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Special Uniform Approximations of Continuous Vector-Valued Functions. Part I: Special Approximations in $C_X(T)$

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In this paper, we give special uniform approximations of functions u from the spaces $C_X(T)$ and $C_\infty(T,X)$, with elements $\bar{u}$ of the tensor products $C_\Gamma(T)\otimes X$, respectively $C_0(T,\Gamma)\otimes X$, for a topological space T and a Γ-locally convex space X. We call an approximation special, if $\bar{u}$ satisfies additional constraints, namely $\operatorname{supp} v \subset u^{-1}(X\setminus\{0\})$ and $\bar{u}(T)\subset \operatorname{co}(u(T))$ (resp. $\subset \operatorname{co}(u(T)\cup\{0\})$). In Section 3, we give three distinct applications, which are due exactly to these constraints: a density result with respect to the inductive limit topology, a Tietze–Dugundji's type extension new theorem and a proof of Schauder–Tihonov's fixed point theorem.

Key Words: uniform approximation; tensor product; extension by continuity.

1. INTRODUCTION

Throughout this paper, T is a topological space, X a locally convex space over the field $\Gamma\in\{\mathbf{R},\mathbf{C}\}$ and $C_X(T)$ the linear space of all X-valued continuous functions on T. Consider the vector subspaces

$$C_{\mathrm{b}}(T,X) := \{u\in C_X(T)\mid u(T) \text{ is bounded}\}\subset C_X(T),$$

$$C_{\mathrm{tb}}(T,X) := \{u\in C_X(T)\mid u(T) \text{ is totally bounded}\}\subset C_{\mathrm{b}}(T,X).$$

Recall that a subset $A\subset X$ is said to be totally bounded iff for every $W\in\mathscr{V}_X(0)$, there exists a finite subset $A_0\subset X$, such that $A\subset A_0+W$ (then, we can choose $A_0\subset A$). If T and X are both Hausdorff spaces, we also use the standard notations

$$C_\infty(T,X) := \{u\in C_X(T)\mid \forall W\in\mathscr{V}_X(0),\ u^{-1}(X\setminus \overset{\circ}{W}) \text{ is compact}\},$$

$$C_0(T,X) := \{u\in C_X(T)\mid \operatorname{supp} u := \overline{u^{-1}(X\setminus\{0\})} \text{ is compact}\}.$$

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It is obvious that $C_0(T,X) \subset C_\infty(T,X) \subset C_{\text{tb}}(T,X) \subset C_{\text{b}}(T,X)$. We have the natural inclusions

$$C_\Gamma(T) \otimes X \subset C_X(T), \qquad C_0(T,\Gamma) \otimes X \subset C_0(T,X).$$

Various results concerning the uniform density of $C_\Gamma(T) \otimes X$ in $C_X(T)$ and Weierstrass–Stone's type theorems are known (see [1, 4–8]). Therefore, we will restrict our attention to special uniform approximations and its applications.

2. SPECIAL APPROXIMATIONS IN $C_X(T)$

2.1. The vector space $(C_\Gamma(T) \otimes X)_{\text{loc}}$

It is easily seen that if E is a Γ-normed space and if $u \in C_{\text{b}}(T,E)$ has the following uniform approximation property:

$$\forall \varepsilon > 0,\ \exists u_\varepsilon \in C_\Gamma(T) \otimes E,\ \text{such that}\ \|u - u_\varepsilon\|_\infty < \varepsilon,$$

then $u \in C_{\text{tb}}(T,E)$. Therefore, to get ε-uniform approximations of u for arbitrary $\varepsilon > 0$, we have to accept $u \in C_{\text{tb}}(T,E)$ (Theorem 1 will prove that this condition is also sufficient, even for special approximations) or to replace the vector subspace $C_\Gamma(T) \otimes E$ of $C_E(T)$ by a larger one. This is a reason for:

DEFINITION 1. Consider the "locally tensor product"

$$(C_\Gamma(T) \otimes X)_{\text{loc}} := \{u : T \to X \mid \forall t \in T,\ \exists V \in \mathscr{V}_T(t),$$

$$\exists v \in C_\Gamma(T) \otimes X,\ \text{such that}\ u_{|_V} = v_{|_V}\}.$$

PROPOSITION 1. (1) *$(C_\Gamma(T) \otimes X)_{\text{loc}}$ is a Γ-vector space and*

$$C_\Gamma(T) \otimes X \subset (C_\Gamma(T) \otimes X)_{\text{loc}} \subset C_X(T).$$

(2) *If T is compact, then* $(C_\Gamma(T) \otimes X)_{\text{loc}} = C_\Gamma(T) \otimes X$.

