Connes–Landi Deformation of Spectral Triples

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Based on:
M. Yamashita. Connes–Landi deformation of spectral triples, *Lett. Math. Phys.* **94**(3):263–291, 2010

1. θ-deformation of spectral triples

2. Cyclic cohomology of one-parameter crossed product

3. Cyclic cohomology of Connes-Landi deformation

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$$f *_\theta g = e^{\pi i \theta (mn' - m'n)} fg$$

when f is a $\mathbb{T}^2$-eigenvector of weight $(m, n) \in \mathbb{Z}^2$, g is a one of weight (m', n').

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$A_\theta = \overline{\mathcal{A}_\theta}$: deformation of $A = \overline{\mathcal{A}}$ by Rieffel (1993)

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$\Rightarrow$ New spectral triple $(\mathcal{A}_\theta, H, D)$

Crossed product presentation of CL-deformation

Strong Morita equivalence

$$A_\theta \simeq (A \otimes C(\mathbb{T}^2_\theta))^{\sigma\otimes\gamma} \simeq_{\mathsf{KK}} \mathbb{T}^2 \ltimes_{\sigma\otimes\gamma} (A \otimes C(\mathbb{T}^2_\theta))$$

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Automorphisms of $\mathbb{Z} \ltimes_{\widehat{\sigma^{(2)}}} \mathbb{T}^2 \ltimes_\sigma A$:

$$\widehat{\sigma^{(1)}}(\sum_{k\in\mathbb{Z}} f_k v^k) = \sum_{k\in\mathbb{Z}} \widehat{\sigma^{(1)}}(f_k) v^k,$$

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an analogue of "noncommutative torus" $\Leftrightarrow$ "Kronecker foliation"

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$$\mathrm{HC}^n(\mathcal{A}) \times K_n(\mathcal{A}) \to \mathbb{C},$$

$$\langle [\phi], [e] \rangle = \phi(e, \dots, e) \quad (\phi : \text{cyclic } 2k\text{-cocycle}, e : \text{projection})$$
$$\langle [\phi], [u] \rangle = \phi(u^*, u, u^*, \dots, u) \quad (\phi : \text{cyclic } 2k+1\text{-cocycle}, u : \text{unitary})$$

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$$S : \mathrm{HC}^n(\mathcal{A}) \to \mathrm{HC}^{n+2}(\mathcal{A}), \mathrm{HP}^*(\mathcal{A}) = \lim_{k\to\infty} \mathrm{HC}^{2k+*}(\mathcal{A})$$

Connes-Thom isomorphism in cyclic cohomology

Theorem (Elliott-Natsume-Nest, 1988)

$\mathcal{A}$: *Fréchet,* $\alpha \colon \mathbb{R} \curvearrowright \mathcal{A}$ *smooth*

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τ: trace on $\mathcal{A}$, invariant under α
$i_X\tau(a, b) = \tau(ah(b))$: cyclic 1-cocycle on $\mathcal{A}$

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$$\Rightarrow \#_{\hat{\alpha}}(\hat{\tau}) = \mathrm{Tr} \otimes i_X \tau \text{ on } \mathbb{R} \ltimes_{\hat{\alpha}}^\infty \mathbb{R} \ltimes_\alpha^\infty \mathcal{A} \simeq \mathcal{K}^\infty \hat{\otimes} \mathcal{A}$$

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- Dual cocycle $\hat{\phi}$ on $\mathbb{R} \ltimes^{\infty}_{\alpha} \mathcal{A}$

$$\hat{\phi}(f^0, \ldots, f^n) = \int_{\sum_{j=0}^{n} t_j = 0} \phi(f^0_{t_0}, \ldots, f^n_{t_n})$$

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- Interior product $i_X\phi$ on $\mathcal{A}$

$$i_X\phi(a_0, \ldots, a_{n+1}) = \sum_{j=1}^{n+1} (-1)^j \tau_\phi(a_0 da_1 \cdots X(a_j) \cdots da_{n+1}).$$

