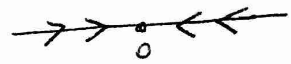
1. $x' = \dfrac{\lambda x - x^3}{1+x^2}$

Critical points: $\dfrac{\lambda x - x^3}{1+x^2} = 0 \iff \lambda x - x^3 = 0 \iff x = 0, \pm\sqrt{\lambda}$

When $\lambda = 0$ one critical point $x=0$, which is stable

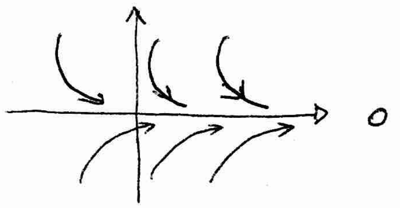


since the function $f(x) = \dfrac{\lambda x - x^3}{1+x^2} = -\dfrac{x^3}{1+x^2}$ is $\begin{cases} \text{positive when } x<0 \\ \text{negative when } x>0. \end{cases}$

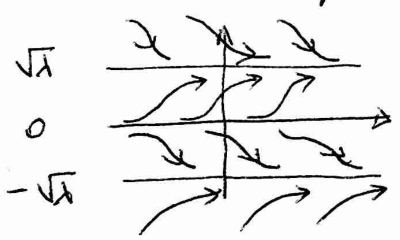
When $\lambda < 0$ one stable critical point $x=0$.

$f(x) = \dfrac{\lambda x - x^3}{1+x^2} = x \cdot \underbrace{\dfrac{\lambda - x^2}{1+x^2}}_{<0}$ is $\begin{cases} \text{positive when } x<0 \\ \text{negative when } x>0. \end{cases}$

Phase Portrait for $\lambda \le 0$

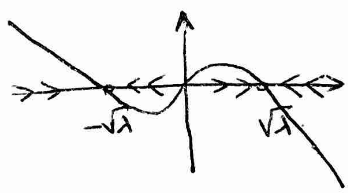


When $\lambda > 0$ three critical points: $x=0$ unstable since the function
$x = \pm\sqrt{\lambda}$ stable

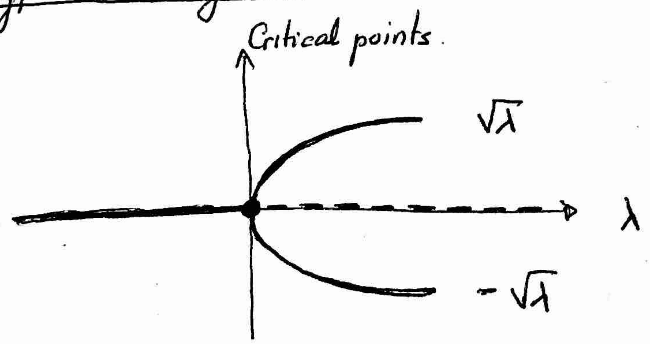


Phase Portrait for $\lambda > 0$.

$\lambda x - x^3$ looks like



Bifurcation Diagram:



Pitchfork Bifurcation.

2. $t^2 y'' + 2t y' - 12 y = 0, \ t>0 \Rightarrow y'' + \overset{=p}{\left(\frac{2}{t}\right)} y' - \frac{12}{t^2} y = 0$

a) $y_1(t) = t^3$ is a solution since $y_1'(t) = 3t^2$, $y_1''(t) = 6t$, and.

$$t^2(6t) + 2t(3t^2) - 12(t^3) = 6t^3 + 6t^3 - 12t^3 = 0$$

b) We look for a solution $y_2(t) = v(t)\, y_1(t)$, where v satisfies the equation

$$y_1 v'' + (2y_1 + p y_1) v' = 0$$

$$\Rightarrow t^3 v'' + \left(6t^2 + \frac{2}{t}\cdot t^3\right) v' = 0 \Rightarrow t v'' + 8 v' = 0.$$

Using the substitution $u = v'$ and separation of variables, we get

$$\frac{du}{u} = -\frac{8}{t}\, dt \Rightarrow \ln|u| = -8 \ln|t| + C_0$$

or $u = 0$.

$\Rightarrow |u(t)| = |t|^{-8} \cdot e^{C_0}$ or $u = 0$ $\Big| \Rightarrow u(t) = t^{-8} \cdot C_1$ where C_1 is $\pm e^{C_0}$ or 0, so C_1 is any random constant.

$$\Rightarrow v(t) = \int t^{-8} C_1 = -\frac{1}{8} t^{-7} C_1 + C_2$$

We can choose $v(t) = t^{-7} \Rightarrow y_2(t) = v y_1 = t^{-7} \cdot t^3 = t^{-4}$

c) $W(y_1, y_2)(t) = \begin{vmatrix} t^3 & t^{-4} \\ 3t^2 & -4t^{-5} \end{vmatrix} = -4t^{-2} - 3t^{-2} = -7t^{-2} < 0$ for all $t > 0$.

$\Rightarrow y_1, y_2$ are linearly independent.

Alternatively if there exist constants a, b such that

$$at^3 + bt^{-4} = 0 \text{ for all } t>0$$

then $at^7 + b = 0 \Rightarrow 7at^6 = 0 \Rightarrow a = 0 \Rightarrow a = b = 0$.

So y_1, y_2 are linearly independent.

