin [11] gave another proof for scalar operators, which was extended by Bucicovschi to operators on vector bundles, or more generally with symbols taking values in a von Neumann algebra. This last method extends to algebras with "symbols" taking values in algebras like the suspended algebras, where the existence of complex powers is known. See also the recent paper [1] where complex powers are constructed in a more general framework. □

Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be the eigenvalues of A. The previous two propositions shows that the map

$$z \mapsto \sum_{j=1}^{\infty} \lambda_j^{-z} = \text{Tr}(A^{-z})$$

is holomorphic for large real parts of z, extends meromorphically to the complex plane with poles on the real axis, and describes the first pole. To deduce the asymptotic behaviour of λ_j we use a result due to Delange [9, Theorem III]:

Lemma 16. *Let $k \in \mathbb{N}^*, a \in \mathbb{R}_+^*$ and $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ be a non-decreasing sequence such that the series*

$$\sum_{j=0}^{\infty} \lambda_j^{-z}$$

is absolutely convergent for $\Re(z) > a$. Let $\zeta(z)$ denote the sum of this series, thus $\zeta(z)$ is holomorphic for $\{\Re(z) > a\}$. Assume that ζ extends to a meromorphic function in a neighborhood of $\{\Re(z) \geq a\}$, regular except at $z = a$ where it has a pole of order k. Let C be the coefficient of $(z - a)^{-k}$ in the Laurent expansion of ζ around $z = a$. Then the counting function $N(\lambda) := \max\{j \in \mathbb{N}; \lambda_j < \lambda\}$ satisfies

$$N(\lambda) \approx \frac{C}{a \cdot (k-1)!} \lambda^a (\log \lambda)^{k-1}.$$

Proof. Set $\alpha(t) := \sum_{j=1}^{\infty} H(t - \log \lambda_j)$, where H denotes the Heaviside function. Then

$$f(z) := \int_0^{\infty} e^{-zt} \alpha(t) dt = \frac{1}{z} \zeta(z)$$

satisfies the hypothesis of [9, Theorem III] with leading coefficient C/a at $z = a$. Thus

$$N(e^t) = \alpha(t) \approx \frac{C/a}{(k-1)!} e^{at} t^{k-1}. \qquad \square$$

We note here that Karamata's method applied to the zeta function gives a weaker result than Theorem 3, in terms of the asymptotics of the spectral function

$$I(n) = \sum_{\lambda_j < n} \frac{1}{\lambda_j}.$$

This is because Karamata's proof does not take into account the behavior of the zeta function outside the real line.

We obtain as a corollary a general result about p-operators.

Theorem 17. *Let X be the interior of a compact manifold with boundary $\overline{X}$, with metric g_p given by (4), $p \in \mathbb{R}$. Let Σ be a hermitian vector bundle over $\overline{X}$, $q > 0$ and $D \in \Psi_c^{1,q}(X, \Sigma)$ a symmetric fully elliptic cusp operator. Then D is essentially self-adjoint on $\mathcal{C}_c^\infty(X, \Sigma)$ with discrete spectrum, and the counting function of its eigenvalues satisfies*

$$N(\lambda) \approx \begin{cases} C\lambda^n & \text{for } 1/n < q, \\ C\lambda^n \log \lambda & \text{for } q = 1/n, \\ C\lambda^{1/q} & \text{for } 0 < q < 1/n. \end{cases}$$

The constant C is explicitly given by Proposition 14 and Lemma 16.

Proof. D_q is isometric to $A := x^{np/2} D x^{-np/2} \in \Psi_c^{1,q}(X, \sigma)$ acting in $L^2(X, \sigma, g_0)$ as in Proposition 7. Now $\mathcal{N}(x^q A) = \mathcal{N}(x^q D)$ by the multiplicativity of the normal operator. So A is also fully elliptic. The theorem follows from Corollary 13 and the results of this section. $\square$

Notably, the invariance of the normal operator under conjugation by powers of x breaks down for b-operators. As a result, the analysis of conical singularities is more delicate.