Proof. Statement (1) is evident. The proof of (2) is immediate, using a partition of unity (p.u.) on T. ∎

Remark 1. If $u \in (C_\Gamma(T) \otimes X)_{\text{loc}}$ and K is a compact subset of T, then $u_{|_K} \in C_\Gamma(K) \otimes X$.

2.2. The uniform density of $(C_\Gamma(T)\otimes X)_{\rm loc}$ in $C_X(T)$

THEOREM 1. *Consider $u\in C_X(T)$. If T or $u(T)$ is paracompact or if $u\in C_{\rm tb}(T,X)$, then for every $W\in\mathscr{V}_X(0)$, there exists an approximant $u_W\in(C_\Gamma(T)\otimes X)_{\rm loc}$, such that:*

(1) $(u-u_W)(T)\subset W,\ u_W(T)\subset \mathrm{co}(u(T)),\ \mathrm{supp}\,u_W\subset u^{-1}(X\backslash\{0\})$,

(2) $u_W=\sum_{i\in I}\varphi_i(\cdot)x_i$, *with* $(x_i)_{i\in I}\subset u(T)$ *and* $(\varphi_i)_{i\in I}$ *p.u. on T. Moreover, if $u\in C_{\rm tb}(T,X)$, then I can be choosen as a finite set and, consequently, $u_W\in C_\Gamma(T)\otimes X$.*

Proof. Fix $W\in\mathscr{V}_X(0)$. We can certainly assume that W is open and convex. If $u\in C_{\rm tb}(T,X)$, then $\exists A_0\subset u(T)$, such that A_0 is finite and $u(T)\subset A_0+2^{-1}W$. Set $A:=A_0$ if $u\in C_{\rm tb}(T,X)$, and $A:=u(T)$, otherwise. Thus, $u(T)\subset A+W$, and so $T=\bigcup_{x\in A}u^{-1}(x+W)$. There are three cases to consider:

(a) If T is paracompact, then $\exists(\varphi_x)_{x\in A}$ p.u. on T, subordinated to the open covering $(u^{-1}(x+W))_{x\in A}$ of T.

(b) If $u(T)$ is paracompact, then $\exists(\psi_x)_{x\in A}$ p.u. on $u(T)$, subordinated to the open covering $((x+W)\cap u(T))_{x\in A}$ of $u(T)$. For $x\in A$, set $\varphi_x:=\psi_x\circ u$. Hence, $\mathrm{supp}\,\varphi_x\subset u^{-1}(\mathrm{supp}\,\psi_x)\subset u^{-1}(x+W)\ \forall x\in A$.

(c) If $u\in C_{\rm tb}(T,X)$, then A is finite and $u(T)\subset A+2^{-1}W$. Define the map $\omega:X\to[0,1],\ \omega(z)=0\vee[1-2p_W(z)]$, where p_W means Minkowski's functional associated to W. Clearly, ω is continuous and $\mathrm{supp}\,\omega\subset 2^{-1}\bar W$. For every $x\in A$, define $\omega_x:u(T)\to[0,1],\ \omega_x(z)=\omega(z-x)$. But $\forall z\in u(T)\subset A+2^{-1}W,\ \exists x\in A$, such that $z\in x+2^{-1}W$, which gives $\omega_x(z)=\omega(z-x)>0$. Since $\sum_{y\in A}\omega_y>0$ on $u(T)$, we can define the map $\psi_x=(\sum_{y\in A}\omega_y)^{-1}\omega_x:u(T)\to[0,1]$, $\varphi_x:=\psi_x\circ u\ \forall x\in A$. Clearly, $\mathrm{supp}\,\psi_x\subset(x+2^{-1}\bar W)\cap u(T)\subset(x+W)\cap u(T)$, $\mathrm{supp}\,\varphi_x\subset u^{-1}(\mathrm{supp}\,\psi_x)\subset u^{-1}(x+W)\ \forall x\in A$.

In all the above three cases, $(\varphi_x)_{x\in A}$ p.u. on T, subordinated to the open covering $(u^{-1}(x+W))_{x\in A}$ of T. Now set $v:=\sum_{x\in A}\varphi_x(\cdot)x\in(C_\Gamma(T)\otimes X)_{\rm loc}$. Obviously, $v(T)\subset\ \mathrm{co}(u(T))$. We next show that $(u-v)(T)\subset W$. Fix $t\in T$ and set $A_t:=\{x\in A\mid\varphi_x(t)\neq 0\}$. Thus, A_t is finite, $\sum_{x\in A_t}\varphi_x(t)=1$ and $\forall x\in A_t$, we have $t\in\ \mathrm{supp}\,\varphi_x\subset u^{-1}(x+W)$, and so $(u-v)(t)=\sum_{x\in A_t}\varphi_x(t)(u(t)-x)\in\sum_{x\in A_t}\varphi_x(t)W=W$. Therefore, $(u-v)(T)\subset W$. We need consider two cases:

(i) If $0\notin u(T)$, then $u^{-1}(X\backslash\{0\})=T\supset\mathrm{supp}\,v$, and so $u_W:=v$ satisfies all required properties.

