Introduction	Free product
Main result	Problem
Sketch of proof	Class C
Open questions	Known results

Radial Multipliers on Reduced Free Products

Sören Möller

IMADA University of Southern Denmark, Odense

28. April 2011

回 と く ヨ と く ヨ と

Introduction	Free product
Main result	Problem
Sketch of proof	Class C
Open questions	Known results

Introduction

Free product Problem Class C Known results

Main result Example

Sketch of proof Equivalence $\Phi_{x,y}^{(i)}$

Open questions

∃ → < ∃ →</p>

Free product Problem

Class C Known results

Freenes

Definition For $(A_i) \subseteq (A, \phi)$ unital C*-subalgebras we say (A_i) are free in A if $\forall n \in \mathbb{N} \quad \forall a_j \in A_{i_j}, 1 \leq j \leq n, i_j \neq i_{j+1}, \phi(a_j) = 0$ we have

$$\phi(a_1a_2\cdots a_n)=0.$$

・ロト ・回ト ・ヨト ・ ヨト

Free product Problem

Class C Known results

Reduced free product

Given unital C*-algebras (A_i, ϕ_i) construct algebra (A, ϕ) such that

- (A_i) free in (A, ϕ)
- $\blacktriangleright \phi|_{A_i} = \phi_i$

Notation

$$(A,\phi)=*_i(A_i,\phi_i)$$

・ロン ・回と ・ヨン ・ヨン

Free product Problem Class C

Properties of reduced free product

$$\forall i : A_i \subseteq B(H_i) \Rightarrow A \subseteq B(*_i H_i)$$
$$*_i C_r^*(G_i) = \frac{C_r^*(*_i G_i)}{C1 \oplus \bigoplus_n \bigoplus_{i_1, \dots, i_n, i_j \neq i_{j+1}} \mathring{\mathcal{A}}_{i_1} \cdots \mathring{\mathcal{A}}_{i_n}}$$

where $\mathcal{A}_i = \ker \phi_i$.

・ロト ・回ト ・ヨト ・ヨト

Э

Free product Problem Class C Known results

Problem

Let $\phi : \mathbb{N} \to \mathbb{C}$ and $\mathcal{A} = *_i \mathcal{A}_i$ and define

$$M_{\phi}(a_1 \dots a_n) = \phi(n)a_1 \dots a_n. \tag{1}$$

- ▶ Is M_{ϕ} welldefined on A?
- When is M_{ϕ} completely bounded?
- For which A_i ?
- For which φ?
- $\blacksquare \|M_{\phi}\|_{cb} = ?$

イロン 不同と 不同と 不同と

Introduction Main result Sketch of proof

Open questions

Free product Problem Class C Known results

$\mathsf{Class}\ \mathcal{C}$

Definition

Let $\mathcal C$ denote the set of functions $\phi:\mathbb N_0\to\mathbb C$ for which the Hankel matrices

$$h = (\phi(i+j) - \phi(i+j+1))_{i,j \in \mathbb{N}_0}$$

$$k = (\phi(i+j+1) - \phi(i+j+2))_{i,j \in \mathbb{N}_0}$$

are of trace class and $c = \lim_{n \to \infty} \phi(n)$ exists. For $\phi \in C$ put

$$\|\phi\|_{\mathcal{C}} = \|h\|_1 + \|k\|_1 + |c|.$$

イロン イヨン イヨン イヨン

Free product Problem Class C Known results

Known results

Theorem (Wysoczanski 1995) Let $G = *_{i \in I} G_i$ and $\phi \in C$ then $M_{\phi} : C_r^*(G) \to C_r^*(G)$ is welldefined and

 $\|M_{\phi}\|_{cb} \leq \|\phi\|_{\mathcal{C}}.$

Theorem (Ricard-Xu 2006) Let $\mathcal{A} = *_i \mathcal{A}_i$ and $\phi(n) = s^n$, $s \in]0, 1[$ then $M_{\phi} : \mathcal{A} \to \mathcal{A}$ is welldefined and $\|M_{\phi}\|_{cb} \leq 1.$

・ロト ・回ト ・ヨト ・ ヨト

Example

Main result

Theorem (Haagerup-M)