6. The Dirac operator of a cusp metric

Let us return to the setting of Section 3. Assume that (X, g_0) is spin, and fix a spin structure $P_{\mathrm{Spin}}(X) \to P_{SO}(X)$. Using Lemma 5 we extend $P_{\mathrm{Spin}}(X)$ to a spin structure for the Riemannian bundle $({}^c T\overline{X}, g_0)$. We claim that the Dirac operator D_0 associated to the cusp metric g_0 given by (2) is a cusp differential operator (of order 1). For this we first need to describe the smooth structure of the spinor bundle over $\overline{X}$. The problem is that g_0 is singular over the boundary of $\overline{X}$. Choose a product decomposition

$$\imath : M \times [0, \epsilon) \to \overline{X}$$

near M such that x becomes the variable in $[0, \epsilon)$, i.e., $x(\imath(y, t)) = t$. Then h^X (see (5)) defines a Riemannian metric on the bundle $TM \times [0, \epsilon)$, such that induced map $\imath : TM \times [0, \epsilon) \to {}^c T\overline{X}$ is an isometric injection.

Choose a local orthonormal frame on $M \times \{0\}$, say $Y_1, \ldots, Y_{n-1}$. Let $Y_1(x), \ldots, Y_{n-1}(x)$ be the orthonormal frame obtained by the Gramm-Schmidt procedure from $Y_1, \ldots, Y_{n-1}$, viewed as a local frame on $M \times \{x\}$. Let Y_0 be inward-pointing unit vector field normal to $M \times \{x\}$ for $0 < x < \epsilon$. Again by (2), Y_0 extends to a smooth cusp vector field down to $x = 0$. Thus we have found a local orthonormal frame in $({}^c T\overline{X}, g_0)$ near $x = 0$ which defines a smooth structure on the orthonormal frame bundle of $({}^c T\overline{X}, g_0)$, and hence on the spin bundle $P_{\mathrm{Spin}}({}^c T\overline{X})$.

An oriented orthonormal frame in $TM \times [0, \epsilon)$ gives rise to a unique oriented orthonormal frame in ${}^c T\overline{X}$ by adding Y_0 as the first component. Thus we get a $SO(n-1)$-map between the principal frame bundles. Define a spin structure on $M \times \{x\}$, $0 \le x < \epsilon$ as the pull-back of the spin bundle of $\overline{X}$ via this map. From the definition there exists a $\mathrm{Spin}(n-1)$-injection $P_{\mathrm{Spin}}(M \times \{x\}) \to P_{\mathrm{Spin}}(\overline{X})$.

Let Σ_n be the spinor representation of $\mathrm{Spin}(n)$. If n is odd, then $\Sigma_n = \Sigma_{n-1} = \Sigma_{n-1}^+ \oplus \Sigma_{n-1}^-$, while for n even, $\Sigma_n = \Sigma_{n-1} \oplus \Sigma_{n-1}$. For n odd, the Clifford action of the extra vector V_0 on Σ_n is given by $c_{\Sigma_n}^0 = \left[\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right]$, and at the same time, $c_{\Sigma_n}^j = c_{\Sigma_n}^0 c_{\Sigma_{n-1}}^j$ for $j = 1, \ldots, n-1$. For n even, V_0 acts as $c_{\Sigma_n}^0 = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, while $c_{\Sigma_n}^j = \begin{bmatrix} 0 & c_{\Sigma_{n-1}}^j \\ c_{\Sigma_{n-1}}^j & 0 \end{bmatrix}$.

Let $\Sigma M \times \{x\}$, $\Sigma \overline{X}$ be the spinor bundles over $M \times \{x\}$, respectively $\overline{X}$. It follows from the above discussion that $\Sigma \overline{X}_{|M \times \{x\}}$ can be identified with $\Sigma M \times \{x\}$ (for n odd), respectively with $\mathbb{R}^2 \otimes \Sigma M \times \{x\}$ (for n even) as smooth bundles over $\overline{X}$. We wish to compute D_0, the Dirac operator of the metric g_0, in terms of the Dirac operators $D^{h_x^X}$ on $M \times \{x\}$. First, like for all Dirac operators, $\sigma_2(D_0^2)(\xi) = g_0(\xi, \xi)$. Lemma 5 implies that D_0^2 (and hence D_0) is cusp-elliptic.