(ii) If $0\in u(T)$, choose $\varepsilon\in(0,1)$. Clearly, $\varepsilon p\bar W\subset\overset{\circ}{W}=W$. Define the maps $\psi:X\to[0,1],\ \psi(x)=(0\vee\frac{p_W(x)-\varepsilon}{1-\varepsilon})\wedge 1,\varphi=\psi\circ u:T\to[0,1]$ and $u_W=\varphi v\in(C_\Gamma(T)\otimes X)_{\rm loc}$. For every $t\in T$, we have the equivalences:

$$\varphi(t)=1\Leftrightarrow p_W(u(t))\geqslant 1\Leftrightarrow u(t)\in X\backslash W\Leftrightarrow t\in T\backslash u^{-1}(W),$$

$$\varphi(t)=0\Leftrightarrow p_W(u(t))\leqslant\varepsilon\Leftrightarrow u(t)\in\varepsilon\bar W\Leftrightarrow t\in u^{-1}(\varepsilon\bar W).$$

Since $v(T) \subset \operatorname{co}(u(T))$, $0 \in u(T)$, $0 \leqslant \varphi \leqslant 1$ and $u_W = \varphi v + (1-\varphi)0$, it follows that $u_W(T) \subset \operatorname{co}(u(T))$, $\operatorname{supp} u_W \subset \operatorname{supp} \varphi = \overline{T \setminus u^{-1}(\varepsilon \bar{W})} \subset T \setminus u^{-1}(\varepsilon W) = u^{-1}(X \setminus \varepsilon W) \subset u^{-1}(X \setminus \{0\})$. It remains to prove that $(u - u_W)(T) \subset W$. We have $u - u_W = u - \varphi v = (1-\varphi)u + \varphi(u - v)$. If $t \in T \setminus u^{-1}(W)$, then $(u - u_W)(t) = (u - v)(t) \in (u - v)(T) \subset W$. If $t \in u^{-1}(W)$, then $(u - u_W)(t) \in (1 - \varphi(t))W + \varphi(t)W = W$. Therefore, $(u - u_W)(T) \subset W$. ∎

Corollary 1. *If T is quasi-compact, then for all $u \in C_X(T)$ and $W \in \mathscr{V}_X(0)$, there exists an approximant $u_W \in C_\Gamma(T) \otimes X$, such that*

$$(u - u_W)(T) \subset W, \; u_W(T) \subset \operatorname{co}(u(T)), \; \operatorname{supp} u_W \subset u^{-1}(X \setminus \{0\}).$$

Corollary 2. *$C_{\mathrm{tb}}(T, \Gamma) \otimes X$ is uniformly dense in $C_{\mathrm{tb}}(T, X)$.*

2.3. *The case of $C_0(T, \Gamma) \otimes X \subset C_\infty(T, X)$*

Theorem 2. *Assume that T and X are Hausdorff. If $u \in C_\infty(T, X)$, then for all $W \in \mathscr{V}_X(0)$ and K a compact subset of T, there exists an approximant $u_{W,K} \in C_0(T, \Gamma) \otimes X$, such that:*

(1) $(u - u_{W,K})(T) \subset W$, $u_{W,K}(T) \subset \operatorname{co}(u(T) \cup \{0\})$, $u_{W,K}(K) \subset \operatorname{co}(u(T))$, $\operatorname{supp} u_{W,K} \subset u^{-1}(X \setminus \{0\})$,

(2) $u_{W,K} = \varphi \cdot \sum_{i \in I} \varphi_i(\cdot) x_i$, *with I finite, $(x_i)_{i \in I} \subset u(T)$, $(\varphi_i)_{i \in I}$ p.u. on T and $\varphi : T \to [0,1]$ a continuous map, such that $\varphi_{|_K} \equiv 1$.*

