Ahead of Print
DOI 10.1515/advgeom-2013-0006

Advances in Geometry

A note on CR quaternionic maps

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(Communicated by G. Gentili)

Abstract. We introduce the notion of *CR quaternionic map* and we prove that any such real-analytic map, between CR quaternionic manifolds, is the restriction of a quaternionic map between quaternionic manifolds. As an application, we prove, for example, that for any submanifold M of dimension $4k-1$ of a quaternionic manifold N such that TM generates a quaternionic subbundle of $TN|_M$ of (real) rank $4k$, there exists, locally, a quaternionic submanifold of N containing M as a hypersurface.

2010 Mathematics Subject Classification. Primary 53C26; Secondary 53C28

Introduction

An *almost quaternionic manifold* is a manifold M whose tangent bundle is a quaternionic vector bundle (see, for example, [2], [5], and, also, Section 2 below). Further, any compatible connection on M induces an almost complex structure on the bundle Z of admissible linear complex structures, whose integrability characterises the integrability of the almost quaternionic structure of M (that is, if $\dim M \geq 8$, the existence of a torsion free compatible connection; see [5, Remark 2.10(2)]); then Z is the *twistor space* of M.

Accordingly, a *quaternionic map* [5] between quaternionic manifolds is, essentially, a map admitting a holomorphic lift between the corresponding twistor spaces. This generalizes the well-known notion of quaternionic submanifold (necessarily, totally geodesic with respect to any compatible torsion free connection [1]; see [12]). However, not all submanifolds of a quaternionic manifold are quaternionic (take, for example, any hypersurface). Nevertheless, the generic submanifold of codimension at most $2k-1$ of a quaternionic manifold of dimension $4k$ inherits a *CR quaternionic structure*; moreover, any real-analytic CR quaternionic structure is obtained this way through a germ unique

*S.M. acknowledges that this work was done under the program of GNSAGA-INDAM of C.N.R. and PRIN07 “Geometria Riemanniana e strutture differenziabili” of MIUR (Italy).

†R. P. acknowledges that this work was supported by a grant of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI, project number PN-II-ID-PCE-2011-3-0362.

embedding into a quaternionic manifold, which is called its *heaven space* [10, Corollary 5.4].

In this note, we strengthen this result by proving that any real-analytic *CR quaternionic map* (that is, any real-analytic map admitting a CR lift between the corresponding twistor spaces) is the restriction of a quaternionic map between quaternionic manifolds (Theorem 3.2). This shows that the CR quaternionic maps are the natural morphisms of the CR Quaternionic Geometry.

We also apply this result to the study of (almost) CR quaternionic submanifolds, thus obtaining Corollaries 3.4 and 3.5. By the latter, for any submanifold M of dimension $4k-1$ of a quaternionic manifold N such that TM generates a quaternionic subbundle of $TN|_M$ of (real) rank $4k$, there exists, locally, a quaternionic submanifold of N containing M as a hypersurface.

1 Brief review of CR quaternionic linear maps

Recall (see [5], [10] and the references therein) that a *linear complex structure* on a (real) vector space U is a linear map $J: U \to U$ such that $J^2 = -\mathrm{Id}_U$. Then the $-\mathrm{i}$ eigenspace C of J satisfies $C \oplus \overline{C} = U^{\mathbb{C}}$; moreover, any such complex vector subspace of the complexification $U^{\mathbb{C}}$ is the $-\mathrm{i}$ eigenspace of a (unique) linear complex structure on U.

More generally, a *linear CR structure* on a vector space U is a complex vector subspace $C \subseteq U^{\mathbb{C}}$ such that $C \cap \overline{C} = \{0\}$. A map $t: (U, C) \to (U', C')$ between CR vector spaces is a *CR linear map* if it is linear and $t(C) \subseteq C'$ (see [10] and [11] for these and, also, for the dual notion of *linear co-CR structure*).

