# Canonical quantization of non-commutative holonomies in 2+1 loop quantum gravity

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# 3-dim Quantum Gravity

## Introduction and motivation

3-dimensional quantum gravity can be defined from a number of different points of view. The first of these was the Ponzano-Regge model of quantum gravity on a triangulated 3-manifold which provides a quantization of the Regge calculus.

The Ponzano-Regge model is a state sum model for 3-dimensional euclidean quantum gravity without cosmological constant using the Lie group *SU(2)*:

- Quantum amplitude for each assignment of SU(2) irreducible representations to each edge of the triangulation;

- Sum of the amplitudes over every possible spin on every edge in the interior of the manifold to give a partition function;

- Since the set of irreducible representations of SU(2) is infinite, the partition function is often a sum with an infinite number of terms, and in many cases diverges;

A regularization of the Ponzano-Regge model is provided by the Turaev-Viro model, where the Lie group $SU(2)$ is replaced by its quantum deformation $U_q\, sl(2)$. When the deformation parameter $q$ is a root of unity, then there are only a finite number of irreducible representations, which means that the edge lengths are not summed up to infinite values, and the partition function is always well-defined.

A very important consequence of this is that the answer obtained is finite, and so the model appears to be a regularized version of the Ponzano-Regge model.

## How the Turaev-Viro state sum is connected to Quantum Gravity?

Witten argued that it was equivalent to a Feynman path integral with the Chern-Simons action for $SU(2)_k \otimes SU(2)_{-k}$. The connection with gravity follows from the fact that Chern-Simons action for this group product is related to the Einstein-Hilbert action for gravity with cosmological constant (Ooguri and Sasakura, Williams) if $k^2 = 4\pi^2/\Lambda$.

# What about LQG?

- In the case $\Lambda = 0$ we have a quantization of the theory [Noui and Perez]:

$$S[e, \omega] = \int_M Tr[e \wedge F(\omega)]$$

Upon the standard 2+1 decomposition, the canonical variables are the 2-dim connection $A^i{}_a$ and the triad field $E^b{}_j$

If one starts from the kinematical Hilbert space $H_{kin}$ spanned by spin network states the only remaining constraint of the theory is the quantum curvature constraint

The physical inner product and the physical Hilbert space $H_{phys}$ of 2+1 gravity with $\Lambda$=0 can be defined by introducing a regularization of the formal expression for the generalized projection operator into the kernel of $F$:

$$P = \text{“}\prod_{x \in \Sigma} \delta(\hat{F}(A(x))\text{”} = \int D[N] exp\left( i \int_\Sigma Tr[N \hat{F}(A)] \right)$$

Noui and Perez showed how, introducing a regularization as an intermediate step for the quantization, this projector can be given a precise definition leading to a rigorous expression for the physical inner product of the theory. Moreover, the constraints algebra is anomaly free in this case.

- In the case $\Lambda \neq 0$ we have NOT a quantization of the theory yet!

But there are strong motivations to the idea that, in the context of LQG, it should be possible to recover the Turaev-Viro amplitudes as the physical transition amplitudes between kinematical spin network states of 2+1 gravity with non-vanishing cosmological constant:

Implementation of the dynamics ($F + \Lambda\, e \wedge e = 0$) → "Emergence" of the quantum group structure

Understanding the relationship between the Turaev-Viro invariants and quantum gravity requires the understanding the dynamical interplay between classical spin-network states and $q$-deformed amplitudes

# 2+1 GRAVITY WITH $\Lambda\neq 0$ IN LQG

## Classical analysis

Space-time $\Sigma = M\times R$

$$S[e,\omega] = \int_M Tr[e\wedge F(\omega) + \frac{\Lambda}{3} e\wedge e\wedge e]$$