Proof. We can assume that $u \not\equiv 0$, that is $\exists t_0 \in T$, with $u(t_0) \neq 0$. Now fix $W \in \mathscr{V}_X(0)$, W convex and $K \subset T$, K compact. Set $M := K$ if $0 \notin u(K)$, and $M := \{t_0\}$ if $0 \in u(K)$. Since $u \in C_\infty(T, X) \subset C_{\mathrm{tb}}(T, X)$, by Theorem 1, $\exists v = \sum_{i \in I} \varphi_i(\cdot) x_i \in C_\Gamma(T) \otimes X$, such that $(u - v)(T) \subset W$, $v(T) \subset \operatorname{co}(u(T))$, $\operatorname{supp} v \subset u^{-1}(X \setminus \{0\})$, with I finite, $(x_i)_{i \in I} \subset u(T)$ and $(\varphi_i)_{i \in I}$ p.u. on T. Since $0 \notin u(M)$ and $u(M)$ is compact, $\exists W_0 \in \mathscr{V}_X(0)$, such that $W_0 \subset W$, W_0 open and convex and $u(M) \cap W_0 = \emptyset$, that is $M \subset u^{-1}(X \setminus W_0)$. Now define $\psi : X \to [0,1], \psi(x) = [0 \vee (2p_{W_0}(x) - 1)] \wedge 1$, $\varphi = \psi \circ u : T \to [0,1]$, $w := \varphi v \in C_\Gamma(T) \otimes X$. For every $t \in T$, we have the following equivalences:

$$\varphi(t) = 1 \Leftrightarrow 2p_{W_0}(u(t)) \geqslant 2 \Leftrightarrow u(t) \notin W_0 \Leftrightarrow t \in u^{-1}(X \setminus W_0),$$

$$\varphi(t) = 0 \Leftrightarrow 2p_{W_0}(u(t)) \leqslant 1 \Leftrightarrow 2u(t) \in \bar{W}_0 \Leftrightarrow t \in u^{-1}(2^{-1} \bar{W}_0).$$

Clearly, $w(T) \subset [0,1] \cdot v(T) \subset \operatorname{co}(u(T) \cup \{0\})$. Since $M \subset u^{-1}(X \setminus W_0)$, we have $\varphi_{|_M} \equiv 1$, and so $w(M) = v(M) \subset v(T) \subset \operatorname{co}(u(T))$ and $\operatorname{supp} \varphi = \overline{u^{-1}(X \setminus 2^{-1} \bar{W}_0)} \subset u^{-1}(X \setminus 2^{-1} W_0)$ compact. Hence, $w \in C_0(T, \Gamma) \otimes X$, $\operatorname{supp} w \subset \operatorname{supp} \varphi \subset u^{-1}(X \setminus \{0\})$. We next show that $(u - w)(T) \subset W$. If

$t \in u^{-1}(X \setminus W_0)$, then $(u - w)(t) = (u - v)(t) \in W$ and if $t \in u^{-1}(W_0)$, then $(u - w)(t) = (1 - \varphi(t))u(t) + \varphi(t)(u - v)(t) \in (1 - \varphi(t))W_0 + \varphi(t)W \subset W$. Hence, $(u - w)(T) \subset W$. If $M \neq K$, then $0 \in u(K)$, and so $w(K) \subset w(T) \subset \mathrm{co}(u(T) \cup \{0\}) = \mathrm{co}(u(T))$ and $w = 1 \cdot (\sum_{i \in I} (\varphi\varphi_i)x_i + (1 - \varphi) \cdot 0)$. We conclude that $u_{W,K} := w$ satisfies all required properties. ∎

3. APPLICATIONS OF SPECIAL APPROXIMATIONS

3.1. The density of $C_0(T, \Gamma) \otimes X$ in $C_0(T, X)$ with respect to the inductive limit topology

The following theorem is due to the constraint on the approximant's support.

THEOREM 3. *Assume that T and X are Hausdorff. If $u \in C_0(T, X)$, then for every $V \in \mathscr{V}_{C_0(T,X)}(0)$ with respect to the inductive limit topology, there exists an approximant $u_V \in C_0(T, \Gamma) \otimes X$, such that:*

(1) $u - u_V \in V$, $u_V(T) \subset \mathrm{co}(u(T))$, $\mathrm{supp}\, u_V \subset u^{-1}(X \setminus \{0\})$,

(2) $u_V = \sum_{i \in I} \varphi_i(\cdot)x_i$, *with I finite, $(x_i)_{i \in I} \subset u(T)$ and $(\varphi_i)_{i \in I}$ p.u. on T.*

