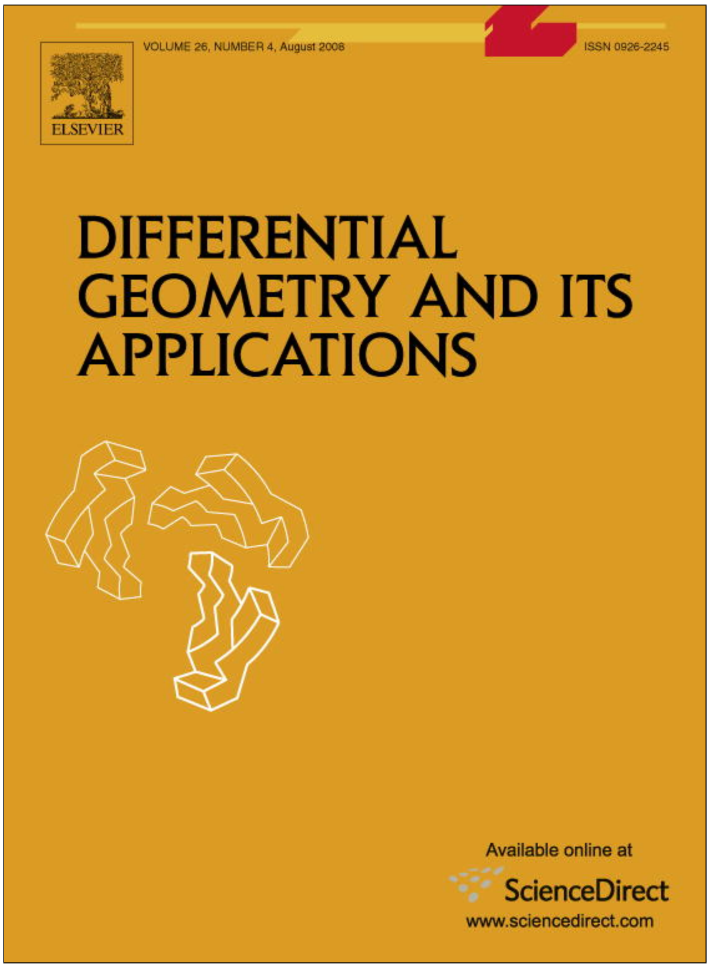
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On a class of twistorial maps

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Abstract

We show that a natural class of twistorial maps gives a pattern for apparently different geometric maps, such as, $(1,1)$-geodesic immersions from $(1,2)$-symplectic almost Hermitian manifolds and pseudo horizontally conformal submersions with totally geodesic fibres for which the associated almost CR-structure is integrable. Along the way, we construct for each constant curvature Riemannian manifold (M,g), of dimension m, a family of twistor spaces $\{Z_r(M)\}_{1\leqslant r<\frac{1}{2}m}$ such that $Z_r(M)$ parametrizes naturally the set of pairs (P,J), where P is a totally geodesic submanifold of (M,g), of codimension $2r$, and J is an orthogonal complex structure on the normal bundle of P which is parallel with respect to the normal connection.

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Introduction

In the complex-analytic category, the *twistor space* of a manifold M, endowed with a twistorial structure, parametrizes the set of certain submanifolds—*the twistors*—of M. For example (see [17] and the references therein), the twistor space of a three-dimensional complex Einstein–Weyl space (M^3,c,D) consists of the (maximal) degenerate surfaces of (M^3,c) which are totally geodesic with respect to D. Also, the twistor space of a four-dimensional anti-self-dual complex-conformal manifold (M^4,c) consists of the self-dual surfaces of (M^4,c) (similar comments apply, for example, to the complex-quaternionic manifolds of dimension at least eight). Further, the space of (unparameterized) isotropic geodesics of a complex-conformal manifold is, in a natural way, a twistor space [9].

In the smooth category, the definition of almost twistorial structure is slightly different [13]; it follows that, *in the smooth category, the twistors are certain submanifolds for which the normal bundle is endowed with a* (*linear*) *CR-structure*. These submanifolds may well be just points. For example, the twistor space Z of a three-dimensional conformal manifold (M^3,c) is a five-dimensional CR-manifold consisting of the orthogonal nontrivial CR-structures on (M^3,c) [10] (assuming (M^3,c) real-analytic, Z is a real hypersurface, endowed with the induced CR-structure,

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in the space of isotropic geodesics of the germ-unique complexification of (M^3, c)). Also, it is well-known (see [13] and the references therein) that the twistor space of a four-dimensional anti-self-dual conformal manifold (M^4, c) is a complex manifold, of complex dimension three, consisting of the positive orthogonal complex structures on (M^4, c) (similar comments apply, for example, to the quaternionic manifolds of dimension at least eight).