There are other ways to describe a linear CR structure. For example, if C is a linear CR structure on U and we denote $E = U^{\mathbb{C}}/C$, and by $\iota: U \to E$ the composition of the inclusion $U \to U^{\mathbb{C}}$ followed by the projection $U^{\mathbb{C}} \to E$, then

(a) ι is injective,
(b) $\operatorname{im}\iota + J(\operatorname{im}\iota) = E$,
(c) $C = \iota^{-1}\big(\ker(J + \mathrm{i})\big)$,

where J is the linear complex structure of E. Moreover, the pair (E, ι) is unique, up to complex linear isomorphisms, with these properties.

Thus, we could define a linear CR structure on a vector space U as a pair (E, ι), where E is a complex vector space and $\iota: U \to E$ is a linear map satisfying properties (a) and (b) above. Furthermore, in this setting, the CR linear maps $t: (U, E, \iota) \to (U', E', \iota')$ are characterised by the fact that there exists a necessarily unique, if $t \neq 0$ complex linear map $\widetilde{t}: E \to E'$ such that $\iota' \circ t = \widetilde{t} \circ \iota$.

Lemma 1.1. *t is injective if and only if $\widetilde{t}$ is injective.*

Proof. The fact that $\widetilde{t}$ injective implies t injective is obvious. For the converse, it is sufficient to consider the case $U' = E'$.

Let F be the complex vector subspace of E' spanned by the image of t. Then $\widetilde{t}$ decomposes into a complex linear isomorphism from E onto F followed by the inclusion of F into E'.

Let $\mathbb{H}$ be the division algebra of quaternions, and note that its automorphism group is $\mathrm{SO}(3)$ acting trivially on $\mathbb{R}$ and canonically on $\mathrm{Im}\mathbb{H}(=\mathbb{R}^3)$.

A *linear quaternionic structure* on a vector space U is an equivalence class of morphisms of associative algebras from $\mathbb{H}$ to $\mathrm{End}(U)$, where two such morphisms σ and τ are equivalent if $\tau = \sigma \circ a$, for some $a \in \mathrm{SO}(3)$.

Note that, if $\sigma : \mathbb{H} \to \mathrm{End}(U)$ defines a linear quaternionic structure on U then the space of *admissible linear complex structures* $Z = \sigma(S^2)$ is well-defined [2].

Let U and U' be quaternionic vector spaces and let Z and Z' be the corresponding spaces of admissible linear complex structures, respectively. A *quaternionic linear map* from U to U' is a linear map $t : U \to U'$ for which there exists a function $T : Z \to Z'$ such that $t \circ J = T(J) \circ t$, for any $J \in Z$; then, if $t \neq 0$, we have that T is unique and an orientation preserving isometry [5].

Definition 1.2 ([10]). 1) A *linear CR quaternionic structure* on a vector space U is a pair (E, ι), where E is a quaternionic vector space and $\iota : U \to E$ is an injective linear map such that $\mathrm{im}\,\iota + J(\mathrm{im}\,\iota) = E$, for any admissible linear complex structure J on E.

2) A *CR quaternionic linear map* $t : (U, E, \iota) \to (U', E', \iota')$ between CR quaternionic vector spaces is a linear map $t : U \to U'$ for which there exists a quaternionic linear map $\widetilde{t} : E \to E'$ such that $\iota' \circ t = \widetilde{t} \circ \iota$.

Any quaternionic vector space of (real) dimension $4k$ is (non-canonically) isomorphic to $\mathbb{H}^k$. More generally (and much less trivial), any CR quaternionic vector spaces is isomorphic to a finite product, unique up to the order of factors, in which each factor is contained in one of the following two classes [10, Corollary 3.7].

Example 1.3. For $k \geq 1$, let V_k be the vector subspace of $\mathbb{H}^k$ formed of all vectors of the form $(z_1, \overline{z_1} + z_2\mathrm{j}, z_3 - \overline{z_2}\mathrm{j}, \dots)$, where $z_1, \dots, z_k$ are complex numbers with $\overline{z_k} = (-1)^k z_k$. Then $(U_k, \mathbb{H}^k)$ is a CR quaternionic vector space, where $U_k = V_k^{\perp}$ and we have used the canonical Euclidean structure of $\mathbb{H}^k$.