Lemma 18. *The Levi-Civita covariant derivative ∇ on (X, g_0) extends to a differential operator with smooth coefficients*

$$\nabla : \mathcal{C}^\infty(\overline{X}, {}^c T\overline{X}) \to \mathcal{C}^\infty(\overline{X}, {}^c T^*\overline{X} \otimes {}^c T\overline{X}).$$

Proof. Use the Koszul formula for ∇ applied to vectors Y_j. Since ${}^c T\overline{X}$ is stable by Lie bracket and g_0 is a true metric on ${}^c T\overline{X}$, the assertion follows immediately. $\qquad\square$

Let $\tilde{Y}$ be a local section in the principal spin bundle over $\overline{X}$ which lifts the local orthonormal frame $(Y_0, \ldots, Y_{n-1})$ constructed previously. Let ϕ be a local section in $\Sigma \overline{X}$. There exists $s : \overline{X} \to \Sigma_n$ smooth

so that $\phi = [\tilde{Y}, s]$, where the square bracket denotes the equivalence class modulo $\mathrm{Spin}(n)$. In other words, ϕ is represented by s in the trivialization given by $\tilde{Y}$. Recall that in such a trivialization, the Levi-Civita covariant derivative of the spinor ϕ takes the form (at least outside the boundary):

$$\nabla_Y \phi = Y(s) + \tfrac{1}{2} \sum_{0 \le i < j \le n-1} g_0(\nabla_Y Y_i, Y_j) c^i c^j \phi.$$

From Lemma 18 it follows that ∇ maps $\mathcal{C}^\infty(\overline{X}, \Sigma\overline{X})$ to $\mathcal{C}^\infty(\overline{X}, \mathcal{T}^*\overline{X} \otimes \Sigma\overline{X})$. For n odd,

(7)
$$\begin{aligned}
D_0 \phi = {}& c^0 (D^h \phi + Y_0(s)) \\
& + \tfrac{1}{2} \sum_{1 \le i < j \le n-1} \langle \nabla_{Y_0} Y_i, Y_j \rangle c^0 c^i c^j \phi \\
& + \tfrac{1}{2} \sum_{1 \le i \le n-1} \langle \nabla_{Y_i} Y_0, Y_i \rangle c^0 \phi - \tfrac{1}{2} c^0 c(\nabla_{Y_0} Y_0) \phi.
\end{aligned}$$

For n even,

$$\begin{aligned}
D_0^+ \phi = {}& D^h + Y_0(s) + \tfrac{1}{2} \sum_{1 \le i < j \le n-1} \langle \nabla_{Y_0} Y_i, Y_j \rangle c^i c^j \phi \\
& + \tfrac{1}{2} \sum_{1 \le i \le n-1} \langle \nabla_{Y_i} Y_0, Y_i \rangle \phi - \tfrac{1}{2} c(\nabla_{Y_0} Y_0) \phi.
\end{aligned}$$

and so $D_0 \in \mathcal{D}_c^1(\overline{X}, \Sigma\overline{X})$.

We make now the additional hypothesis that the metric g_0 satisfies (3), i.e., it is an exact cusp metric [25].

Lemma 19. *If g_0 is an exact cusp metric then*

- *for n odd, $\mathcal{N}(D_0)(\xi) = c^0(D^h + i\xi)$;*
- *for n even, $\mathcal{N}(D_0)(\xi) = \begin{bmatrix} 0 & D^h - i\xi \\ D^h + i\xi & 0 \end{bmatrix}$.*

Thus D_0 is fully elliptic if and only if D^h is invertible.

Proof. Recall that $\mathcal{I}$ is the ideal $x \cdot {}^c\mathcal{V}(\overline{X}) = \ker \mathcal{N}$ inside ${}^c\mathcal{V}(\overline{X})$. From the hypothesis (3), $Y_0 = x^2 \partial_x + \mathcal{I}$. It follows that $\mathcal{N}(Y_0) = i\xi$ and also $[Y_0, Y_j] \in \mathcal{I}$ for all j. The Koszul formula shows that $\nabla_{Y_0} Y_i, \nabla_{Y_i} Y_0$ and $\nabla_{Y_0} Y_0$ all belong to $\mathcal{I}$. Since $\mathcal{N}(\mathcal{I}) = 0$, formula (7) gives the desired formula. Then clearly $\mathcal{N}(D_0)(\xi)$ is invertible for all $\xi \in \mathbb{R}$ if and only if D^h is invertible. $\square$

For example, if $\mathrm{scal}_h(M) \ge 0$ and does not vanish identically on any connected component of M then, by Lichnerowicz's formula, D_0 is fully elliptic.