Let $\mathcal{A} = *_{i \in I}(\mathcal{A}_i, \omega_i)$ be the reduced free product of unital C*-algebras $(A_i)_{i \in I}$ with respect to states $(\omega_i)_{i \in I}$ for which the GNS-representation π_{ω_i} is faithful for all $i \in I$. If $\phi : \mathbb{N}_0 \to \mathbb{C}$ belongs to \mathcal{C} , then there is an unique linear completely bounded map

 $M_{\phi}: \mathcal{A} \to \mathcal{A}$

such that $M_{\phi}(1) = \phi(0)1$ and

$$M_{\phi}(a_1a_2\ldots a_n)=\phi(n)a_1a_2\ldots a_n$$

whenever $a_i \in \check{A}_{i_i} = ker(\omega_{i_i})$ and $i_1 \neq i_2 \neq \cdots \neq i_n$. Moreover $\|M_{\phi}\|_{cb} \leq \|\phi\|_{\mathcal{C}}$. Sören Möller

Example

Example

Let
$$\mathbb{D} = \{s \in \mathbb{C} | |s| < 1\}$$
. For every $s \in \mathbb{D}$
 $\phi_s(n) = s^n$ (2)

defines a radial multiplier M_{ϕ} on $\mathcal{A} = *_{i \in I}(\mathcal{A}_i, \omega_i)$ with

$$\|M_{\phi_s}\|_{cb} \le \frac{|1-s|}{1-|s|}.$$
(3)

イロン イボン イヨン イヨン 三日

Equivalence $\Phi_{x,y}^{(i)}$

Strategy

- Uniqueness of M_{ϕ}
- Reduce to $A_i = B(H_i)$
- Equivalent description of M_{ϕ}
- Construct $\Phi_{x,y}^i$
- Construct T_1, T_2, T
- Show T is M_{ϕ}
- ► Estimate $||T||_{cb}$

- - 4 回 ト - 4 回 ト

Equivalence $\Phi_{x,y}^{(i)}$

Notation

$$H = \mathbb{C}\Omega \oplus \bigoplus_{n=0}^{\infty} \bigoplus_{i_1 \neq \cdots \neq i_n} \mathring{H}_{i_1} \otimes \cdots \otimes \mathring{H}_{i_n}.$$
 (4)

and denote basis by

$$\Lambda = \{\Omega\} \cup \bigcup_{n=1}^{\infty} \{\gamma_1 \otimes \cdots \otimes \gamma_n | \gamma_j \in \mathring{\Gamma}_{i_j}, i_1 \neq \cdots \neq i_n\}.$$
 (5)

For $\gamma \in H$, define $L_{\gamma} \in B(H)$ as

$$L_{\gamma}(\chi) = \begin{cases} \gamma \otimes \chi & \text{if } i \neq i_1 \\ 0 & \text{if } i = i_1 \end{cases}$$

For $\eta, \xi \in H$ let case 2 if $\eta_{|\eta|}, \xi_{|\xi|} \in H_i$ and case 1 otherwise.

Equivalence $\Phi_{x,y}^{(\prime)}$

Equivalent description of M_{ϕ}

Lemma

Let $T : B(H) \to B(H)$ be a bounded linear normal map, and let $\phi : \mathbb{N}_0 \to \mathbb{C}$ be a function on \mathbb{N}_0 . TFAE (a) $T(1) = \phi(0)1$ and

$$T(a_1a_2\ldots a_n)=\phi(n)a_1a_2\ldots a_n \tag{6}$$

whenever $a_j \in B(\mathring{H}_{i_j}) = ker(\omega_{i_j})$ and $i_1 \neq i_2 \neq \cdots \neq i_n$. (b) For all $k, l \in \mathbb{N}_0$ and $\xi \in \Lambda(k), \eta \in \Lambda(l)$ we have

$$T(L_{\xi}L_{\eta}^{*}) = \begin{cases} \phi(k+l)L_{\xi}L_{\eta}^{*} & \text{in case } 1\\ \phi(k+l-1)L_{\xi}L_{\eta}^{*} & \text{in case } 2. \end{cases}$$
(7)

