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Harmonic morphisms on heaven spaces

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Abstract

We prove that any (real or complex) analytic horizontally conformal submersion from a three-dimensional conformal manifold (M^3, c_M) to a two-dimensional conformal manifold (N^2, c_N) can be, locally, 'extended' to a unique harmonic morphism from the $\mathcal{H}$(eaven)-space (H^4, g) of (M^3, c_N) to (N^2, c_N). Moreover, any positive harmonic morphism with two-dimensional fibres from (H^4, g) is obtained in this way.

Introduction

Harmonic morphisms are maps between Riemannian manifolds (or, more generally, Weyl spaces) that pull back (local) harmonic functions to harmonic functions. By a basic result, harmonic morphisms are characterised as harmonic maps that are horizontally weakly conformal (see [**2**, **10**]).

In the complex analytic category, a *twistor* on a manifold M (that is, a point of a twistor space of M) is a submanifold of M (see [**13**]). Accordingly, a map φ between manifolds M and N endowed with twistor spaces Z_M and Z_N, respectively, is *twistorial* if there exists a map Z_φ from some submanifold $Z_{M,\varphi} \subseteq Z_M$ to Z_N such that if $P_z \subseteq M$ is the submanifold corresponding to any $z \in Z_{M,\varphi}$ then $Z_\varphi(z) \in Z_N$ corresponds to $\varphi(P_z)$; see [**13**] for the general definition and [**11**] for the corresponding theory in the smooth category (see, also, [**12**] for more information on twistorial maps).

In [**15**] (see [**10**]) it is proved that any (submersive) harmonic morphism φ with two-dimensional fibres from an orientable four-dimensional Einstein manifold (M^4, g) is twistorial. In this context, this means that one of the two almost Hermitian structures on (M^4, g), with respect to which φ is holomorphic, is integrable; moreover, this Hermitian structure J is parallel along the fibres of φ. Then we say that φ is *positive*, with respect to the orientation on M^4 determined by J. Thus if, further, (M^4, g) is anti-self-dual then J determines a complex surface S in the twistor space of (M^4, g) which, if J is nowhere Kähler, is foliated by the holomorphic distribution determined by the Levi–Civita connection of g. It follows then, that, φ is completely determined by S. This gives the twistorial construction of all positive harmonic morphisms with two-dimensional fibres from a four-dimensional Einstein anti-self-dual manifold with non-zero scalar curvature [**15**].

In [**8**], it is proved that any analytic three-dimensional conformal manifold (N^3, c) is the conformal infinity of a unique (up to homotheties) four-dimensional Einstein anti-self-dual manifold (M^4, g) with non-zero scalar curvature; the fact that (N^3, c) is the conformal infinity of (M^4, g) means that $N^3 \sqcup M^4$ is a manifold with boundary (equal to N^3) on which c and g determine a conformal structure with respect to which g has a pole along N^3. The constructed Einstein anti-self-dual manifold (M^4, g) (which can be assumed to be a punctured collar neighbourhood of N^3) is called the *$\mathcal{H}$(eaven)-space* of (M^3, c) (see [**8**]).

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The two constructions mentioned above raise the following questions. Given a positive harmonic morphism with two-dimensional fibres on the $\mathcal{H}$-space of a three-dimensional analytic conformal manifold (N^3, c), can it be extended over N^3? If so, is the resulting map horizontally conformal? Conversely, can any analytic horizontally conformal submersion, with one-dimensional fibres on (N^3, c) be, locally, 'extended' to a harmonic morphism on its $\mathcal{H}$-space? In this paper we answer all of these questions in the affirmative, thus generalising the results of [1] in which the same conclusions are obtained for four-dimensional constant curvature Riemannian manifolds.

In Section 1 we review a few facts on harmonic morphisms between Weyl spaces. In Section 2 we present the basic properties of the space of isotropic geodesics of a (three-dimensional) complex-conformal manifold. Here, the emphasis is on the induced contact structure [8, 9]. We end Section 2 by illustrating how this theory can be used to construct (essentially all) horizontally conformal submersions with one-dimensional fibres from the Euclidean space (Example 2.6; cf. [1]). This also provides examples for the main result of the paper (Theorem 3.2), which is proved in Section 3. There, we also explain how other examples can be obtained by using a construction of [3].

1. *Harmonic morphisms between Weyl spaces*

This paper mainly deals with (real or complex) analytic manifolds and maps. Nevertheless, the results presented in this section also hold in the category of smooth manifolds.

