MAT324: Real Analysis – Fall 2016

Assignment 6

Due Thursday, **November 3**, in class.

Problem 1: Let $f : E \to [0, \infty)$ be a Lebesgue integrable function and suppose $\int_E f\,dm = C$ and $0 < C < \infty$. Prove that

$$\lim_{n\to\infty} \int_E n \ln\left(1 + \left(\frac{f(x)}{n}\right)^{\alpha}\right) dm = \begin{cases} \infty, & \text{for } \alpha \in (0,1) \\ C, & \text{for } \alpha = 1 \\ 0, & \text{for } 1 < \alpha < \infty. \end{cases}$$

Hint: For $\alpha = 1$, use the inequality $e^x \geq x + 1$, for all $x \geq 0$. For $\alpha > 1$, use $(1 + x)^{\alpha} \geq 1 + x^{\alpha}$. DCT and the Fatou Lemma might prove useful.

Problem 2: Consider the sequence of functions

$$f_n(x) = \frac{1}{\sqrt{x}} \chi_{(0, \frac{1}{n}]}(x), \quad n \geq 1.$$

a) Is f_n in $L^1(0, 1]$?

b) Is the sequence Cauchy in $L^1(0, 1]$?

c) Is f_n in $L^p(0, 1]$ for $p \geq 4$?

Problem 3: Consider the sequence $f_n = n\chi_{[n+\frac{1}{n^3}, n+\frac{2}{n^3}]}$, $n \geq 1$. Determine whether the following are true or false and explain your answers.

a) $(f_n)_{n\geq 1}$ is Cauchy as a sequence of $L^1(0, \infty)$.

b) $f(x) = \sum_{n=2}^{\infty} f_n(x)$ belongs to $L^1(\mathbb{R})$.

b) $f(x) = \sum_{n=2}^{\infty} f_n(x)$ belongs to $L^2(\mathbb{R})$.

c) $f_n \in L^2(\mathbb{R})$ for each $n \geq 1$.

Problem 4: Let $(X, \|\cdot\|)$ be a normed vector space. Show that X is complete if and only if whenever $\sum_{j=1}^{\infty} \|x_j\| < \infty$, then $\sum_{j=1}^{\infty} x_j$ converges to an element $x^* \in X$.

Hint: Rework the proof of the completeness theorem for L^1.