Upon the standard 2+1 decomposition, the phase space variables are the 2-dim *su(2)* Lie algebra valued connection $A^i{}_a$ and triad field $e^j{}_b$. The symplectic structure is defined by

$$\{A^i_a(x)\,,\, e^j_b(y)\} = \epsilon_{ab}\,\delta^i_j\delta^{(2)}(x,\,y)$$

Smeared constraints

$$G(\alpha) = \int_\Sigma \alpha_i d_A e^i = 0$$

$$C(N) = \int_\Sigma N_i(F^i(A) + \Lambda\epsilon^i{}_{jk}e^j\wedge e^k) = 0$$

Constraints algebra

$$\begin{aligned} \{C(N),C(M)\} &= \Lambda\, G([N,M]) \\ \{G(\alpha),G(\beta)\} &= G([\alpha,\beta]) \\ \{C(N),G(\alpha)\} &= C([N,\alpha]) \end{aligned}$$

local symmetry

$$su(2)\oplus su(2)$$

# 2+1 GRAVITY WITH $\Lambda \neq 0$ IN LQG

## QUANTUM ANALYSIS: KINEMATICAL HILBERT SPACE

Basic kinematical observables:
holonomy of the connection and
smeared functionals of the triad field $e$

→ unique representation on the kinematical Hilbert space $H_K$, with a diffeomorphism invariant inner product:

$$h_\gamma[A] = P\exp(-\int_\gamma dx\, A) \;\in SU(2)$$

$$E(\eta) = \int e^i_a \tau_i \frac{d\eta^a}{dt} dt = \int E^{ai}\tau_i n_a dt \;\in su(2)$$

$$n_a \equiv \epsilon_{ab}\frac{d\eta^a}{dt}$$

flux of $E$ across the curve $\eta$

space of cylindrical functions *Cyl*

finite graph $\quad \Gamma \subset \Sigma$

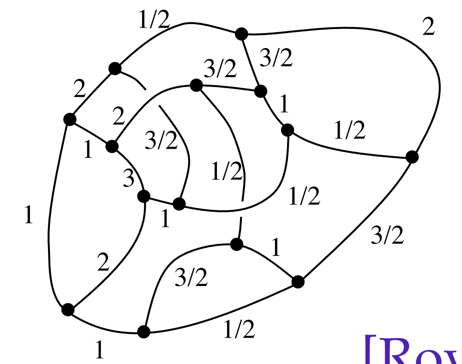
$$f : SU(2)^{N_\ell(\Gamma)} \to \mathbb{C}$$

$$\Psi_{\Gamma,f}[A] = f(h_{\gamma_1}[A], \cdots, h_{\gamma_{N_\ell(\Gamma)}}[A])$$

the Ashtekar-Lewandowski measure

$$< \Psi_{\Gamma_1,f}, \Psi_{\Gamma_2,g} > \equiv \mu_{AL}(\overline{\Psi_{\Gamma_1,f}[A]}\Psi_{\Gamma_2,g}[A]) =$$

$$= \int \prod_{i=1}^{N_{\ell_{\Gamma_{12}}}} dh_i \overline{f(h_{\gamma_1}, \cdots, h_{\gamma_{N_\ell(\Gamma_{12})}})} g(h_{\gamma_1}, \cdots, h_{\gamma_{N_\ell(\Gamma_{12})}})$$




[Rovelli, Smolin]

holonomy → operator acting by multiplication in $H_K$

$$\hat{h}_\gamma[A]\Psi[A] = h_\gamma[A]\Psi[A]$$

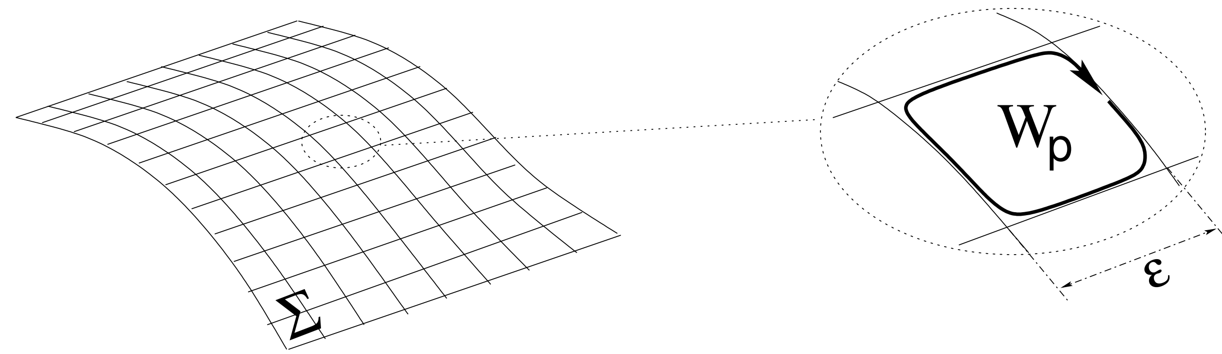
triad field → derivative operator in $H_K$

$$\hat{E}(\eta) \triangleright h_\gamma = -\frac{i\hbar}{2}\begin{cases} o(p)\tau_i h_\gamma & \text{if } \gamma \text{ ends at } \eta \\ o(p) h_\gamma \tau_i & \text{if } \gamma \text{ starts at } \eta \end{cases}$$