A *conformal structure* on a manifold M is a line subbundle c of $\otimes^2 T^*M$ which, on open sets of M, is generated by Riemannian metrics [9]; then any such metric is a *local representative* of c.

If c is a conformal structure on M then (M, c) is a *conformal manifold*. A *conformal connection* on (M, c) is a linear connection that preserves the sheaf of sections of c. A *Weyl connection* is a torsion-free conformal connection. If D is a Weyl connection on (M, c) then (M, c, D) is a *Weyl space*. For more information on Weyl spaces see [3], where although the discussion is placed in the setting of smooth manifolds it can be easily extended to complex manifolds (see [10]). Whilst in the category of complex manifolds, we shall assume that all maps between conformal manifolds have nowhere degenerate fibres (see [10] for a few facts on harmonic morphisms with degenerate fibres).

Definition 1.1 (cf. [2]). Let $\varphi : (M, c_M) \to (N, c_N)$ be a map between conformal manifolds. Then φ is *horizontally weakly conformal* if, outside the set where its differential is zero, φ is a Riemannian submersion with respect to suitable local representatives of c_M and c_N.

Next, we recall the definitions of harmonic maps and morphisms (see [2, 10]).

Definition 1.2. A map $\varphi : (M, c_M, D^M) \to (N, c_N, D^N)$ between Weyl spaces is *harmonic* if the trace, with respect to c_M, of the covariant derivative of $d\varphi$ is zero.

A *harmonic morphism* between Weyl spaces is a map that pulls back (local) harmonic functions to harmonic functions.

The following result is basic for harmonic morphisms (see [10] and its references, and also cf. [2]).

Theorem 1.3. *A map between Weyl spaces is a harmonic morphism if and only if it is a harmonic map which is horizontally weakly conformal.*

See [11, 10] for more information on harmonic morphisms between Weyl spaces and [2] for harmonic morphisms in the setting of Riemannian manifolds; also see [4] for an up-to-date list of papers on harmonic morphisms.

2. *Spaces of isotropic geodesics*

In this section, with the exception of Example 2.6, all the manifolds and maps are assumed to be complex analytic.

Let (M^3, c_M) be a three-dimensional (complex-)conformal manifold, and let $\pi : P \to M$ be the bundle of isotropic directions tangent to (M^3, c_M). Then P is a locally trivial bundle with typical fibre the conic $Q_1 \cong \mathbb{C}P^1$ in $\mathbb{C}P^2$.

Let D be a Weyl connection, locally defined, on (M^3, c_M). For any $p \in P$ let $\mathscr{F}_p \subseteq T_pP$ be the horizontal lift, with respect to D, of $p \subseteq T_{\pi(p)}M$. Then $\mathscr{F}$ is (the tangent bundle of) a foliation on P; moreover, $\mathscr{F}$ does not depend on D.

The quadruple $\tau = (P, M, \pi, \mathscr{F})$ is a twistorial structure on M (see [13]), its twistor space Z (that is, the leaf space of $\mathscr{F}$) is the *space of isotropic geodesics* (see [8, 9]) of (M^3, c_M). Furthermore, locally, τ is simple (that is, by passing, if necessary, to an open set of M there exists a necessarily unique, complex structure on Z with respect to which the projection $\pi_Z : P \to Z$ is a submersion with connected fibres and each of its fibres intersects the fibres of π at most once); then the fibres of π are mapped by π_Z diffeomorphically onto the *skies* (see [8, 9]) of (M^3, c_M).

From now on, in this section, we shall assume that τ is simple.

Let $\mathscr{D}^P$ be the distribution on P such that $\mathscr{D}^P_p$ is the sum of $\ker d\pi_p$ and the horizontal lift, with respect to D, of $p^\perp \subseteq T_{\pi(p)}M$, for each $p \in P$. Then $\mathscr{D}^P$ does not depend on D. Moreover, $\mathscr{D}^P$ is projectable with respect to $\mathscr{F}$. To prove this, consider the bundle E of conformal frames on (M^3, c_M). Let p_0 be a fixed isotropic direction on $\mathbb{C}^3$, endowed with its canonical conformal structure, and let G be the subgroup of $\mathrm{CO}(3, \mathbb{C})$ that preserves p_0. Then $P = E/G$, and let $\mathscr{F}^E$ and $\mathscr{D}^E$ be the preimages of $\mathscr{F}$ and $\mathscr{D}^P$, respectively, through the differential of the projection $E \to P$.