7. Proofs of the main results

In this section (3) is assumed to hold.

Proof of Theorem 1. By Proposition 7, we see that for each p, D_p is isometric to an operator $A_p \in \mathcal{D}_c^{1,p}(\overline{X}, \Sigma)$ with the property that $A_p - x^{-p}D_0 \in \mathcal{D}_c^{0,p-1}(\overline{X}, \Sigma)$. This means that A_p is fully elliptic if and only D_0 is. Lemmata 10 and 19 end the proof. $\qquad\square$

Proof of Theorem 2. Assume that D^h is invertible, so $D_p \in \Psi_c^{1,p}(\overline{X}, S(\overline{X}))$ is fully elliptic by Lemma 19. The result follows from Corollary 13. $\qquad\square$

Proof of Theorem 3. We assume that $p > 0$ and D^h is invertible. Then A_p is fully elliptic and has discrete spectrum, and moreover $P_{\ker A_p}$, the orthogonal projection onto the kernel of A_p in $L^2(X, \Sigma, g_0)$, belongs to $\Psi_c^{-\infty,-\infty}(X, \Sigma)$. We apply Proposition 14 to the analytic family $A(z) := (A_p^2 + P_{\ker A_p})^{z/2} \in \Psi_c^{z,pz}(\overline{X}, \Sigma)$ to find that the zeta function of A_p^2 has either a simple or a double leading pole. Note that $\sigma_1(A(1)) = \|\cdot\|_{g_p}$. From Lemma 19 we get

$$\mathcal{N}(x^{-1}A(1)^{-1/p})(\xi) = ((D^h)^2 + \xi^2)^{-\frac{1}{2p}},$$

for n odd, respectively

$$\mathcal{N}(x^{-1}A(1)^{-1/p})(\xi) = \begin{bmatrix} ((D^h)^2 + \xi^2)^{-\frac{1}{2p}} & 0 \\ 0 & ((D^h)^2 + \xi^2)^{-\frac{1}{2p}} \end{bmatrix}$$

for n even. Thus

$$\operatorname{tr} \sigma_{-n} A(-n)_{|^cS^*\overline{X}} = \dim \Sigma(n) = 2^{[n/2]};$$
$$\operatorname{tr} \sigma_{-n}(\mathcal{N}(x^{-1}A(-n)))_{|^cS^*\overline{X}_{|M}} = \dim \Sigma(n) = 2^{[n/2]};$$
$$\operatorname{tr} \mathcal{N}(x^{-1}A(-1/p))(\xi) = \epsilon_n \operatorname{tr}((D^h)^2 + \xi^2)^{-\frac{1}{2p}}.$$

The last identity implies that

$$\int_{\mathbb{R}} \operatorname{Tr} \mathcal{N}(x^{-1}A(-1/p))(\xi)d\xi = \epsilon_n \operatorname{Tr}((D^h)^2)^{-\frac{1}{2p}+\frac{1}{2}} \int_{\mathbb{R}} (1 + \xi^2)^{-\frac{1}{2p}} d\xi$$

$$= \epsilon_n \zeta(D^h, \tfrac{1}{p} - 1) \frac{\sqrt{\pi}\Gamma\left(\frac{1}{2p} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2p}\right)}$$

where the integral was evaluated using [27, equation (4)]. The result follows from these formulae, Proposition 14 and Lemma 16. $\qquad\square$

Theorem 3 applies with minimal modifications to Dirac operators twisted by a bundle $W \to \overline{X}$ with smooth connection down to $x = 0$,

provided that the induced twisted Dirac operator $D^{M,W}$ over M is invertible. Namely, for $p \geq 1/n$ we have to multiply by $\dim(W)$ the coefficient of λ^n, resp. $\lambda^n \log \lambda$, while for $0 < p < 1/n$ we must replace D^h by $D^{M,W}$ inside the zeta function. If M is connected then the obstruction to the invertibility of $D^{M,W}$ given by index theory vanishes, as a consequence of cobordism invariance. Thus, by Anghel's result [2], $D^{M,W}$ is invertible for generic connections on W, at least if $\dim(M) \leq 4$.