Example 1.4. Let $V_0' = \{0\}(\subseteq \mathbb{H})$ and, for $k \geq 1$, let V_k' be the vector subspace of $\mathbb{H}^{2k+1}$ formed of all vectors of the form $(z_1, \overline{z_1} + z_2\mathrm{j}, z_3 - \overline{z_2}\mathrm{j}, \dots, \overline{z_{2k-1}} + z_{2k}\mathrm{j}, -\overline{z_{2k}}\mathrm{j})$, where $z_1, \dots, z_{2k}$ are complex numbers. Then $(U_k', \mathbb{H}^{2k+1})$ is a CR quaternionic vector space, where $U_k' = (V_k')^{\perp}$.

It follows that if E is a quaternionic vector space, $\dim E = 4k$, and $U \subseteq E$ is a generic vector subspace of codimension at most $2k-1$, then (U, E) is a CR quaternionic vector space [10].

2 CR quaternionic manifolds

An *almost CR structure* on a manifold M is a complex vector subbundle $\mathcal{C}$ of $T^{\mathbb{C}}M$ such that $\mathcal{C} \cap \overline{\mathcal{C}} = 0$. An (*integrable almost*) *CR structure* is an almost CR structure whose space of sections is closed under the usual bracket.

If M is a (real) hypersurface in a complex manifold (N, J) then $T^{\mathbb{C}}M \cap \ker(J+\mathrm{i})$ is a CR structure on M. This admits a straightforward generalization to higher codimensions. Furthermore, any real-analytic CR structure is obtained this way through a germ unique embedding into a complex manifold [3].

Let $\mathfrak{a}$ be a (finite-dimensional) associative algebra. A *bundle of associative algebras, with typical fibre* $\mathfrak{a}$, is a vector bundle A, with typical fibre $\mathfrak{a}$, whose structural group is the automorphism group of $\mathfrak{a}$; in particular, each fibre of A is an associative algebra (non-canonically) isomorphic to $\mathfrak{a}$. A *morphism of bundles of associative algebras* is a vector bundle morphism whose restriction to each fibre is a morphism of associative algebras.

A *quaternionic vector bundle* is a vector bundle E endowed with a pair (A, σ), where A is a bundle of associative algebras, with typical fibre $\mathbb{H}$, and $\sigma : A \to \mathrm{End}(E)$ is a morphism of bundles of associative algebras. Then the bundle Z of *admissible linear complex structures* on E is the sphere bundle of $\sigma(\mathrm{Im}A)$; in particular, the typical fibre of Z is the Riemann sphere. A *compatible connection* on E is a linear connection ∇ on E such that the connection induced by ∇ on $\mathrm{End}(E)$ restricts to a connection on the image of σ; note that ∇, also, induces a connection on Z.

An *almost CR quaternionic structure* [10] on a manifold M is a pair (E, ι), where E is a quaternionic vector bundle over M and $\iota : TM \to E$ is an injective vector bundle morphism such that (E_x, ι_x) is a linear CR quaternionic structure on T_xM, for any $x \in M$ (see [11] for the dual notion of *almost co-CR quaternionic structure*).

Let (E, ι) be an almost quaternionic structure on M and let ∇ be a compatible connection on E. Denote by $\mathcal{B}$ the complex vector subbundle of $T^{\mathbb{C}}Z$ whose fibre, at each $J \in Z$, is the horizontal lift of $\iota^{-1}_{\pi(J)}\big(\ker(J+\mathrm{i})\big)$, where $\pi : Z \to M$ is the projection. Then $\mathcal{C} = \mathcal{B} \oplus (\ker \mathrm{d}\pi)^{0,1}$ is an almost CR structure on Z.

