MAT324: Real Analysis – Fall 2016
Assignment 2 – Solutions

Problem 1: Suppose $E_1, E_2 \subseteq \mathbb{R}$ are measurable sets. Show that

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

Solution. Notice that $E_1 \cup E_2$ can be expressed as a union of disjoint measurable sets

$$E_1 \cup E_2 = (E_1 \setminus E_2) \cup (E_1 \cap E_2) \cup (E_2 \setminus E_1).$$

Additivity implies that

$$\begin{aligned} m(E_1 \cup E_2) &= m(E_1 \setminus E_2) + m(E_1 \cap E_2) + m(E_2 \setminus E_1) \\ &= m(E_1 \cap E_2) + m(E_1 \cap (E_2)^c) + m(E_2 \cap (E_1)^c) \end{aligned}$$

Hence

$$\begin{aligned} m(E_1 \cup E_2) + m(E_1 \cap E_2) &= [m(E_1 \cap E_2) + m(E_1 \cap (E_2)^c)] + [m(E_2 \cap (E_1)^c) + m(E_1 \cap E_2)] \\ &= m(E_1) + m(E_2) \end{aligned}$$

□

Problem 2: Construct a *Cantor-like* closed set $\mathcal{C} \subset [0,1]$ so that at the k^{th} stage of the construction one removes 2^{k-1} centrally situated open intervals each of length ℓ_k, with

$$\ell_1 + 2\ell_2 + \ldots + 2^{k-1}\ell_k < 1.$$

Suppose ℓ_k are chosen small enough so that $\sum_{k=1}^{\infty} 2^{k-1}\ell_k < 1$.

a) Show that $m(\mathcal{C}) = 1 - \sum_{k=1}^{\infty} 2^{k-1}\ell_k$ and conclude that $m(\mathcal{C}) > 0$.

b) Give an example of a sequence $(\ell_k)_{k\geq 1}$ that verifies the hypothesis.

Solution.

a) The intervals removed are disjoint. If C_n denotes what is left in $[0,1]$ after the n-th process, then $m(C_n) = 1 - \sum_{k=1}^{n} 2^{k-1} l_k$. Furthermore, by construction we have $C_{n+1} \subset C_n$. Apply Theorem 2.19.

b) Let $l_k = 4^{-k} = 2^{-2k}$. Then

$$\sum_{k=1}^{\infty} 2^{k-1} 2^{-2k} = \sum_{k=1}^{\infty} 2^{-1-k} = \frac{1}{2}$$

□

Problem 3: Let $E_1, E_2, \ldots, E_{2014} \subset [0,1]$ be measurable sets such that $\sum_{k=1}^{2014} m(E_k) > 2013$. Show that $m\left(\bigcap_{k=1}^{2014} E_k\right) > 0$.

Solution. Let $F_n = [0,1] \setminus E_n$, for each $1 \leq n \leq 2014$. Notice that

$$m\left(\bigcup_{n=1}^{2014} F_n\right) = 1 - m\left(\bigcap_{n=1}^{2014} E_n\right).$$

Use subadditivity and the inequality provided to show that $m\left(\bigcup_{n=1}^{2014} E_n\right) < 1$. Combined with the result in the previous paragraph, $m\left(\bigcup_{n=1}^{2014} E_n\right) > 0$. □

Problem 4: Suppose $A \in \mathcal{M}$ and $m(A \Delta B) = 0$. Show that $B \in \mathcal{M}$ and $m(A) = m(B)$.

Solution. See page 36 in the textbook. □

Problem 5: Suppose $A \subset E \subset B$ where A and B are measurable sets of finite measure. Show that if $m(A) = m(B)$, then E is measurable.

Solution. Notice that

$$m(B) = m(A) + m(B \setminus A)$$

Since $m(A) = m(B) < \infty$, we can subtract this on both sides to get $m(B \setminus A) = 0$. Since $E \setminus A \subset B \setminus A$, completeness of the Lebesgue measure shows that $E \setminus A$ is measurable, but then so is $E = A \cup (E \setminus A)$. □

Problem 6: Suppose $E \in \mathcal{M}$ and $m(E) > 0$. Prove that there exists an open interval I such that

$$m(E \cap I) > 0.99 \cdot m(I).$$

Hint: Argue by contradiction, using the regularity of Lebesgue measure. See Theorems 2.17, 2.29.

Solution. We'll show that in fact a more general result holds.

Claim 1 *If $E \in \mathcal{M}$ and $m(E) > 0$, then for any $0 < \alpha < 1$, there exists an interval I such that*

$$m(E \cap I) > \alpha \cdot m(I).$$

Proof of Claim 1. We'll use a slight modification of Theorem 2.29, which the reader can prove as an exercise. This is the

Lemma 1 *If $E \in \mathcal{M}$, then*

$$m(E) = \sup\{m(K) \mid K \subset E,\ K \text{ is compact}\}.$$

With this the reader can easily prove that if E has finite measure, we can find a finite union of disjoint open intervals $A = \bigcup_{n=1}^{N} I_n$ such that $m(E\Delta A) < \epsilon$ (consider a suitable open cover of K by open intervals, and extract a finite subcover).

Let $\epsilon = (1-\alpha)m(E)$, and ket A be the set given by the lemma. Since A is a measurable set,

$$\begin{aligned} m(E) &= m(E \cap A) + m(E \cap A^c) \\ m(E) &\leq m(E \cap A) + m(E \Delta A^c) \\ m(E) &< m(E \cap A) + (1-\alpha)m(E) \\ \alpha m(E) &< m(E \cap A) \end{aligned}$$

Since E is a measurable set,

$$\begin{aligned} m(A) &= m(A \cap E) + m(A \cap E^c) \\ m(A) &\leq m(A \cap E) + (1-\alpha)m(E) \\ m(A) &< m(A \cap E) + \frac{1-\alpha}{\alpha} m(E \cap A) \\ m(A) &< \frac{1}{\alpha} m(E \cap A) \end{aligned}$$

Now we notice that

$$\begin{aligned} m(A) &= \sum_{n=1}^{N} m(I_n) \\ m(A \cap E) &= \sum_{n=1}^{N} m(E \cap I_n) \end{aligned}$$

This yields,

$$\sum_{n=1}^{N} m(I_n) < \frac{1}{\alpha} \left(\sum_{n=1}^{N} m(E \cap I_n) \right)$$

And this proves the claim if $m(E) < \infty$ (argue by contradiction). Now if $m(E) = +\infty$, take $E' \subset E$ with $m(E') < \infty$ and proceed in the same way to get the result for E'. Apply monotonicity to get the claim in its general form. $\square$