8. WEYL LAWS FOR GENERAL CUSP METRICS

Let us consider the case of a general cusp metric. Write g_0 in the form (5) and fix a product decomposition near M so that θ^X and h^X are uniquely defined. Let $\theta^\#$ be the dual vector field to θ^X relative to h^X. Set
$$F := 1/\|x^2\partial_x - \theta^\#\|_{g_0}, \quad f := F_{|M} \in \mathcal{C}^\infty(M).$$

Proposition 20. *The normal operator of the Dirac operator corresponding to g_0 is given by*

$$(8) \qquad \begin{aligned} \mathcal{N}(D_0)(\xi) = c^0(D^h + if\xi - f\nabla_{\theta^\#} \\ + \tfrac{1}{4}fc(d\theta) - \tfrac{1}{4}f\operatorname{tr}(L_{\theta^\#}h(0)) - \tfrac{1}{2}c(df)/f) \end{aligned}$$

for n odd, respectively

$$\begin{aligned} \mathcal{N}(D_0^+)(\xi) = D^h + if\xi - f\nabla_{\theta^\#} \\ + \tfrac{1}{4}fc(d\theta) - \tfrac{1}{4}f\operatorname{tr}(L_{\theta^\#}h(0)) - \tfrac{1}{2}c(df)/f \end{aligned}$$

for n even.

Proof. The idea is to expand formula (7) for D_0. Note that $Y_0 = F(x^2\partial_x - \theta^\#)$, and therefore

$$\begin{aligned} (9) \qquad [Y_i, Y_0] &= \frac{Y_i(F)}{F}Y_0 + F(-[Y_i, \theta^\#] - x^2\partial_x(Y_i)) \\ &\equiv \frac{Y_i(f)}{f}Y_0 - f[Y_i, \theta^\#] + \mathcal{I}^2 \end{aligned}$$

(recall that $\mathcal{I}$ is the ideal $x^c\mathcal{V}(\overline{X})$ annulated by $\mathcal{N}$). Thus

$$\begin{aligned} \mathcal{N}(D_0)(\xi) = c^0 \cdot (D^h + if\xi - f\nabla_{\theta^\#}) \\ + \tfrac{1}{2}\sum_{1\leq i<j\leq n-1}\langle\nabla_{fx^2\partial_x}Y_i, Y_j\rangle_{|x=0}c^0c^ic^j \\ + \tfrac{1}{2}\sum_{1\leq i\leq n-1}\langle\nabla_{Y_i}Y_0, Y_i\rangle_{|x=0}c^0 \\ - \tfrac{1}{2}\sum_{j=1}^{n-1}\langle\nabla_{Y_0}Y_0, Y_j\rangle_{|x=0}c^0c^j. \end{aligned}$$

We already see here the first three terms from (8). Koszul's formula gives

$$2\langle \nabla_{x^2\partial_x} Y_i, Y_j \rangle = Y_i(\theta_j) - Y_j(\theta_i) - \theta([Y_i, Y_j]) = d\theta(Y_i, Y_j)$$

so by summing over $i < j$ we get the fourth term. Since Y_i has constant length 1, it follows by (9) that

$$\langle \nabla_{Y_i} Y_0, Y_i \rangle = \langle [Y_i, Y_0], Y_i \rangle = -f\langle [Y_i, \theta^\#], Y_i \rangle + \mathcal{I}^2.$$

Use

$$\sum_{i=1}^{n-1} \langle [Y_i, \theta^\#], Y_i \rangle = \tfrac{1}{2}\operatorname{tr}(L_{\theta^\#} h(0)) = \frac{L_{\theta^\#} dh(0)}{dh(0)}$$

to get the fifth term. As for the last term, again by (9) we have

$$\langle \nabla_{Y_0} Y_0, Y_j \rangle = -\langle Y_0, \nabla_{Y_0} Y_j \rangle = \langle Y_0, [Y_j, Y_0] \rangle \equiv Y_j(f)/f + \mathcal{I}^2.$$