3.

$$y'' + w_0^2 y = F_0 \cos(wt).$$

Homogeneous Equation: $y'' + w_0^2 y = 0$. Characteristic equation $r^2 + w_0^2 = 0$ with complex conjugate roots $r_{1,2} = \pm i w_0$

General Solution $y_h(t) = C_1 \cos(w_0 t) + C_2 \sin(w_0 t)$

a) $\underline{w \neq w_0}$ We look for a particular solution $y_p(t) = A\cos(wt)$. There is no need to consider $y_p(t) = A\cos(wt) + B\sin(wt)$ since y' is not present in the differential equation.

$$y_p'' + w_0^2 y_p = F_0 \cos(wt)$$

$$\Rightarrow -Aw^2\cos(wt) + Aw_0^2\cos(wt) = F_0\cos(wt) \Rightarrow A = \frac{F_0}{w_0^2 - w^2}$$

General Solution: $y(t) = y_h(t) + y_p(t) = C_1\cos(w_0 t) + C_2 \sin(w_0 t) + \dfrac{F_0}{w_0^2 - w^2}\cos(wt)$

Initial Conditions $\begin{cases} y(0) = 0 \Rightarrow C_1 = -\dfrac{F_0}{w_0^2 - w^2} \\ y'(0) = 0 \Rightarrow C_2 = 0 \end{cases}$

$$\Rightarrow y(t) = \frac{F_0}{w_0^2 - w^2}\left(\cos(wt) - \cos(w_0 t)\right)$$

b) $\underline{w = w_0}$ We look for a particular solution $y_p(t) = t\left(A\cos(w_0 t) + B\sin(w_0 t)\right)$

We compute $y_p' = \left(-Aw_0\sin(w_0 t) + Bw_0\cos(w_0 t)\right)t + \left(A\cos(w_0 t) + B\sin(w_0 t)\right)$

$$y_p'' = \left(-Aw_0^2\cos(w_0 t) - Bw_0^2\sin(w_0 t)\right)\cdot t + \left(-Aw_0\sin(w_0 t) + Bw_0\cos(w_0 t)\right) + \left(-Aw_0\sin(w_0 t) + Bw_0\cos(w_0 t)\right)$$

$$\Rightarrow y_p'' + w_0^2 y_p = -2Aw_0\sin(w_0 t) + 2Bw_0\cos(w_0 t) = F_0\cos(w_0 t)$$

$$\Rightarrow \left.\begin{array}{l} A = 0 \\ B = \dfrac{F_0}{2w_0} \end{array}\right| \Rightarrow y_p(t) = t \cdot \frac{F_0}{2w_0}\sin(w_0 t)$$

General Solution $y(t) = y_h(t) + y_p(t) = C_1\cos(w_0 t) + C_2\sin(w_0 t) + \dfrac{F_0}{2w_0} t\sin(w_0 t)$

Initial Conditions $\begin{cases} y(0) = 0 \Rightarrow C_1 = 0 \\ y'(0) = 0 \Rightarrow C_2 = 0 \end{cases}$ $\Rightarrow y(t) = \dfrac{F_0}{2w_0} t\sin(w_0 t)$

4. $y'' - 2y' + (1+\lambda)y = 0 \qquad y(0)=0\ ,\quad y(\pi)=0$

Characteristic Equation: $\quad r^2 - 2r + (1+\lambda) = 0$ with roots $r_{1,2} = \dfrac{2 \pm \sqrt{4-4(1+\lambda)}}{2}$

$\Rightarrow r_{1,2} = 1 \pm \sqrt{-\lambda}$

$\underline{\lambda=0}$ One repeated root $r=1$

General Solution $y(x) = c_1 e^x + c_2 x e^x$

BVP Conditions $\begin{cases} y(0)=0 \Rightarrow c_1 = 0 \\ y(\pi)=0 \Rightarrow c_2 \pi e^{\pi} = 0 \Rightarrow c_2 = 0 \end{cases}$ $\Rightarrow$ $y=0$ is the only solution $\Rightarrow \lambda=0$ is not an eigenvalue

$\underline{\lambda>0}$ Complex Conjugate Roots $r_{1,2} = 1 \pm i\mu$

$\lambda = \mu^2$

General Solution $y(x) = c_1 e^x \cos(\mu x) + c_2 e^x \sin(\mu x)$.

BVP Conditions $\begin{cases} y(0)=0 \Rightarrow c_1 = 0 \\ y(\pi)=0 \Rightarrow c_2 \sin(\mu\pi) = 0 \Rightarrow \mu\pi = k\pi,\ k\in\mathbb{Z} \end{cases}$

or $c_2 = 0$.

$\Rightarrow \mu = k \Rightarrow \lambda = k^2,\ k \in \mathbb{Z}^*$

$\underline{\lambda<0}$ ~~Complex~~ Distinct Real Roots $r_{1,2} = 1 \pm \mu$

$\lambda = -\mu^2$

General Solution $y(x) = c_1 e^{(1+\mu)x} + c_2 e^{(1-\mu)x}$

BVP Conditions $\quad y(0)=0 \Rightarrow c_1 + c_2 = 0$

$y(\pi)=0 \Rightarrow c_1 e^{(1+\mu)\pi} + c_2 e^{(1-\mu)\pi} = 0 \Rightarrow$

$\Rightarrow c_1 e^{2\mu\pi} + c_2 = 0$

$\Rightarrow c_1 = c_2 = 0 \Rightarrow$ the only solution is $y(x)=0$.

b) Non-trivial Solutions corresponding to $\lambda = k^2,\ k = \cancel{0}, 1, 2, 3, \dots$

not an eigenvalue.

$y_\lambda(x) = c_2 e^x \sin(kx)$.