# 2+1 gravity with $\Lambda \neq 0$ in LQG

## Quantum analysis: constraints

Quantum constraints:




$$G[\alpha] \triangleright \Psi = \int_\Sigma \mathrm{Tr}[\alpha\, \mathrm{d}_A e] \triangleright \Psi = 0$$

$$C_\Lambda[N] \triangleright \Psi = \int_\Sigma \mathrm{Tr}\left[N\left(F(A) + \Lambda\, e \wedge e\right)\right] \triangleright \Psi = 0$$

- The $\Lambda = 0$ case:

path integral representation of the theory from the canonical picture → Relationship between physical inner product of 2+1 gravity and Ponzano-Regge amplitudes

introduction of a regulator: cellular decomposition $\Delta_\Sigma$ of $\Sigma$

$$C_0(N) = \int_\Sigma \mathrm{Tr}[N\, F(A)] = \lim_{\epsilon \to 0} \sum_{p \in \Delta_\Sigma} \mathrm{Tr}[N_p\, W_p(A)]$$

$$W_p(A) = 1 + \epsilon^2 F(A) + O(\epsilon^2) \in SU(2)$$

background independence and anomaly-free quantum constraints algebra

definition of a physical scalar product by means of a projector operator into the kernel of $C_0(N)$

# 2+1 GRAVITY WITH $\Lambda\neq 0$ IN LQG

## QUANTUM ANALYSIS: CONSTRAINTS

Let us define $A_\pm = A \pm \sqrt{\Lambda} e$ and replace $W_p(A) \longrightarrow W_p(A_\pm)$

at the classical level we get

$$C_\Lambda\left[N\right] = \lim_{\epsilon \to 0} \sum_{p \in \Delta_\Sigma} \mathrm{Tr}\left[N_p\, W_p\left(A_\pm\right)\right] - \cancel{\mathcal{G}\left[\pm\sqrt{\Lambda} N\right]}$$

on gauge-invariant states

candidate background independent regularization of the curvature constraint $C_\Lambda[N]$

$\Downarrow$

quantization of the holonomy of $A_\pm$

❖ As a first step toward the quantization of $C_\Lambda[N]$, we are now going to quantize the holonomy of the general connection $A_\lambda = A + \lambda e$

# Quantization of non-commutative holonomies

Quantization of $h_\eta\,[A_\lambda] = P\,\mathrm{e}^{-\int_\eta A+\lambda e}$ as an operator on the kinematical Hilbert space of 2+1 LQG

- action on the vacuum:

$$h_\eta[A_\lambda]|0\rangle = h_\eta[A]|0\rangle$$

simply creates a Wilson line excitation

- action on a transversal Wilson line in the fundamental representation:

quantization of each term in the series expansion of $h_\eta[A\lambda]$ in powers of $\lambda$ → quantization of products of $e$ operators potentially ill-defined due to factor ordering ambiguities

$$h_\eta\,(A_\lambda)\,h_\gamma\,(A_\lambda)\,|0\rangle = h_\eta\,(A_\lambda)\,h_\gamma\,(A)\,|0\rangle = \left(1+\sum_{1\leq n}(-1)^n\int_0^1 \mathrm{d}t_1\cdot\cdot\int_0^{t_{n-1}}\mathrm{d}t_n\,A_\lambda\,(\eta\,(t_1))\cdot\cdot A_\lambda\,(\eta\,(t_n))\right)\triangleright$$
$$\left(1+\sum_{1\leq m}(-1)^m\int_0^1 \mathrm{d}s_1\cdot\cdot\int_0^{s_{m-1}}\mathrm{d}s_m\,A\,(\gamma\,(s_1))\cdot\cdot A\,(\gamma\,(s_m))\right)|0\rangle$$