If $\mathcal{C}$ is integrable then (E, ι, ∇) is a *CR quaternionic structure*, (M, E, ι, ∇) is a *CR quaternionic manifold* and $(Z, \mathcal{C})$ is its *twistor space*. Also, $(M, Z, \pi, \mathcal{C})$ is the *twistorial structure* [8] of (M, E, ι, ∇).

Note that, if ι is a vector bundle isomorphism then we retrieve the usual notion of quaternionic structure/manifold (see [5, Remark 2.10(2)]).

Let M be a hypersurface in a quaternionic manifold N. Then $(TN|_M, \iota, \nabla|_M)$ is a CR quaternionic structure on M, where $\iota : TM \to TN|_M$ is the inclusion and ∇ is any quaternionic connection on N (that is, ∇ is a torsion free compatible connection on N). Further, this admits a straightforward generalization to higher codimensions. Moreover, any real-analytic CR quaternionic structure is obtained this way through a germ unique embedding into a quaternionic manifold, which is called its *heaven space* [10, Corollary 5.4].

3 CR quaternionic maps

Let (M, E_M, ι_M) and (N, E_N, ι_N) be almost CR quaternionic manifolds and let $\pi_M : Z_M \to M$ and $\pi_N : Z_N \to N$ be the corresponding bundles of admissible linear complex structures, respectively. Also, let $\varphi : M \to N$ and $\Phi : Z_M \to Z_N$ be maps.

We say that $\varphi : (M, E_M, \iota_M) \to (N, E_N, \iota_N)$ is an *almost CR quaternionic map, with respect to Φ*, if $\mathrm{d}\varphi_x$ is CR quaternionic, with respect to Φ_x, at each point $x \in M$.

Suppose that E_M and E_N are endowed with compatible connections ∇^M and ∇^N, respectively, with respect to which M and N become CR quaternionic manifolds; denote by $\mathcal{C}^M$ and $\mathcal{C}^N$ the corresponding CR structures on Z_M and Z_N, respectively.

The map $\varphi : (M, E_M, \iota_M, \nabla^M) \to (N, E_N, \iota_N, \nabla^N)$ is *CR quaternionic, with respect to Φ*, if $\pi_N \circ \Phi = \varphi \circ \pi_M$ and $\Phi : (Z_M, \mathcal{C}^M) \to (Z_N, \mathcal{C}^N)$ is a CR map.

The notions of (*almost*) *CR quaternionic immersion/submersion/diffeomorphism* are defined accordingly.

Note that the CR quaternionic maps are just twistorial maps (see [8] for the definition of the latter). Also, for maps of rank at least one between quaternionic manifolds, the two notions coincide [5, Theorem 3.5]. That is why we call *quaternionic maps* the (almost) CR quaternionic maps, between (almost) quaternionic manifolds.

Furthermore, any CR quaternionic map is, obviously, almost CR quaternionic. However, the converse does not hold, as the following example shows.

Example 3.1. Let (M, c) be a three-dimensional conformal manifold and denote by $L = (\Lambda^3 TM)^{1/3}$ the line bundle of M. Then $E = L \oplus TM$ is oriented and c induces on it a linear conformal structure. Therefore E is a quaternionic vector bundle and (M, E, ι) is an almost CR quaternionic manifold, where $\iota : TM \to E$ is the inclusion.

Furthermore, if D is a conformal connection on (M, c) and $\nabla = D^L \oplus D$, where D^L is the connection induced by D on L, then (M, E, ι, ∇) is a CR quaternionic manifold (this is, for example, a straightforward consequence of [10, Theorem 4.6]). Let D' be another conformal connection on (M, c) and let $\nabla' = (D')^L \oplus D'$, where $(D')^L$ is the connection induced by D' on L. Then $\mathrm{Id}_M : (M, E, \iota, \nabla) \to (M, E, \iota, \nabla')$ is a CR quaternionic map if and only if the trace-free self-adjoint part of $*T$ is zero, where $*$ is the Hodge star-operator of (M, c) and T is the difference between the torsion tensors of D and D'.