It is clear that $\mathscr{D}^P$ is projectable with respect to $\mathscr{F}$ if and only if $\mathscr{D}^E$ is projectable with respect to $\mathscr{F}^E$. Furthermore, we can define $\mathscr{F}^E$ and $\mathscr{D}^E$, alternatively, as follows. Let ξ_0 be a non-zero element of p_0 and let $B(\xi_0)$ be the standard horizontal vector field corresponding to ξ_0; that is, $B(\xi_0)$ is the vector field on E which, at each $u \in E$, is the horizontal lift of $u(\xi_0) \in TM$, with respect to D. Then, by using the same notation for elements of $\mathfrak{co}(3, \mathbb{C})$ and the corresponding fundamental vector fields on E, the distribution $\mathscr{F}^E$ is generated by $B(\xi_0)$ and all $A \in \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G. Similarly, $\mathscr{D}^E$ is generated by all $B(\xi)$ with $\xi \in {p_0}^\perp$ and all $A \in \mathfrak{co}(3, \mathbb{C})$.

From Cartan's structural equations and [6, Chapter III, Proposition 2.3] it follows quickly that $\mathscr{D}^E$ is projectable with respect to $\mathscr{F}^E$ and, therefore, $\mathscr{D}^P$ is projectable with respect to $\mathscr{F}$. Moreover, Proposition 2.3 in [6, Chapter III] also implies that $\mathscr{D}^E$ is non-integrable; that is, nowhere integrable. Hence, also, the induced distribution $\mathscr{D} = d\pi_Z(\mathscr{D}^P)$ on Z is non-integrable. As $\dim Z = 3$, this is equivalent to the fact that $\mathscr{D}$ is a contact distribution on Z.

REMARK 2.1. (1) The skies are tangent to the contact distribution $\mathscr{D}$. Moreover, at each point, $\mathscr{D}$ is generated by the tangent spaces to the skies.

(2) Any surface embedded in Z corresponds to a foliation by isotropic geodesics on some open set of M^3.

All of the above can be quickly generalised to conformal manifolds of any dimension, thus obtaining the results of [9]. Next, we concentrate on features specific to dimension three.

PROPOSITION 2.2 ([8, 9], also cf. [5]). *Let $\mathscr{V}$ be a foliation by isotropic geodesics on (M^3, c_M). Then $\mathscr{V}^\perp$ is integrable. Furthermore, if we denote by $S_{\mathscr{V}}$ the embedded surface in Z corresponding to $\mathscr{V}$ then the foliations induced by $\mathscr{V}$, on the leaves of $\mathscr{V}^\perp$, correspond to the curves determined by $\mathscr{D}$ on $S_{\mathscr{V}}$.*

Proof. The fact that $\mathscr{V}^\perp$ is integrable follows quickly from the fact that D is torsion-free.
Let p be the section of P corresponding to $\mathscr{V}$. Then $\mathscr{F}$ induces a foliation on $p(M)$ (whose leaves are mapped by π on the leaves of $\mathscr{V}$). As $p(M)$ is an embedded submanifold of P foliated by the fibres of π_Z, we have that $S_{\mathscr{V}} = \pi_Z(p(M))$ is an embedded surface in Z.
The last assertion follows from the fact that $dp(\mathscr{V}^\perp) = T(p(M)) \cap \mathscr{D}^P$. □

REMARK 2.3. In Proposition 2.2, the fact that $\mathscr{V}^\perp$ is integrable also follows from the fact that $d(\pi_Z \circ p)(\mathscr{V}^\perp) = TS_{\mathscr{V}} \cap \mathscr{D}$ is one-dimensional and hence integrable.

Recall that, under the identification of any sky with $\mathbb{C}P^1$, its normal bundle in Z is $\mathcal{O}(1) \oplus \mathcal{O}(1)$ (see [8]). Thus, by applying [7] we obtain that, locally, the skies are contained by a locally complete, family of projective lines, parametrised by a four-dimensional manifold N^4 that contains M^3 as a hypersurface. Let $H^4 = N^4 \setminus M^3$. Furthermore, if we denote by c_N the anti-self-dual conformal structure on N^4 with respect to which Z is the twistor space (see [13]) of (N^4, c_N), then:

(i) (M^3, c_M) is a totally umbilical hypersurface of (N^4, c_N); furthermore, under the identification of the twistor space of (N^4, c_N) with the space of isotropic geodesics of (M^3, c_M) any self-dual surface S on (N^4, c_N) corresponds to the isotropic geodesic $S \cap M$ of (M^3, c_M) (see [8]),
(ii) $\mathscr{D}|_H$ determines an Einstein representative g of $c_N|_H$, unique up to homotheties, with non-zero scalar curvature [14].

