MAT324: Real Analysis – Fall 2014
Assignment 4

Due Thursday, **October 6**, in class.

Problem 1: Let $\{f_n\}$ be a sequence of measurable functions defined on $\mathbb{R}$. Show that the sets

$$\begin{aligned} E_1 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) \text{ exists and is finite}\} \\ E_2 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) = \infty\} \\ E_3 &= \{x \in \mathbb{R} \mid \lim_{n\to\infty} f_n(x) = -\infty\} \end{aligned}$$

are measurable.

Problem 2: Let $\mathcal{C} \subset [0,1]$ be the Cantor middle-thirds set. Suppose that $f : [0,1] \to \mathbb{R}$ is defined by $f(x) = 0$ for $x \in \mathcal{C}$ and $f(x) = k$ for all x in each interval of length 3^{-k} which has been removed from $[0,1]$ at the k^{th} step of the construction of the Cantor set. Show that f is measurable and calculate $\int_{[0,1]} f dm$.

Problem 3: Let E be a measurable set. For a function $f : E \to \mathbb{R}$ we define the *positive part* $f^+ : E \to \mathbb{R}$, $f^+(x) = \max(f(x), 0)$, and the *negative part* $f^- : E \to \mathbb{R}$, $f^-(x) = \min(f(x), 0)$. Prove that f is measurable if and only if both f^+ and f^- are measurable.

Problem 4: Prove that if f is integrable on $\mathbb{R}$ and $\int_E f(x)dm \geq 0$ for every measurable set E, then $f(x) \geq 0$ a.e. x.
Hint: Show that the set $F = \{x \mid f(x) < 0\}$ is null.

Problem 5: Let E be a measurable set. Suppose $f \geq 0$ and let $E_k = \{x \in E \mid 2^k < f(x) \leq 2^{k+1}\}$ for any integer k. If f is finite almost everywhere, then

$$\bigcup_{k=-\infty}^{\infty} E_k = \{x \in E \mid f(x) > 0\},$$

and the sets E_k are disjoint.

(a) Prove that f is integrable if and only if $\sum_{k=-\infty}^{\infty} 2^k m(E_k) < \infty$.

(b) Let $a > 0$ and consider the function

$$f(x) = \begin{cases} |x|^{-a} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Use part a) to show that f is integrable on $\mathbb{R}$ if and only if $a < 1$.