Proof. We can certainly assume that $0 \in u(T)$, since otherwise T is compact and the conclusion is given by Theorem 1. Fix $V \in \mathscr{V}_{C_0(T,X)}(0)$ and set $K := \mathrm{supp}\, u$, $C_0(T, X)_K := \{w \in C_0(T, X) \mid \mathrm{supp}\, w \subset K\}$. Since $V \cap C_0(T, X)_K$ is a neighborhood of the origin in $C_0(T, X)_K$ with respect to the uniform convergence topology, $\exists W \in \mathscr{V}_X(0)$, such that $\{w \in C_0(T, X)_K \mid w(T) \subset W\} \subset V \cap C_0(T, X)_K$. Now Theorem 2 shows that $\exists v = \varphi \cdot \sum_{i \in I} \varphi_i(\cdot)x_i \in C_0(T, \Gamma) \otimes X$, with I finite, $(x_i)_{i \in I} \subset u(T)$, $(\varphi_i)_{i \in I}$ p.u. on T, $\varphi : T \to [0, 1]$ continuous, $\varphi_{|_K} \equiv 1$, and such that $(u - v)(T) \subset W$, $v(T) \subset \mathrm{co}(u(T))$, $\mathrm{supp}\, v \subset u^{-1}(X \setminus \{0\}) \subset K$. We thus get $u - v \in C_0(T, X)_K$, $(u - v)(T) \subset W$, and so $u - v \in V$. Since $v = \sum_{i \in I} (\varphi\varphi_i)x_i + (1 - \varphi) \cdot 0$, $u_V := v$ satisfies all required properties. ∎

COROLLARY 3. *If T, X are Hausdorff, then $C_0(T, \Gamma) \otimes X$ is dense in $C_0(T, X)$ with respect to the inductive limit topology. Moreover, if X is metrizable, then this density is sequential.*

Proof. The proof is immediate, with Theorem 3. ∎

3.2. A Tietze–Dugundji's type extension theorem

In this subsection, T denotes a topological space and X a Γ-locally convex Hausdorff space. The following two lemmas emphasize the existing connection between approximation and extension theorems.

LEMMA 1. *Assume that X is also a Fréchet space. Then, for every subset $F \subset T$, the following two statements are equivalent*:

(1) *For all* $u \in C_X(F)$ and $W \in \mathscr{V}_X(0)$, *there exists* $u_W \in C_X(T)$, *such that* $(u - u_W)(F) \subset W$.

(2) *For every* $u \in C_X(F)$, *there exists* $\tilde{u} \in C_X(T)$, such that $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \overline{\mathrm{co}(u(F))}$.

Proof. It is to prove (1) $\Rightarrow$ (2). Let us first show that $\forall u \in C_X(F)$, $\forall W \in \mathscr{V}_X(0), \exists v \in C_X(T)$, with $(u - v)(F) \subset W, v(T) \subset \mathrm{co}(u(F)) + W$. Fix $u \in C_X(F)$, $W \in \mathscr{V}_X(0)$ and choose $W_0 \in \mathscr{V}_X(0)$, W_0 balanced, convex, such that $2W_0 \subset W$. According to our hypothesis, $\exists w \in C_X(T)$, with $(u - w)(F) \subset W_0$. It follows that $\overline{w(F)} \subset w(F) + W_0 \subset \mathrm{co}\,(u(F)) + 2W_0$, and so $\mathrm{co}(\overline{w(F)}) \subset \mathrm{co}(u(F)) + 2W_0 \subset \mathrm{co}(u(F)) + W$. Now define the map $f : \overline{w(F)} \to X$, $f(x) = x$. By Dugundji's theorem, $\exists \tilde{f} \in C_X(X)$, such that $\tilde{f}_{|_{\overline{w(F)}}} = f$ and $\tilde{f}(X) \subset \mathrm{co}(\overline{w(F)})$. Set $v := \tilde{f} \circ w \in C_X(T)$. It follows that $v_{|_F} = \tilde{f} \circ w_{|_F} = w_{|_F}$, $(u - v)(F) = (u - w)(F) \subset W_0 \subset W$, $v(T) \subset \tilde{f}(X) \subset \mathrm{co}(\overline{w(F)}) \subset \mathrm{co}(u(F)) + W$. To finally prove (2), fix again $u \in C_X(F)$. Since X is metrizable, we can choose $(W_n)_{n \in \mathbf{N}}$ a fundamental system of convex neighborhoods of the origin in X, with $2W_{n+1} \subset W_n \; \forall n \in \mathbf{N}$. For $u_0 := u \in C_X(F)$, $\exists v_0 \in C_X(T)$, such that $(u_0 - v_0)(F) \subset W_0$ and $v_0(T) \subset \mathrm{co}(u_0(F)) + W_0$. Set $u_1 := u_0 - v_0|_F \in C_X(F)$. Thus, we can inductively define $(u_n)_{n \in \mathbf{N}} \subset C_X(F)$, $(v_n)_{n \in \mathbf{N}} \subset C_X(T)$, such that $\forall n \in \mathbf{N}$, we have: $(u_n - v_n)(F) \subset W_n$, $v_n(T) \subset \mathrm{co}(u_n(F)) + W_n$, $u_{n+1} := u_n - v_n|_F$. Hence, $\forall n \in \mathbf{N}, u_{n+1}(F) = (u_n - v_n)(F) \subset W_n$, $v_{n+2}(T) \subset \mathrm{co}(u_{n+2}(F)) + W_{n+2} \subset W_{n+1} + W_{n+2} \subset 2W_{n+1} \subset W_n$. Therefore, $u_n \overset{\mathrm{u.}}{\to} 0$ on F and $\sum_{n \geqslant 0} v_n$ is uniformly convergent on T. Set $v := \sum_{n=0}^{\infty} v_n \in C_X(T)$. But $\forall t \in F$, we have $v(t) = \lim_{n\to\infty} \sum_{j=0}^{n} v_j(t) = \lim_{n\to\infty} \sum_{j=0}^{n} (u_j(t) - u_{j+1}(t)) = u(t)$, and so $v_{|_F} = u$. As before, $\exists \tilde{g} \in C_X(X)$, with $\tilde{g}(x) = x \; \forall x \in \overline{u(F)}$ and $\tilde{g}(X) \subset \mathrm{co}(\overline{u(F)})$. For $\tilde{u} := \tilde{g} \circ v \in C_X(T)$, we clearly have $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \overline{\mathrm{co}(u(F))}$. ∎