DEFINITION 2.4 ([8]). The Einstein anti-self-dual manifold (H^4, g) is called the *$\mathcal{H}$-space* of (M^3, c_M).

In [8] it is also proved that g has a pole of order two along M^3.

EXAMPLE 2.5 (cf. [8]). By identifying, as usual, $\mathbb{C}^3$ with a subspace of $\mathbb{C}^4$ we obtain that the space of isotropic lines (equivalently, geodesics) on $\mathbb{C}^3$ is isomorphic to the space of self-dual planes on $\mathbb{C}^4$ (any self-dual plane of $\mathbb{C}^4$ intersects $\mathbb{C}^3$ along an isotropic line and any isotropic line of $\mathbb{C}^3$ is obtained in this way). Therefore the space Z of isotropic lines of $\mathbb{C}^3$ is $\mathbb{C}P^3 \setminus \mathbb{C}P^1$ (this can also be proved directly).
From Remark 2.1(1), it follows that the contact structure on Z is induced by the one-form $\theta = z_1\,dz_3 - z_3\,dz_1 - z_2\,dz_4 + z_4\,dz_2$, where $(z_1, \ldots, z_4)$ are homogeneous coordinates on $\mathbb{C}P^3$.
On the other hand, on $\mathbb{C}^4 \setminus \mathbb{C}^3$, the contact form θ induces, up to homotheties, the well-known metric of constant curvature $g = (1/{x_4}^2)({dx_1}^2 + \ldots + {dx_4}^2)$.
Thus $(\mathbb{C}^4 \setminus \mathbb{C}^3, g)$ is the $\mathcal{H}$-space of $\mathbb{C}^3$.

We conclude this section by illustrating how Remark 2.1(2) and Example 2.5 can be used to construct (essentially all) horizontally conformal submersions with one-dimensional fibres on the real Euclidean space.

EXAMPLE 2.6 (cf. [1]). With the same notation as in Example 2.5, let S be a complex surface in Z given, in homogeneous coordinates, by $z_j = z_j(u, v)$, $j = 1, \ldots, 4$. Suppose that the foliation on S given by $u=$ constant is tangent to the contact distribution; equivalently, suppose that the following relation holds:

$$z_1\frac{\partial z_3}{\partial v}-z_3\frac{\partial z_1}{\partial v}=z_2\frac{\partial z_4}{\partial v}-z_4\frac{\partial z_2}{\partial v}.$$

Then the map $x=x(u,v)$, into $\mathbb{R}^4$, given by $x=(z_1+z_2j)^{-1}(z_3+z_4j)$ is (locally) a real analytic diffeomorphism. Thus, under this diffeomorphism, the projection $(u,v)\mapsto u$ corresponds to a submersion $\widetilde{\varphi}$ (locally defined) on $\mathbb{R}^4$. We claim that $\varphi=\widetilde{\varphi}|_{\{x_4=0\}}$ is a horizontally conformal submersion on $\mathbb{R}^3$; moreover, any horizontally conformal submersion with one-dimensional fibres on $\mathbb{R}^3$ is, locally, of this form.

Indeed, any real analytic horizontally conformal submersion with one-dimensional fibres on $\mathbb{R}^3$ determines, by complexification, a (germ-unique) complex analytic horizontally conformal submersion with one-dimensional fibres on $\mathbb{C}^3$. Now, any such submersion on $\mathbb{C}^3$ (more generally, on any three-dimensional complex-conformal manifold) is determined by the two foliations by isotropic geodesics that are orthogonal to its fibres (see the proof of Theorem 3.2 below). In our case, under the correspondence of Remark 2.1(2), these two foliations are determined by S and its conjugate $\overline{S}$, given by $w_j=w_j(u,v)$, $j=1,\dots,4$, where $w_1=-\overline{z_2}$, $w_2=\overline{z_1}$, $w_3=-\overline{z_4}$, $w_4=\overline{z_3}$.