developing in powers of $\lambda$ the coefficient at order $p$ is

$$\sum_{n\geq p}\sum_{m\geq p}(-1)^{m+n}\sum_{1\leq k_1<\cdots<k_p\leq n}\int_0^1 \mathrm{d}t_1\cdots\int_0^{t_{n-1}}\mathrm{d}t_n\int_0^1\mathrm{d}s_1\cdots\int_0^{s_{m-1}}\mathrm{d}s_m$$
$$\left[A\,(\eta\,(t_1))\cdots E(\eta(t_{k_1}))\cdots E(\eta(t_{k_p}))\cdots A\,(\eta\,(t_n))\right]\triangleright A\,(\gamma\,(s_1))\cdots A\,(\gamma\,(s_m))$$

Let us concentrate on the action of the derivation operators on the connection along γ:

$$\int_0^1 \mathrm{d}s_1 \cdots \int_0^{s_{m-1}} \mathrm{d}s_m \; E(\eta(t_{k_1})) \cdots E(\eta(t_{k_p})) \triangleright A(\gamma(s_1)) \cdots A(\gamma(s_m))$$

one now uses

$$\begin{aligned} E(\eta(t)) \triangleright A(\gamma(s)) &= \left(\epsilon_{ab}\dot{\gamma}^a(s_*)\,\dot{\eta}^b(t^*)\right)\delta(\gamma(s)-\eta(t)) \\ &= o\,\delta(s-s_*)\,\delta(t-t_*) \end{aligned}$$

where $o$ is the orientation of the intersection between $\eta$ and $\gamma$

and the fact that only those terms containing $p$ consecutive graspings $E$'s acting on $p$ consecutive $A$'s remain to get, after rearranging of integration variables

$$\frac{(-io\hbar\lambda)^p}{p!} \sum_{k_1 \geq 1} (-1)^{k_1-1} \int_{t_*}^1 \mathrm{d}t_1 \cdots \int_{t_*}^{t_{k_1-2}} \mathrm{d}t_{k_1-1}\, A(\eta(t_1)) \cdots A(\eta(t_{k_1-1}))$$

$$\tau^{i_{k_1}} \cdots \tau^{i_{k_p}} \sum_{v \geq 0} (-1)^v \int_0^{t_*} \mathrm{d}\tilde{t}_1 \cdots \int_0^{t_{v-1}} \mathrm{d}\tilde{t}_v\, A(\eta(\tilde{t}_1)) \cdots A(\eta(\tilde{t}_v)) \otimes$$

$$\sum_{\alpha_{k_1} \geq 1} (-1)^{\alpha_{k_1}-1} \int_{s_*}^1 \mathrm{d}s_1 \cdots \int_{s_*}^{s_{\alpha_{k_1}-2}} \mathrm{d}s_{\alpha_{k_1}-1}\, A(\gamma(s_1)) \cdots A(\gamma(s_{\alpha_{k_1}-1}))$$

$$\tau_{(i_{k_1}} \cdots \tau_{i_{k_p})} \sum_{u \geq 0} (-1)^u \int_0^{s_*} \mathrm{d}\tilde{s}_1 \cdots \int_0^{s_{u-1}} \mathrm{d}\tilde{s}_u\, A(\gamma(\tilde{s}_1)) \cdots A(\gamma(\tilde{s}_u))$$

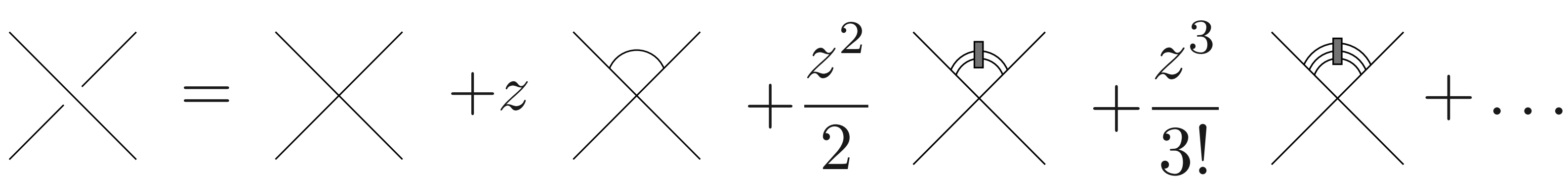
$$\tau_{(i_1} \cdots \tau_{i_p)}$$

ordering ambiguites in the product of generators due to the non-commutativity of grasping operators

symmetrized quantization map

$$Q_S: \quad E_{i_1} E_{i_2} \cdots E_{i_p} \quad \to \quad \frac{1}{p!} \sum_{\pi \in S(p)} \tau_{i_{\pi(1)}} \tau_{i_{\pi(2)}} \cdots \tau_{i_{\pi(p)}}$$

graphical notation for the action of $h_\eta[A\lambda]$ $\qquad z = -io\hbar\lambda$