This settles the case n odd. The case n even is done in the same way. $\qquad\square$

We wish now to give conditions on g_0 so that $\mathcal{N}(D_0)(\xi)$ is invertible for all $\xi \in \mathbb{R}$. If $\theta = 0$ this condition was seen to be equivalent to $D^{M,h(0)}$ being invertible. For the general case, we rewrite (8) as

$$\mathcal{N}(D_0)(\xi)$$
$$= c^0 f \left(f^{-\frac{1}{2}} D^h f^{-\frac{1}{2}} + i\xi - \left(\nabla_{\theta^\#} + \tfrac{1}{2}\frac{L_{\theta^\#} dh(0)}{dh(0)} \right) + \tfrac{1}{4} c(d\theta) \right).$$

It is not clear if a reasonable condition on h, f and θ exists so that the above operator is invertible for all $\xi \in \mathbb{R}$. Recall that the metric h is not canonically determined by g_0, since it depends on the choice of x inside the fixed cusp structure. Thus, we rewrite $\mathcal{N}(D_0)$ in terms of the canonical metric g^M on M, see Lemma 6, although we must still fix x in order to define $\mathcal{N}$.

Proposition 21. *Let g_0 be a cusp metric on X and α, g^M, q defined in Lemma 6. Then for n odd,*

$$\mathcal{N}(D_0)(\xi) = c^0 q^{-\frac{n+1}{2}} (D^{g^M} + i\xi(1 - c(\alpha)) + \tfrac{1}{4}c(d\alpha)) q^{-\frac{n-1}{2}}$$

while for n even,

$$\mathcal{N}(D_0^+)(\xi) = q^{-\frac{n+1}{2}} (D^{g^M} + i\xi(1 - c(\alpha)) + \tfrac{1}{4}c(d\alpha)) q^{-\frac{n-1}{2}}.$$

Proof. Recall that in the expression (5) for g_0, the tensors θ^X, h^X are well-defined only up to $\mathcal{I}^2$. Fix a product decomposition of $\overline{X}$ near M, thus removing the ambiguity.

First, assume that $a_{00} = 1$ near M, so $g_0 = (dx/x^2 + \theta^X)^2 + h^X - \theta^X \otimes \theta^X$. Let $\tilde{Y}_0 := x^2 \partial_x$. Define an isometric embedding

$$(TM \times [0, \infty), h^X) \to ({}^c T\overline{X}, g_0), \quad Y \mapsto \tilde{Y} := Y - \theta(Y)\tilde{Y}_0.$$

This allows us to compare spinors on X and on M. Choose a local orthonormal frame $Y_1, \ldots, Y_{n-1}$ on M, transport it to X using the product decomposition and re-orthonormalize it using Gramm-Schmidt. Note that the frame $\{Y_j\}$ is different from the frame with the same name from Section 6, since they are orthonormal with respect to different metrics. Then $\tilde{Y}_0, \ldots, \tilde{Y}_{n-1}$ is a (smooth) orthonormal frame on $T\overline{X}$. We use i, j, k to denote a subscript in $\{1, \ldots, n-1\}$. Notice that

$$[\tilde{Y}_i, \tilde{Y}_j] = \widetilde{[Y_i, Y_j]} - d\theta(Y_i, Y_j) + \mathcal{I}^2$$
$$[\tilde{Y}_0, \tilde{Y}_i] \in \mathcal{I}^2$$

so

$$(\nabla_{\tilde{Y}_i}\tilde{Y}_j, \tilde{Y}_k) = (\nabla_{Y_i}Y_j, Y_k)$$
$$\nabla_{\tilde{Y}_0}\tilde{Y}_0 \in \mathcal{I}^2$$
$$(\nabla_{\tilde{Y}_i}\tilde{Y}_0, \tilde{Y}_i) \in \mathcal{I}^2$$
$$(\nabla_{\tilde{Y}_0}\tilde{Y}_j, \tilde{Y}_k) = \tfrac{1}{2}d\theta(\tilde{Y}_i, \tilde{Y}_j) + \mathcal{I}^2.$$

Using (7), under the assumption that $a_{00} = 1$ we get immediately in the case n is odd

$$\mathcal{N}(D_0)(\xi) = c^0(D^{g^M} + i\xi(1 - c(\theta)) + \tfrac{1}{4}c(d\theta))).$$

Remove now the assumption that $a_{00} = 1$ in a neighborhood of M. Using the formula for the conformal change for Dirac operators (6), we get the desired expression. The case n even is entirely similar. $\square$

As a corollary, we give a condition for the full ellipticity of D_p in some cases where g_0 is not exact.