As any conformal connection is determined by its torsion and the connection it induces on the line bundle of the manifold, we can therefore easily construct many almost CR quaternionic maps which are not CR quaternionic.

A straightforward generalization shows that the same applies to any almost CR quaternionic manifold (M, E, ι) with $\dim M = 2k + 1$ and $\operatorname{rank} E = 4k$, for some nonzero natural number k (this follows from the fact that then for any compatible connection ∇ on E we have that (M, E, ι, ∇) is a CR quaternionic manifold).

Theorem 3.2. *Let M and N be real-analytic CR quaternionic manifolds and let $\widetilde{M}$ and $\widetilde{N}$ be the corresponding heaven spaces, respectively. If $\varphi : M \to N$ is a real-analytic map whose differential is nowhere zero then the following assertions are equivalent.*

(i) *φ is CR quaternionic (with respect to some real-analytic lift between the twistor spaces of M and N).*
(ii) *φ is the restriction of a germ unique quaternionic map $\widetilde{\varphi}$ defined on some open neighbourhood of M in $\widetilde{M}$ and with values in $\widetilde{N}$.*

Proof. Let $Z_{\widetilde{M}}$ and $Z_{\widetilde{N}}$ be the twistor spaces of $\widetilde{M}$ and $\widetilde{N}$, respectively. If we denote by Z_M and Z_N their restrictions to M and N we obtain (generic) CR submanifolds which are the twistor spaces of M and N, respectively. Denote by $\pi_{\widetilde{M}} : Z_{\widetilde{M}} \to \widetilde{M}$ and $\pi_{\widetilde{N}} : Z_{\widetilde{N}} \to \widetilde{N}$ the projections and by π_M and π_N their restrictions to Z_M and Z_N, respectively.

Suppose that (i) holds and let $\Phi : Z_M \to Z_N$ be a real-analytic CR map with respect to which φ is CR quaternionic. Note that, locally, any real-analytic CR function on Z_M is the restriction of a unique holomorphic function on $Z_{\widetilde{M}}$. Thus, if f is a holomorphic function, locally defined on $Z_{\widetilde{N}}$, then $f \circ \Phi$ is the restriction of some holomorphic function, locally defined on $Z_{\widetilde{N}}$. Consequently, Φ is the restriction of some germ unique holomorphic map $\widetilde{\Phi}$ defined on some open neighbourhood Z of Z_M in $Z_{\widetilde{M}}$.

As Φ restricted to each fibre of π_M is a holomorphic diffeomorphism (in fact, an orientation preserving isometry), by passing if necessary to an open subset of Z, we may suppose that $\widetilde{\Phi}$ restricted to each twistor line is a holomorphic diffeomorphism. Then $\widetilde{\Phi}$ maps the family of twistor lines contained by Z into a family of complex projective lines which, obviously, contains the fibres of π_N. As the families of twistor lines (on the twistor spaces of quaternionic manifolds) are locally complete, we obtain that $\widetilde{\Phi}$ maps twistor lines to twistor lines.

Let $\tau_{\widetilde{M}}$ and $\tau_{\widetilde{N}}$ be the conjugations on $Z_{\widetilde{M}}$ and $Z_{\widetilde{N}}$, respectively, which on the fibres of the projections $\pi_{\widetilde{M}}$ and $\pi_{\widetilde{N}}$ are given by the antipodal map. Obviously, $\tau_{\widetilde{N}} \circ \widetilde{\Phi} \circ \tau_{\widetilde{M}} : \tau_{\widetilde{M}}(Z) \to Z_{\widetilde{N}}$ is, also, a holomorphic extension of Φ. Hence, $\tau_{\widetilde{N}} \circ \widetilde{\Phi} = \Phi \circ \tau_{\widetilde{M}}$ on $Z \cap \tau_{\widetilde{M}}(Z)$ (which, obviously, contains Z_M).

