MAT324: Real Analysis – Fall 2016
Assignment 10

Due Thursday, **December 8**, in class.

Problem 1: Let λ_1, λ_2 and μ be measures on a measurable space $(X, \mathcal{F})$. Show that if $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$ then $(\lambda_1 + \lambda_2) \ll \mu$.

Problem 2: Let $X = [0,1]$ with Lebesgue measure and consider probability measures μ and ν given by densities f and g as follows

$$\nu(E) = \int_E f\,dm \quad \text{and} \quad \mu(E) = \int_E g\,dm,$$

for every measurable subset $E \subset [0,1]$. Suppose $f(x), g(x) > 0$ for every $x \in [0,1]$. Is ν absolutely continuous with respect to μ (that is $\nu \ll \mu$)? If it is, determine the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$. Is μ absolutely continuous with respect to ν (that is $\mu \ll \nu$)?

Problem 3: Suppose μ is a σ-finite measure on $([0,1], \mathcal{F})$ and $E_1, E_2, \ldots, E_{2016}$ are measurable subsets of $[0,1]$. Define ν on $\mathcal{F}$ by $\nu(E) = \sum_{k=1}^{2016} \mu(E \cap E_k)$. Show that $\nu \ll \mu$ and find the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$.

Problem 4: Let $\lambda_1, \lambda_2, \mu$ be measures on a σ-algebra $\mathcal{F}$. Show that

a) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$ then $(\lambda_1 + \lambda_2) \perp \mu$.

b) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$ then $\lambda_2 \perp \lambda_1$.

Problem 5: For a point x, define the Dirac measure δ_x to be

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

For a fixed set B define the Lebesgue measure restricted to B by $m_B(A) = m(A \cap B)$. Let $\mu = \delta_1 + m_{[2,4]}$ and $\nu = \delta_0 + m_{(1,2)}$. Show that $\nu \perp \mu$.