Corollary 22. *Let g_0 be a cusp metric with α closed (by Lemma 6, this condition is independent on the boundary-defining function). Then D_p is fully elliptic if and only if D^{g^M} is invertible.*

Proof. By Proposition 21, we must check the invertibility of the family of operators

$$P(\xi) := D^{g^M} - i\xi c(\alpha) + i\xi$$

for all $\xi \in \mathbb{R}$. But $P(\xi)$ is invertible if and only if $P^*(\xi)P(\xi) = (D^{g^M} - i\xi c(\alpha))^2 + \xi^2$ is invertible. This holds automatically for $\xi \neq 0$, and for $\xi = 0$ it is equivalent to D^{g^M} being invertible. $\square$

Product cylindrical metrics have been studied by several authors starting with [3]. Such metrics can be considered either as b- or as cusp metrics; they can actually be treated by elementary methods using separation of variables. Melrose [23] has successfully studied exact b-metrics,

which are in some sense only asymptotically cylindrical. Corollary 22 allows even more geometric structure to be embedded in the metric. If $H^1(M, \mathbb{R}) \neq 0$, it simply says that our results on Weyl laws hold for some metrics (i.e., closed cusp metrics) which are not exact.

Let us finally state our most general result about Dirac eigenvalues of cusp metrics.

Theorem 23. *Assume that the normal operator of D_0, computed in two different ways in Propositions 20, 21, is invertible. Then for $p > 0$, D_p is essentially self-adjoint with pure point spectrum accumulating towards infinity. The rate of growth is given by Theorem 3 if $p \geq 1/n$, and by Theorem 17 if $0 < p < 1/n$.*

$\square$

This theorem has the drawback that the invertibility of $\mathcal{N}(D_0)$ must be assumed. The only case where we have found a reasonable condition for invertibility is for closed cusp metrics (Corollary 22). We are therefore led to the following

Problem. Let (M, g^M) be a connected compact Riemannian spin manifold and θ a 1-form on M. Find conditions on θ, g^M so that the family of operators

$$D^{g^M} + i\xi(1 - c(\theta)) + \tfrac{1}{4}c(d\theta))$$

is invertible for all $\xi \in \mathbb{R}$.

Another variant of the problem would be to assume additionally that M vanishes in the Spin bordism ring. Mere invertibility of D^{g^M} is not enough in general, as shown by the following example (courtesy of Andrei Moroianu, see also [6]):

Example 24. Let $M = S^2$ with an arbitrary metric h and D the associated Dirac operator. Then D^h is invertible, since the Dirac operator of the standard metric is invertible, and any two metrics on S^2 are conformally isometric. Nevertheless, there exist a metric h and $\theta \in \Lambda^1(S^2)$ so that $D^h + c(\theta)$ is not invertible. For this, take h to be the metric induced from an immersion of S^2 in $\mathbb{R}^3$ with total mean curvature 0. First, there exists such an immersion, since we can deform S^2 by stretching cylinders of positive or negative mean curvature, thus increasing or decreasing the total mean curvature at will. Secondly, take $\tilde{\phi}$ to be a constant spinor on $\mathbb{R}^3$, and pull it back to a spinor $\phi = (\phi^+, \phi^-)$ on M. Then by [10, Proposition 2],

$$D^h\phi^+ = -iH\phi^-, \qquad\qquad D^h\phi^- = iH\phi^+,$$

where H is the mean curvature function of (M, h). The form Hdh has volume 0, thus it is exact, i.e., $Hdh = d\theta$ for some $\theta \in \Lambda^1(M, \mathbb{R})$. Recall moreover that $c(dh)$ acts by $\pm i$ on $\Sigma^\pm(M)$. Thus ϕ is a solution

of the equation

$$(D^h + c(d\theta))\phi = 0.$$

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