Therefore $\widetilde{\Phi}$ maps real twistor lines to real twistor lines. But the real twistor lines are just the fibres of $\pi_{\widetilde{M}}$ and $\pi_{\widetilde{N}}$. Hence, $\widetilde{\Phi}$ descends to a quaternionic map, as required.

Suppose, now, that (ii) holds and let $\widetilde{\varphi} : \widetilde{M} \to \widetilde{N}$ be a quaternionic extension of φ. Then by [5, Theorem 3.5] we have that $\widetilde{\varphi}$ admits a holomorphic lift $\widetilde{\Phi} : Z_{\widetilde{M}} \to Z_{\widetilde{N}}$ which, obviously, restricts to a CR map $\Phi : Z_M \to Z_N$ which is a lift of φ. Thus, φ is CR quaternionic.

For applications we shall, also, need the following:

Lemma 3.3. *Let M be an almost CR quaternionic submanifold of a CR quaternionic manifold N; denote by E the corresponding quaternionic vector bundle on M. Then there exists a connection on E with respect to which M is a CR quaternionic submanifold of N.*

Proof. Let Z be the bundle of admissible linear complex structures on E. Then the connection on the corresponding quaternionic vector bundle over N induces a connection on Z.

Locally, we can write $E^{\mathbb{C}} = H \otimes F$, with H and F complex vector bundles of ranks 2 and $2k$, and structural groups $\mathrm{Sp}(1)$ and $\mathrm{GL}(k, \mathbb{H})$, respectively, where $\operatorname{rank} E = 4k$.

Then $Z = PH$ and the connection on Z corresponds to a connection ∇^H on H. Thus, if ∇^F is any connection on F (compatible with its structural group) then $\nabla^H \otimes \nabla^F$ is the complexification of a connection on E, which is as required.

More can be said about the CR quaternionic submanifolds of a quaternionic manifold.

Corollary 3.4. *Let (M,E) be a real-analytic almost CR quaternionic submanifold of a quaternionic manifold N. Then the following assertions hold:*

(i) *The inclusion from M into N extends to a germ unique quaternionic immersion from a quaternionic manifold P to N, with $\dim P = \operatorname{rank} E$.*

(ii) *Any quaternionic connection on N induces, by restriction, a connection on E with respect to which M is a CR quaternionic submanifold of N.*

Proof. Assertion (i) is an immediate consequence of Theorem 3.2 and Lemmas 1.1 and 3.3. Assertion (ii) is an immediate consequence of (i) and the fact that the quaternionic submanifolds are totally geodesic with respect to any quaternionic connection on the ambient space [1] (see [5]).

We end with the following result (cf. [9, Proposizione 12.1]).

Corollary 3.5. *Let $\varphi : M \to N$ be an immersion from a manifold of dimension $4k-1$ to a quaternionic manifold N of dimension $4n$. Suppose that, for any $x \in M$, the quaternionic vector subspace of $T_{\varphi(x)}N$ spanned by $\mathrm{d}\varphi(T_xM)$ has dimension $4k$.*

Then φ extends to a germ unique quaternionic immersion from a quaternionic manifold, of dimension $4k$, to N; in particular, the pull-back of any quaternionic connection on N preserves the quaternionic vector bundle generated by TM.

Proof. Let Z_M and Z_N be the twistor spaces of M and N, respectively. By [3, Proposition 1.10], in an open neighbourhood U of each point of Z_M there exists a complex manifold Z^U, with $\dim_{\mathbb{C}}(Z^U) = 2k+1$, containing U as a CR submanifold. Moreover, we may suppose that any CR function on U can be locally (and uniquely) extended to a holomorphic function on Z^U (use [6, Proposition III.2.3] to show that [4, Theorem 14.1.1] can be applied; cf. [7]). But this is sufficient for the proof of the global embeddability theorem [3] to work, thus showing that there exists a complex manifold Z, with $\dim_{\mathbb{C}} Z = 2k+1$, which contains Z_M as an embedded CR submanifold. Consequently, also Theorem 3.2 extends to this setting.

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Received 23 August, 2011

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