MAT 303 - HW10 - Additional Problems
due Friday, December 2
All homework problems are mandatory!

Exercise 1. Consider the system of differential equations $x' = Ax$, where $A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ and λ is an arbitrary real number.

a) Compute A^2 and A^3. Use an inductive argument to show that $A^n = \begin{pmatrix} \lambda^n & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}$.

b) Determine the exponential e^{At} using the computations in part a).

c) Find the exponential e^{At} by first writing A as a sum of a diagonal matrix and a nilpotent matrix $A = \lambda I_3 + C$, then computing e^{At} as a product of two exponential matrices $e^{\lambda I_3 t} e^{Ct}$.

d) Find the general solution of the system $x' = Ax$ using the exponential e^{At}.

e) Find the eigenvalues and eigenvectors of the matrix A.

f) Find the general solution of the system $x' = Ax$ using part e), then compare it to the answer you got for part d).

Exercise 2. Consider the system of differential equations $x' = Bx$, where $B = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$ and λ is an arbitrary real number.

a) Compute B^2, B^3. Use an inductive argument to show that $B^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}$.

b) Determine the exponential e^{Bt} using part a).

c) Find the exponential e^{Bt}, using a different approach: write B as a sum of a scalar multiple of the identity matrix and a nilpotent matrix $B = \lambda I_3 + C$. Then use the fact that $e^{Bt} = e^{\lambda I_3 t} e^{Ct}$.

d) Find the general solution of the system $x' = Bx$ using the exponential matrix e^{Bt}.

e) Find the eigenvalues and eigenvectors of the matrix B.

f) Find the general solution of the system $x' = Bx$ using part e), then compare it to the answer you got for part d).