MAT324: Real Analysis – Fall 2016
Assignment 6 – Solutions

Problem 1: Let $f : E \to [0, \infty)$ be a Lebesgue integrable function and suppose $\int_E f\, dm = C$ and $0 < C < \infty$. Prove that

$$\lim_{n\to\infty} \int_E n \ln\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) dm = \begin{cases} \infty, & \text{for } \alpha \in (0,1) \\ C, & \text{for } \alpha = 1 \\ 0, & \text{for } 1 < \alpha < \infty. \end{cases}$$

Hint: For $\alpha = 1$, use the inequality $e^x \geq x + 1$, for all $x \geq 0$. For $\alpha > 1$, use $(1 + x)^\alpha \geq 1 + x^\alpha$. DCT and the Fatou Lemma might prove useful.

Solution. There are three cases to consider. First, suppose that $0 < \alpha < 1$. Then, by Fatou's Lemma, we get

$$\lim_{n\to\infty} \int_E n \ln\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) dm \geq \int_E \lim_{n\to\infty} n \ln\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) dm = \infty,$$

because

$$\lim_{n\to\infty} n \ln\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) = \lim_{n\to\infty} n^{1-\alpha} f(x)^\alpha = \infty.$$

Suppose $\alpha = 1$. Then $n \ln\left(1 + \frac{f(x)}{n}\right) \leq n\frac{f(x)}{n} = f(x)$, which is integrable. We can apply DCT and get

$$\lim_{n\to\infty} \int_E n \ln\left(1 + \frac{f(x)}{n}\right) dm = \int_E \lim_{n\to\infty} n \ln\left(1 + \frac{f(x)}{n}\right) dm = \int_E f(x) dx = C.$$

Finally, suppose $\alpha > 1$. Then, using the inequality $(1 + x)^\alpha \geq 1 + x^\alpha$, we get

$$n \ln\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) \leq \alpha n \ln\left(1 + \frac{f(x)}{n}\right) \leq \alpha f(x),$$

which is integrable. The last inequality follows from the fact that the sequence $\left(1 + \frac{f(x)}{n}\right)^n$ is increasing to $e^{f(x)}$ so $\ln\left(1 + \frac{f(x)}{n}\right) \leq f(x)$. We can therefore apply DCT and interchange the integral and the limit. We get that the limit is 0 because

$$\lim_{n\to\infty} n \ln\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) = \lim_{n\to\infty} n^{-(\alpha-1)} f(x)^\alpha = 0.$$

□

Problem 2: Consider the sequence of functions

$$f_n(x) = \frac{1}{\sqrt{x}} \chi_{(0,\frac{1}{n}]}(x), \quad n \geq 1.$$

a) Is f_n in $L^1(0, 1]$?

b) Is the sequence Cauchy in $L^1(0,1]$?

c) Is f_n in $L^p(0,1]$ for $p \geq 4$?

Solution.

a) Yes, it is. Notice that $\|f_n\|_1 = \frac{1}{2\sqrt{n}}$.

b) Yes, it is. In fact, it converges to the zero function.

c) No. If $p \geq 4$, then

$$\int_{(0,1)} |f_n(x)|^p dx = \int_{(0,\frac{1}{n}]} x^{-\frac{p}{2}} dx \geq \int_{(0,\frac{1}{n}]} x^{-2} dx = +\infty$$

□

Problem 3: Consider the sequence $f_n = n\chi_{[n+\frac{1}{n^3}, n+\frac{2}{n^3}]}$, $n \geq 1$. Determine whether the following are true or false and explain your answers.

a) $(f_n)_{n\geq 1}$ is Cauchy as a sequence of $L^1(0,\infty)$.

b) $f(x) = \sum_{n=2}^{\infty} f_n(x)$ belongs to $L^1(\mathbb{R})$.

c) $f(x) = \sum_{n=2}^{\infty} f_n(x)$ belongs to $L^2(\mathbb{R})$.

d) $f_n \in L^2(\mathbb{R})$ for each $n \geq 1$.

Solution.

a) Yes, the sequence is Cauchy. Note that $\|f_n\|_1 = \frac{1}{n^2}$.

b) Yes, $f \in L^1(\mathbb{R})$. Note that $\|f\|_1$ behaves as $\sum \frac{1}{n^2}$, which converges.

c) No, $f \notin L^2(\mathbb{R})$. Note that $\|f\|_2$ behaves as $\sum \frac{1}{n}$, which diverges.

d) Yes, each f_n belongs to $L^2(\mathbb{R})$.

□

Problem 4: Let $(X, \|\cdot\|)$ be a normed vector space. Show that X is complete if and only if whenever $\sum_{j=1}^{\infty} \|x_j\| < \infty$, then $\sum_{j=1}^{\infty} x_j$ converges to an element $x^* \in X$.
Hint: Rework the proof of the completeness theorem for L^1.

Solution. Suppose that X is complete and $\sum_{j=1}^{\infty} \|x_j\| < \infty$. Let $y_n = \sum_{j=1}^{n} x_j$. Then $(y_n)_{n\geq 1}$ is Cauchy because the tail $\sum_{j=n}^{\infty} \|x_j\|$ can be made arbitrarily small. Since X is complete, we get that y_n converges to, say, x^* and x^* belongs to X, which proves the claim. The converse implication is the same as the proof of Theorem 5.5 from the textbook. □