LEMMA 2. *Assume that X is also a Fréchet space. Then, for every subset $F \subset T$, the following two statements are equivalent*:

(1) *For all $u \in C_{\mathrm{tb}}(F, X)$ and $W \in \mathscr{V}_X(0)$, there exists $u_W \in C_{\mathrm{tb}}(T, X)$, such that $(u - u_W)(F) \subset W$.*

(2) *For every $u \in C_{\mathrm{tb}}(F, X)$, there exists $\tilde{u} \in C_{\mathrm{tb}}(T, X)$, such that $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \mathrm{co}(\overline{u(F)})$.*

Proof. It is to show (1) $\Rightarrow$ (2). The proof is similar to that of Lemma 1, observing first that $\forall u \in C_{\mathrm{tb}}(F, X), \forall W \in \mathscr{V}_X(0), \exists v \in C_{\mathrm{tb}}(T, X)$, with $(u - v)(F) \subset W$, $v(T) \subset \mathrm{co}(u(F)) + W$. Consequently, $u - v \in C_{\mathrm{tb}}(F, X)$. Thus, the same construction as in the previous proof finally leads to the maps $v := \sum_{n=0}^{\infty} v_n \in C_{\mathrm{tb}}(T, X)$ and $\tilde{u} := \tilde{g} \circ v \in C_{\mathrm{tb}}(T, X)$. ∎

The following lemma is a variant of Theorem 1.

LEMMA 3. *Assume that T is normal. Consider a closed subset $F \subset T$ and $u \in C_X(F)$. If T is paracompact or $u \in C_{\text{tb}}(F, X)$, then for every $W \in \mathscr{V}_X(0)$, there is a $u_W \in (C_\Gamma(T) \otimes X)_{\text{loc}}$, with $(u - u_W)(F) \subset W$ and $u_W(T) \subset$ $\text{co}(u(F))$. Moreover, if $u \in C_{\text{tb}}(F, X)$, then we can find $u_W \in C_\Gamma(T) \otimes X$.*