(1) マン・ション・ (1) マン・

Equivalence $\Phi_{x,y}^{(i)}$

Construction of maps

For $x, y \in l^2(\mathbb{N}_0)$ and $a \in B(H)$ put

$$\Phi_{x,y}^{(1)}(a) = \sum_{n=0}^{\infty} D_{(S^*)^{n_x}} a D_{(S^*)^{n_y}}^* + \sum_{n=1}^{\infty} D_{S^{n_x}} \rho^n(a) D_{S^{n_y}}^*$$

$$\Phi_{x,y}^{(2)}(a) = \sum_{n=0}^{\infty} D_{(S^*)^{n_x}} a D_{(S^*)^{n_y}}^* + \sum_{n=1}^{\infty} D_{S^{n_x}} \rho^{n-1}(\epsilon(a)) D_{S^{n_y}}^*$$

$$T_1 = \sum_{i=1}^{\infty} \Phi_{x_i,y_i}^{(1)} \quad \text{for } h = \sum_{i=1}^{\infty} x_i \odot y_i$$

$$T_2 = \sum_{i=1}^{\infty} \Phi_{z_i,w_i}^{(2)} \quad \text{for } k = \sum_{i=1}^{\infty} z_i \odot w_i$$

$$T = T_1 + T_2 + cI.$$

・ロト ・回ト ・ヨト ・ヨト

Э



Properties of
$$\Phi_{x,y}^{(i)}$$

Lemma If $\xi \in \Lambda(k), \eta \in \Lambda(I)$ then

$$\Phi_{x,y}^{(1)}(L_{\xi}L_{\eta}^*) = \left(\sum_{t=0}^{\infty} x(k+t)\overline{y(l+t)}\right) L_{\xi}L_{\eta}^*$$

and

$$\Phi_{x,y}^{(2)}(L_{\xi}L_{\eta}^{*}) = \begin{cases} \sum_{t=0}^{\infty} x(k+t)\overline{y(l+t)}L_{\xi}L_{\eta}^{*} & \text{in case 1} \\ \sum_{t=0}^{\infty} x(k+t-1)\overline{y(l+t-1)}L_{\xi}L_{\eta}^{*} & \text{in case 2.} \end{cases}$$

・ロン ・雪 ・ ・ ヨ ・ ・ ヨ ・ ・



Properties of $\phi \in \mathcal{C}$

Lemma With

$$\phi(n) = \psi_1(n) + \psi_2(n) + c$$

$$\psi_1(k+l) = \sum_{i=1}^{\infty} \sum_{t=0}^{\infty} x_i(k+t)\overline{y_i(l+t)}$$

$$\psi_2(k+l) = \sum_{i=1}^{\infty} \sum_{t=0}^{\infty} z_i(k+t)\overline{w_i(l+t)}.$$
(8)

 \cdots Hence T has the right behavior

・ロト ・回ト ・ヨト ・ヨト

Э

Equivalence $\Phi_{x,y}^{(i)}$

Estimate $||T||_{cb}$

$$\begin{split} \|\Phi_{x_{i},y_{i}}^{(1)}\|_{cb} &\leq \|x_{i}\|_{2}\|y_{i}\|_{2} \\ \|\Phi_{z_{i},w_{i}}^{(2)}\|_{cb} &\leq \|z_{i}\|_{2}\|w_{i}\|_{2} \\ \|T_{1}\|_{cb} &\leq \sum_{i=1}^{\infty} \|\Phi_{x_{i},y_{i}}^{(1)}\|_{cb} \leq \|h\|_{1} \\ \|T_{2}\|_{cb} &\leq \sum_{i=1}^{\infty} \|\Phi_{z_{i},w_{i}}^{(2)}\|_{cb} \leq \|k\|_{1} \\ \|T\|_{cb} &\leq \|T_{1}\|_{cb} + \|T_{2}\|_{cb} + \|cId\|_{cb} \leq \|\phi\|_{c} \end{split}$$

< □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □ > < □

Open questions

- ► For which $(A_i, \omega_i)_{i \in I}$ holds $||M_{\phi}||_{cb} = ||\phi||_{C}$ for all $\phi \in C$?
- Find other examples for $\phi \in \mathcal{C}$ with calculable $\|\phi\|_{\mathcal{C}}$
- Find $\phi \in \mathcal{C}$ with finite support
- Find $\phi_n \in \mathcal{C}$ where $\phi_n \to 1$ pointwise

イロト イポト イヨト イヨト