# The Duflo map

The Duflo map is a generalization of the universal quantization map proposed by Harish-Chandra for semi-simple Lie algebras. The latter provides a prescription to quantize polynomials of commuting variables (the classical triad fields) which after quantization acquire Lie algebra commutation relations (the flux operators).

Given a set of commuting variables $E_i$ on the dual space $\mathbf{g}_*$ of the algebra $\mathbf{g}$, they generate the commutative algebra of polynomials, called the symmetric algebra over $\mathbf{g}$ and denoted ***Sym*(g)**. If now we want to map this algebra into the one generated by non-commutative variables $\tau_i$ which satisfy the commuta tion relations $[\tau_i, \tau_j] = f_{ijk}\, \tau_k$, we run into ordering problem since the commutative algebra *Sym* (g) must be mapped to the non-commutative *universal enveloping algebra* ***U*(g)**. A natural quantization map introduced by Harish-Chandra is the so-called symmetric quantization, defined by its action on monomials, namely

$$Q_S : \; E_{i_1} E_{i_2} \cdots E_{i_n} \quad \rightarrow \quad \frac{1}{n!} \sum_{\pi \in S_n} \tau_{i_{\pi(1)}} \tau_{i_{\pi(2)}} \cdots \tau_{i_{\pi(n)}}$$

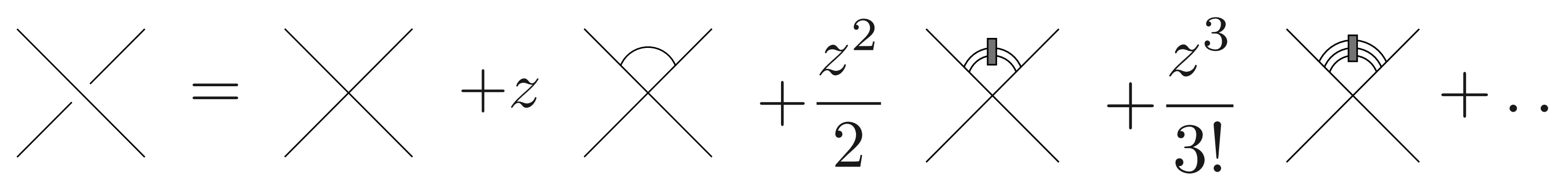
A generalization of the previous map was provided by Duflo by composing it with a differential operator **$j_{1/2}$ (∂)** on *Sym*(g), where $\partial \equiv \partial / \partial E$ represents derivatives with respect to the generators of *Sym* (g). In the case of the Lie algebra ***su*(2)**, the Duflo map $Q_D$ reads

$$Q_D = Q_S \circ j^{\frac{1}{2}}(\partial) = Q_S \circ \left( 1 + \frac{1}{12} \partial_i \partial_i + \cdots \right)$$

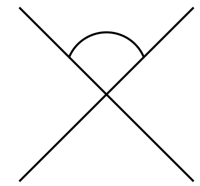
Given two Casimir elements A and B: $Q_D(A)\, Q_D(B) = Q_D(AB)$ ➔ Duflo map is an isomorphism between the invariant sub-algebras ***Sym*(g)$_\mathbf{g}$** and ***U*(g)$_\mathbf{g}$**

# Quantization in terms of flux operators

Quantization of flux operators
+
Duflo map




- First order term: $E$ acts as LIV on $\gamma$ source, as RIV on $\gamma$ target