Proof. Fix $W \in \mathscr{V}_X(0)$, with W open and convex. If $u \in C_{\text{tb}}(F, X)$, then $\exists A_0 \subset u(F)$, with A_0 finite and $u(F) \subset A_0 + W$. Set $A := A_0$ if $u \in C_{\text{tb}}(F, X)$, and $A := u(F)$, otherwise. Therefore, $u(F) \subset A + W$, $F = u^{-1}(A + W) = \bigcup_{x \in A} u^{-1}(x + W)$. For every $x \in A$, $u^{-1}(x + W)$ is open in F, and so $\exists U_x$ open in T, with $u^{-1}(x + W) = U_x \cap F$. Thus, $T = (T \setminus F) \cup \bigcup_{x \in A} U_x = \bigcup_{x \in B} U_x$, where $B := A \cup \{A\}$, $U_A := T \setminus F$. If $u \in C_{\text{tb}}(F, X)$, then $(U_x)_{x \in B}$ is a finite open covering of T, which is a normal space. If $u \notin C_{\text{tb}}(F, X)$, then $(U_x)_{x \in B}$ is an open covering of the paracompact space T. In both cases, $\exists (\varphi_x)_{x \in B}$ p.u. on T, subordinated to $(U_x)_{x \in B}$. Hence, $\varphi_{_A} = 1 - \sum_{x \in A} \varphi_x$, $\text{supp}\, \varphi_{_A} \subset T \setminus F$. Now choose $z \in u(F)$ and consider $u_W := \sum_{x \in A} \varphi_x(\cdot)x + \varphi_{_A}(\cdot)z \in (C_\Gamma(T) \otimes X)_{\text{loc}}$. Obviously, $u_W(T) \subset \text{co}(u(F))$. We next show that $(u - u_W)(F) \subset W$. Fix $t \in F$ and set $J := \{x \in A \mid \varphi_x(t) \neq 0\}$. Thus, $\sum_{x \in J} \varphi_x(t) = 1$, $(u - u_W)(t) = \sum_{x \in J} \varphi_x(t) \times (u(t) - x)$. But $\forall x \in J$ we have $\varphi_x(t) \neq 0$, and so $t \in \text{supp}\, \varphi_x \subset U_x$, $u(t) \in x + W$. We thus get $(u - u_W)(t) \in \sum_{x \in J} \varphi_x(t) W = W$. Hence, u_W satisfies all required properties. ∎

THEOREM 4. *Assume that T is normal and X is Fréchet. Consider a closed subset $F \subset T$ and $u \in C_X(F)$. If T is paracompact or $u \in C_{\text{tb}}(F, X)$, then there exists $\tilde{u} \in C_X(T)$, such that $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \text{co}(\overline{u(F)})$.*

Proof. By Lemma 3, F satisfies condition (1) of Lemma 1 or 2. In both cases, $\exists \tilde{u} \in C_X(T)$, such that $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \text{co}(\overline{u(F)})$. ∎

The above theorem is a strengthening of Theorem 3.6 [6], since u need not be a compact map (if T is paracompact) and X need not be a Banach space. For a metrizable space T, we recover Dugundji's extension theorem (see [2, 3]). The following corollary is known (see [6, Corollary 3.5, p. 54]).

COROLLARY 4. *Assume that T is completely regular and that X is metrizable. Consider a compact subset $F \subset T$ and $u \in C_X(F)$. Then, there exists $\tilde{u} \in C_X(T)$, such that $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \text{co}(u(F))$.*

Proof. Since T is completely regular, $\exists \tilde{T}$ a compact topological space, such that T is a dense subspace of $\tilde{T}$. Let $\tilde{X}$ denote the completion of X. Since $u \in C_{\tilde{X}}(F)$ and F is compact in $\tilde{T}$, Theorem 4 shows that $\exists v \in C_{\tilde{X}}(\tilde{T})$, such that $v_{|_F} = u$, $v(\tilde{T}) \subset \operatorname{co}(\overline{u(F)}) = \operatorname{co}(u(F)) \subset X$. We thus get $\tilde{u} := v_{|_T} \in C_X(T)$, $\tilde{u}_{|_F} = u$, $\tilde{u}(T) \subset \operatorname{co}(u(F))$. ∎

COROLLARY 5. *Assume that T is σ-compact and that X is metrizable. Consider a closed subset $F \subset T$ and $u \in C_X(F)$. Then, there exists $\tilde{u} \in C_X(T)$, such that $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \operatorname{co}(u(F))$.*