Note that, from Theorem 3.2 below, it will follow that $\widetilde{\varphi}|_{\{x_4>0\}}$ is a harmonic morphism from the four-dimensional real hyperbolic space.

3. *Harmonic morphisms on $\mathcal{H}$-spaces*

In this section all the manifolds and maps are assumed to be complex analytic; by a *real manifold/map* we mean a manifold/map which is the (germ-unique) complexification of a real analytic manifold/map. We shall further assume that all the even-dimensional conformal manifolds are *oriented*; that is, they are endowed with a reduction of the bundle of conformal frames to the identity component of the group of conformal transformations.

Next, we recall two examples of twistorial maps (see [**10**, **13**]).

EXAMPLE 3.1. (1) Let $\varphi:(M^4,c_M)\to(N^2,c_N)$ be a horizontally conformal submersion between conformal manifolds of dimensions four and two. Let J^N be the positive Hermitian structure on (N^2,c_N), and let J^M be the (unique) almost Hermitian structure on (M^4,c_M) with respect to which $\varphi:(M^4,J^M)\to(N^2,J^N)$ is holomorphic. Then φ is twistorial if J^M is integrable.

(2) Let $\varphi:(M^4,c_M)\to(N^3,c_N,D^N)$ be a horizontally conformal submersion from a four-dimensional conformal manifold to a three-dimensional Weyl space. Let $\mathscr{V}=\ker d\varphi$ and let $\mathscr{H}=\mathscr{V}^\perp$. Also, let D be the Weyl connection of $(M^4,c_M,\mathscr{V})$ (see [**10**]; D is characterised by the property $\mathrm{trace}_{c_M}(D\mathscr{V})=0$).

Let $D_+=D+*_{\mathscr{H}}I^{\mathscr{H}}$, where $I^{\mathscr{H}}$ is the integrability tensor of $\mathscr{H}$ (by definition, $I^{\mathscr{H}}(X,Y)=-\mathscr{V}[X,Y]$, for any horizontal vector fields X and Y) and $*_{\mathscr{H}}$ is the Hodge star-operator of $(\mathscr{H},c_M|_{\mathscr{H}})$. Then φ is twistorial if the partial connections on $\mathscr{H}$, over $\mathscr{H}$, induced by $\varphi^*(D^N)$ and D_+ are equal.

Any real harmonic morphism with two-dimensional fibres from an oriented four-dimensional Einstein manifold (more generally, Einstein–Weyl space) is twistorial, possibly with respect to the opposite orientation on its domain [**15**] (see [**10**]); we say that the harmonic morphism is *positive* if it is twistorial with respect to the given orientation on its domain and *negative* otherwise.

Let (M^3, c_M) be a three-dimensional conformal manifold and let (H^4, g) be its $\mathcal{H}$-space. We shall use the same notation as in the previous section.

THEOREM 3.2. *Any horizontally conformal submersion $\varphi : (M^3, c_M) \to (Q^2, c_Q)$ can be, locally, extended to a unique twistorial map $\widetilde{\varphi} : (N^4, c_N) \to (Q^2, c_Q)$.*

Moreover, $\widetilde{\varphi}|_H : (H^4, g) \to (Q^2, c_Q)$ is a harmonic morphism, and any positive harmonic morphism with two-dimensional fibres on (H^4, g) is obtained in this way.

Proof. Recall that a submersion $\varphi : (M^3, c_M) \to (Q^2, c_Q)$ is horizontally conformal if and only if $(\ker d\varphi)^\perp$ is umbilical (see [2]); equivalently, $D_Y Y$ is horizontal for any isotropic horizontal vector field Y, where D is any Weyl connection (locally defined) on (M^3, c_M). As the distribution $\left(\ker d\varphi\right)^\perp$ has dimension two, there are just two isotropic horizontal foliations $(\mathscr{V}_j)_{j=1,2}$, each having dimension one. Therefore φ is horizontally conformal if and only if $(\mathscr{V}_j)_{j=1,2}$ are geodesic. Moreover, as $\ker d\varphi = \left(\mathscr{V}_1 \oplus \mathscr{V}_2\right)^\perp$, we have that φ is determined by these two foliations, up to a conformal diffeomorphism of its codomain.

Let Z be the space of isotropic geodesics of (M^3, c_M), and let $\mathscr{H} \subseteq TZ$ be the corresponding contact distribution.