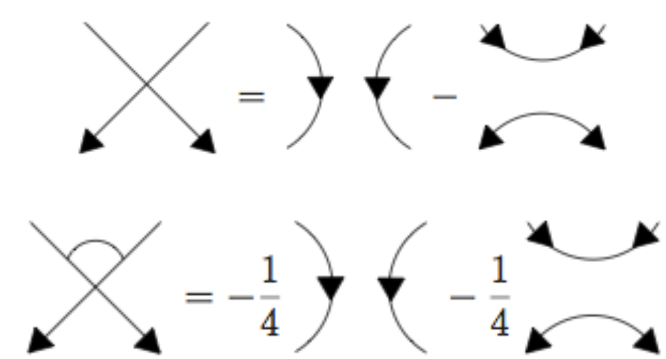
no ambiguity

- Second order term: action of two flux operators at the same point → Duflo map to write $\left(\tau_j\tau_k\right)$

$$
\begin{aligned}
Q_D[E_jE_k] &= Q_S \circ \left(1 + \frac{1}{12}\partial_i\partial_i + \cdots\right)[E_jE_k] \\
&= \frac{1}{2}(\tau_j\tau_k + \tau_k\tau_j) + \frac{1}{6}\delta_{jk}
\end{aligned}
$$

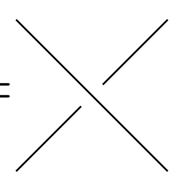

- Third order term proportional to the first order and so on

from representation theory of $SU(2)$
(Penrose notation)

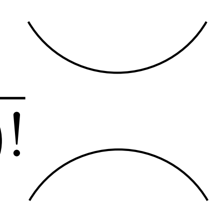



general expression for arbitrary order:

$$h_\eta\left(A_\lambda\right)\, h_\gamma\left(A_\lambda\right)\, |0\rangle =$$

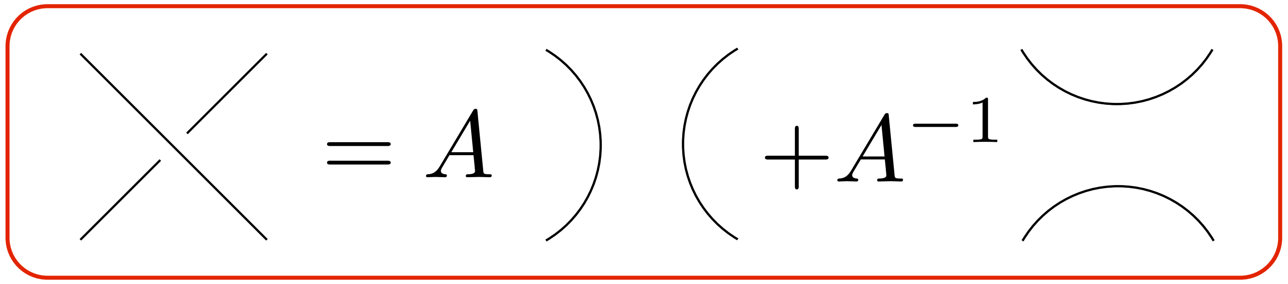


$$= \sum_{n\geq 0}\frac{(-z)^n}{4^n(n)!}\Bigg)\Bigg(-\sum_{n\geq 0}\frac{(z)^n}{4^n(n)!}$$



# Final result

Using Pensrose's convention $\epsilon_{AB} \to i\epsilon_{AB}$ and $\epsilon^{AB} \to i\epsilon^{AB}$



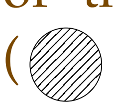

Kauffman's $q$-deformed binor identity for $q = \exp i\lambda/2$

where $A = e^{\frac{io\hbar\lambda}{4}}$

# DISCUSSION

❖ We have shown that the holonomy of $A_\lambda = A + \lambda\, e$ in the fundamental representation can be quantized in the LQG formalism, leading to the Kauffman-like algebraic structure for the action of the quantum holonomy defining a crossing. This result is expected if a relationship between Turaev-Viro amplitudes and physical amplitudes in canonical LQG formulation exists.

❖ The recovering of the Kauffman bracket related to the $q$-deformed crossing identity is a remarkable result since it was obtained starting from the standard $SU(2)$ kinematical Hilbert space of LQG and combining the flux operators representation of the theory together with a mathematical input coming from the Duflo isomorphism.

❖ However, the full link between the role of quantum groups in 3d gravity with $\Lambda \neq 0$ and its canonical quantization can only be established if the dynamical input from the implementation of the curvature constraints is brought in: Reidermeister moves and quantum dimension (  $= -A^2 - A^{-2}$) are only to be found through dynamical considerations.