Proof. By hypothesis, $\exists (K_n)_{n\in\mathbf{N}}$ a family of compact subsets of T, such that $T = \bigcup_{n\in\mathbf{N}} K_n$ and $K_n \subset \overset{\circ}{K}_{n+1}$ $\forall n \in \mathbf{N}$. Set $F_n := F \cap K_n$ $\forall n \in \mathbf{N}$. Since F_0 is compact, K_0 is normal and $u_{|_{F_0}} \in C_X(F_0)$, by Corollary 4, $\exists u_0 \in C_X(K_0)$, with $u_0|_{F_0} = u_{|_{F_0}}$, $u_0(K_0) \subset \operatorname{co}(u(F_0))$. For fixed $n \in \mathbf{N}$, assume that $\exists u_n \in C_X(K_n)$, such that $u_n|_{F_n} = u_{|_{F_n}}, u_n(K_n) \subset \operatorname{co}(u(F_n))$. Define $v \in C_X(K_n \cup F_{n+1})$ by $v_{|_{K_n}} = u_n$, $v_{|_{F_{n+1}}} = u_{|_{F_{n+1}}}$. By Corollary 4, it follows that $\exists u_{n+1} \in C_X(K_{n+1})$, such that $u_{n+1}|_{(K_n\cup F_{n+1})} = v$ and $u_{n+1}(K_{n+1}) \subset \operatorname{co}(v(K_n \cup F_{n+1}))$. This easily gives $u_{n+1}|_{F_{n+1}} = u_{|_{F_{n+1}}}$, $u_{n+1}(K_{n+1}) \subset \operatorname{co}(u_n(K_n) \cup u(F_{n+1})) = \operatorname{co}(u(F_{n+1}))$ and $u_{n+1}|_{K_n} = u_n$. Therefore, we can inductively define $u_n \in C_X(K_n)$, such that $\forall n \in \mathbf{N}$, we have $u_{n+1}|_{K_n} = u_n$, $u_n|_{F_n} = u_{|_{F_n}}$ and $u_n(K_n) \subset \operatorname{co}(u(F_n))$. It follows that $\exists \tilde{u} \in C_X(T)$, defined by $\tilde{u}_{|_{K_n}} = u_n$ $\forall n \in \mathbf{N}$. Obviously, $\tilde{u}_{|_F} = u$ and $\tilde{u}(T) \subset \operatorname{co}(u(F))$. ∎

3.3. A proof of Schauder–Tihonov's fixed point theorem

LEMMA 4. *Let T be a topological space, Y a Γ-topological vector space and $(u_\delta)_{\delta\in\Delta} \subset C_Y(T), (t_\delta)_{\delta\in\Delta} \subset T$ nets, such that $t_\delta \to t \in T$ and $u_\delta \xrightarrow{\text{u.}} u$, $u : T \to Y$. Then, $u_\delta(t_\delta) \to u(t)$.*

Proof. The proof is standard. ∎

The following proof is due to the constraint on the approximant's range.

THEOREM 5 (Schauder–Tihonov). *Let X be a Γ-locally convex Hausdorff space, M a closed convex subset of X and $U : M \to M$ a completely continuous operator. Then, U has a fixed point.*

Proof. Since U is completely continuous, $K := \overline{U(M)}$ is compact in X. Clearly, $K \subset M$. On $\mathscr{W} := \{W \in \mathscr{V}_X(0) \mid W \text{ is balanced}\}$, we consider the usual order relation, given by: $W_1 \preceq W_2 \Leftrightarrow W_1 \supset W_2$. Now fix $W \in \mathscr{W}$. Since $U \in C_{\text{tb}}(M, X)$, by Theorem 1, $\exists U_W = \sum_{i\in I} \varphi_i(\cdot)x_i \in C_\Gamma(M) \otimes X$, such that $(U - U_W)(M) \subset W$, $(x_i)_{i\in I} \subset U(M), (\varphi_i)_{i\in I}$ p.u. on M for some finite set I. Set $K_W := \operatorname{co}(\{x_i \mid i \in I\}) \subset \operatorname{co}(U(M)) \subset M$, $X_W := \operatorname{Sp}(K_W)$. But X_W is clearly normable since it is Hausdorff and has finite dimension, K_W is convex and compact and $U_W(K_W) \subset U_W(M) \subset K_W$. Now Brouwer's fixed point theorem shows that $\exists x_W \in K_W$, such that $U_W(x_W) = x_W$. This leads to $x_W = U_W(x_W) \in U_W(M) \subset U(M) + W \subset K + W$, and so $\exists y_W \in K$, such that $x_W - y_W \in W$. We thus get $(U_W)_{W\in\mathscr{W}} \subset C_X(M)$, $(x_W)_{W\in\mathscr{W}} \subset M$ and $(y_W)_{W\in\mathscr{W}} \subset K$, nets with the above properties. But $(U - U_W)(M) \subset W$ and $x_W - y_W \in W\ \forall W \in \mathscr{W}$, lead to $U_W \overset{\text{u.}}{\to} U$ and $x_W - y_W \to 0$. As K is compact, $(y_W)_{W\in\mathscr{W}} \subset K$ has a subnet $(y_{\varphi(\delta)})_{\delta\in\Delta}$, convergent to an element $\xi \in K$. Therefore, $U_{\varphi(\delta)} \overset{\text{u.}}{\to} U$ and $x_{\varphi(\delta)} \to \xi$. Now Lemma 4 gives $x_{\varphi(\delta)} = U_{\varphi(\delta)}(x_{\varphi(\delta)}) \to U(\xi)$, and so $U(\xi) = \xi$. ∎

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