Suppose that $\varphi : (M^3, c_M) \to (Q^2, c_Q)$ is a horizontally conformal submersion. From Proposition 2.2, it follows that φ corresponds to a pair of disjoint surfaces $S_1 \sqcup S_2 \subseteq Z$ endowed with the one-dimensional foliations $TS_j \cap \mathscr{H}$ $(j = 1, 2)$.

However, Z is also the twistor space of (N^4, c_N). Then $S_1 \sqcup S_2$ and the endowed foliations determine a twistorial map $\widetilde{\varphi} : (N^4, c_N) \to (Q^2, c_Q)$ (see [13]). Obviously, $\widetilde{\varphi}|_M = \varphi$ and so its uniqueness is a consequence of analyticity.

Conversely, from [15] (see [13]), it follows that any positive harmonic morphism $\eta : (H^4, g) \to (Q^2, c_Q)$ determines a pair of foliations by self-dual surfaces on (N^4, c_N). Then the leaves of these two foliations intersect M^3 to determine a pair of foliations $(\mathscr{V}_j)_{j=1,2}$ by isotropic geodesics on (M^3, c_M). Let $\varphi : (M^3, c_M) \to (Q^2, c_Q)$ be such that $\ker d\varphi = \left(\mathscr{V}_1 \oplus \mathscr{V}_2\right)^\perp$. Then φ is horizontally conformal and $\eta = \widetilde{\varphi}|_H$. □

REMARK 3.3. (1) If we apply Theorem 3.2 to the particular case when (H^4, g) has constant curvature, then we obtain the results of [1].

(2) With the same notation as above, if D^M is an Einstein–Weyl connection on (M^3, c_M) with respect to which $\varphi : (M^3, D^M) \to (Q^2, c_Q)$ is a harmonic morphism, then $\widetilde{\varphi}|_H = \varphi \circ \psi$, where $\psi : N^4 \to M^3$ is the retract of $M^3 \hookrightarrow N^4$ corresponding to D^M (see [5]). It follows that $\psi|_H : (H^4, g) \to (M^3, c_M, D^M)$ is a harmonic morphism (see [10]).

Note that, φ and $\widetilde{\varphi}$ of Example 2.6 satisfy the assertions of Theorem 3.2. Other examples can be built based on the following construction.

EXAMPLE 3.4 ([3]). Let (M^3, c_M, D^M) be a three-dimensional Einstein–Weyl space. Let h be a (local) representative of c_M and let α be the Lee form of D^M with respect to h. By passing to an open set of M^3, if necessary, let I be an open set of $\mathbb{C}$ containing 0 such that $t^2 S - 6 \neq 0$, on $I \times M$, where S is the scalar curvature of D^M with respect to h.

Let $N = I \times M$ and $H^4 = N^4 \setminus M^3$, where M^3 is embedded in N^4 through $M^3 \hookrightarrow N^4$, $x \mapsto (0, x)$, with $x \in M^3$. Define the Riemannian metric g on H^4 by

$$g = \frac{1}{t^2}\left(\left(1 - \frac{1}{6}t^2 S\right) h + \left(1 - \frac{1}{6}t^2 S\right)^{-1}\left(dt + t\alpha + \frac{1}{2}t^2 *_M d\alpha\right)^2\right).$$

Then $t^2 g$ defines an anti-self-dual conformal structure on N^4. Moreover, (H^4, g) is the $\mathcal{H}$-space of (M^3, c_M) and the projection $\psi : N^4 \to M^3$, $(t, x) \mapsto x$, where $(t, x) \in N^4$, is the retract of $M^3 \hookrightarrow N^4$ corresponding to D^M.

Note that, if (M^3, c_M, D^M) is the Euclidean space and h is its canonical metric, then we obtain the $\mathcal{H}$-space of Example 2.5. Furthermore, the resulting retract is (the complexification of) a well-known (see, for example, [**2**]) harmonic morphism of warped-product type.

By Remark 3.3(2) and with the same notation as in Example 3.4, for any harmonic morphism $\varphi : (M^3, c_M, D^M) \to (Q^2, c_Q)$ we have that $\widetilde{\varphi} = \varphi \circ \psi$ is as in Theorem 3.2. Such morphisms φ can be obtained, for example, by using for (M^3, c_M, D^M) the R. S. Ward and C. R. LeBrun construction (see [**3**] and the references therein).

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