Invariant cocycles and ENN-isomorphism

Theorem (Y., 2010)

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$$\mathbb{R} \curvearrowright M_2(\mathbb{R} \ltimes_{\hat{\sigma}} \mathbb{R} \ltimes_{\sigma} \mathcal{A}) \quad \text{(with generator } Y\text{)}$$

$\Rightarrow$ Two embeddings $\Psi_1, \Psi_2\colon \mathbb{R} \ltimes_{\hat{\sigma}} \mathbb{R} \ltimes_{\sigma} \mathcal{A} \to M_2(\mathbb{R} \ltimes_{\hat{\sigma}} \mathbb{R} \ltimes_{\sigma} \mathcal{A})$

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$$\Psi_1^*(i_Y(\phi \otimes \mathrm{Tr}_{M_2})) = i_{\tilde{X}}\phi \qquad \Psi_2^*(i_Y(\phi \otimes \mathrm{Tr}_{M_2})) = i_X\phi$$

Invariant cocycles and ENN-isomorphism, cont.

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- $\partial\colon K_{*+1}(\mathbb{Z} \ltimes_\alpha \mathcal{B}) \to K_*(\mathcal{B})$ connecting map in the Pimsner–Voiculescu sequence
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$$\Rightarrow \langle i_D\phi, x\rangle = \langle \phi, \partial(x)\rangle$$

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e: minimal projection in $\mathcal{K}^\infty$, embedding

$$\Psi \colon \mathcal{A}_\theta \to \mathcal{A} \hat{\otimes} \mathcal{K}^\infty \simeq \mathbb{R} \ltimes_{\widehat{\sigma^{(1)}}, \sigma^{(2)}_{\theta t}} \mathbb{R} \ltimes_{\sigma^{(1)}} \mathcal{A},$$

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Rem. ϕ is a trace on $\mathcal{A} \Rightarrow \phi^{(\theta)}(x^{(\theta)}) = \phi(x)$

Comparison of cocycles

$$\#_{\widehat{\sigma^{(1)}},\sigma^{(2)}_{\theta t}} \circ \#_{\sigma^{(1)}} \colon \mathsf{HP}(\mathcal{A}) \to \mathsf{HP}(\mathbb{R} \ltimes_{\widehat{\sigma^{(1)}},\sigma^{(2)}_{\theta t}} \mathbb{R} \ltimes_{\sigma^{(1)}} \mathcal{A}) \simeq \mathsf{HP}(\mathcal{A}_\theta)$$

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$$[\Xi^{(\theta)}(\phi^{(\theta)})] = [\phi] + \theta[i_{X_1} i_{X_2} \phi] \in \mathrm{HC}^{n+2}(\mathcal{A})$$

Invariance of Chern-Connes character

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$(\mathcal{A}_\theta, H, D)$ *deformed triple*

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$(\mathcal{A}_\theta, H, D)$ deformed triple

$$\langle \mathrm{ch}_{(\mathcal{A},H,D)}, x \rangle = \langle \Xi^{(\theta)}(\mathrm{ch}_{(\mathcal{A}_\theta,H,D)}), x \rangle,$$

i.e. "$\Xi^{(\theta)}(\mathrm{ch}_{(H,D)}) = \mathrm{ch}_{(H,D)}$."

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$\because$

$$\langle \Xi^{(\theta)}(\mathrm{ch}_{(\mathcal{A}_\theta,H,D)}), x \rangle = \langle \mathrm{ch}_{(\mathcal{A},H,D)}, x \rangle + \theta \langle i_{X_1} i_{X_2} \mathrm{ch}_{(\mathcal{A},H,D)}, x \rangle$$

and

$$\forall \theta : \langle \Xi^{(\theta)}(\mathrm{ch}_{(\mathcal{A}_\theta,H,D)}), x \rangle \in \mathbb{Z}.$$