MAT324: Real Analysis – Fall 2016
ASSIGNMENT 9 – SOLUTIONS

Problem 1: Let $\mathcal{N} \subset [0,1]$ be a non-measurable set and let $\mathcal{C} \subset [0,1]$ be the Cantor middle-thirds set. Decide whether the following are true or false and explain your answer.

a) $\mathcal{N} \times \mathcal{C}$ is a Borel set;

b) $\mathcal{N} \times \mathcal{C}$ is a Lebesgue measurable set;

c) $\mathcal{N} \times \mathcal{C}$ is not measurable with respect to m_2, the Lebesgue measure on $\mathbb{R}^2$.

SOLUTION. Part a) is false. Part b) is true since the set is null. Part c) is false since b) is true. □

Problem 2: Consider the function

$$g(x,y) = \begin{cases} \frac{1}{x^2} & \text{if } 0 < y < x < 1 \\ -\frac{1}{y^2} & \text{if } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $\displaystyle\int_0^1 \int_0^1 g(x,y)\,dxdy = -1$ and $\displaystyle\int_0^1 \int_0^1 g(x,y)\,dydx = 1$. Is g an integrable function?

SOLUTION. Direct calculation shows that

$$\begin{aligned} \int_0^1 \left[\int_0^1 g(x,y)dx\right] dy &= \int_0^1 \left[\int_0^y \left(\frac{-1}{y}^2\right) dx + \int_y^1 \frac{1}{x}^2 dx\right] dy \\ &= \int_0^1 (-1) dy \\ &= -1 \end{aligned}$$

and

$$\begin{aligned} \int_0^1 \left[\int_0^1 g(x,y)dy\right] dx &= \int_0^1 \left[\int_0^x \frac{1}{x}^2 dy + \int_x^1 \left(\frac{-1}{y}^2\right) dy\right] dx \\ &= \int_0^1 1 dx \\ &= 1 \end{aligned}$$

However, the function g is not integrable since its positive part $g^+ = f$ from Problem 3, HW8 is not integrable. □

Problem 3: Compute

$$\int_{(0,\infty)\times(0,1)} y \sin(x) e^{-xy}\,dxdy,$$

and explain why Fubini's theorem is applicable.

SOLUTION. Notice that

$$\int_{(0,\infty)\times(0,1)} |y\sin(x)e^{-xy}|dxdy \leq \int_{(0,\infty)\times(0,1)} ye^{-xy}dxdy$$

The integrand on the right-hand side is measurable, so we can apply Tonelli's Theorem:

$$\begin{aligned}\int_{(0,\infty)\times(0,1)} ye^{-xy}dxdy &= \int_{(0,1)}\left[\int_{(0,\infty)} ye^{-xy}dx\right]dy \\ &= \int_{(0,1)}\left[\int_{(0,\infty)} \frac{d(e^{-xy})}{dx}dx\right]dy \\ &= \int_{(0,1)} 1dy = 1\end{aligned}$$

Therefore the integrand of $\int_{(0,\infty)\times(0,1)} |y\sin(x)e^{-xy}|dxdy$ is in L^1, and we can apply Fubini's theorem:

$$\begin{aligned}\int_{(0,\infty)\times(0,1)} y\sin(x)e^{-xy}dxdy &= \int_0^1 y\left[\int_0^\infty \sin(x)e^{-xy}dx\right]dy \\ &= \int_0^1 \left(\frac{y}{y^2+1}\right)dy = \frac{1}{2}\log(2)\end{aligned}$$

□