On the other hand (see [13] and the references therein), the twistor space Z of a three-dimensional Einstein–Weyl space (M^3, c, D) is, locally, a complex surface consisting of the pairs (γ, J), where γ is a geodesic of D and J is an orthogonal complex structure on the normal bundle of γ (obviously, if M^3 is oriented then Z is just the space of oriented geodesics of D). In Section 5, below, we generalize this example by constructing for each constant curvature Riemannian manifold (M, g), of dimension $m \geqslant 3$, a family of twistor spaces $\{Z_r(M)\}_{1 \leqslant r < \frac{1}{2}m}$ such that, locally, $Z_r(M)$ is a complex manifold, of complex dimension $r(2m - 3r + 1)/2$, consisting of the pairs (P, J), where P is a totally geodesic submanifold of (M, g), of codimension $2r$, and J is an orthogonal complex structure on the normal bundle of P which is parallel with respect to the normal connection (the particular case $r = 1$ is due to [1]). Moreover, we prove that, in dimension at least four, the constant curvature Riemannian manifolds give all the Weyl spaces for which this construction works (Theorem 5.4).

A map $\varphi : M \to N$ between manifolds endowed with twistorial structures is *twistorial* if it maps consistently (some of the) twistors on M to twistors on N (see Section 2 for a definition suitable for this paper and [13] for a more general definition; cf. [17]).

In this paper, we show that apparently different geometric maps are examples of such twistorial maps:

- in Section 3, we study twistorial immersions between even dimensional oriented Weyl spaces endowed with the *associated nonintegrable almost twistorial structures* (see Example 3.1; cf. [6]),
- in Section 4, we study $(1, 1)$-geodesic immersions from $(1, 2)$-symplectic almost Hermitian manifolds,
- in Section 5, we study pseudo horizontally conformal submersions with totally geodesic fibres for which the associated almost CR-structure is integrable.

In Section 1, we prove a powerful integrability result (Theorem 1.1; cf. [14]) which can be applied to all of the examples of almost twistorial structures known to us. Here, we use this result to give the necessary and sufficient conditions for the integrability of several almost CR-structures and almost f-structures (see, for example, Theorems 4.1 and 5.4), related to the twistorial maps we consider.

See [12,17] and [13] for more information on almost twistorial structures and twistorial maps.

1. An integrability result

Let F_j be a complex submanifold of the Grassmannian manifold $\mathrm{Gr}_{r_j}(m_j, \mathbb{C})$, $1 \leqslant r_j \leqslant m_j$, $(j = 1, 2)$. Suppose that there exists a complex Lie subgroup G_j of $\mathrm{GL}(m_j, \mathbb{C})$ whose canonical action on $\mathrm{Gr}_{r_j}(m_j, \mathbb{C})$ induces a transitive action on F_j; thus, $F_j = G_j/H_j$, as complex manifolds, where H_j is the isotropy group of G_j at some point of F_j, $(j = 1, 2)$.

Let (P_j, M, G_j) be a (smooth) principal bundle endowed with a connection ∇_j, $(j = 1, 2)$, where M is a (smooth connected) manifold, $\dim M = m_1$. We suppose that (P_1, M, G_1) is a subbundle of the complex frame bundle of $T^{\mathbb{C}}M$. Denote $Q_j = P_j \times_{G_j} F_j$ and let $\mathscr{H}^j \subseteq TQ_j$ be the connection induced by ∇_j on Q_j; note that, $Q_j = P_j/H_j$, $(j = 1, 2)$.

Denote $Q = \iota^*(Q_1 \times Q_2)$ where $\iota : M \to M \times M$ is defined by $\iota(x) = (x, x)$, for any $x \in M$, and let $\pi : Q \to M$ be the projection. Obviously, $\ker d\pi$ is a complex vector bundle.

The connections ∇_1 and ∇_2 induce a connection $\mathscr{H}(\subseteq TQ)$ on Q and let $\mathscr{G}_0 \subseteq \mathscr{H}^{\mathbb{C}}$ be the subbundle characterized by $d\pi(\mathscr{G}_0)(p, q) = p$, for all $(p, q) \in Q$. Define

$$\mathscr{G} = \mathscr{G}_0 \oplus (\ker d\pi)^{0,1},$$

$$\mathscr{G}' = \mathscr{G}_0 \oplus (\ker d\pi)^{1,0}.$$

Let T be the torsion of ∇_1 and let R_j be the curvature form of ∇_j $(j = 1, 2)$.

Theorem 1.1. *(Cf. [6,14].) If* $\dim_{\mathbb{C}} F_1 \geqslant 1$ *then* $\mathscr{G}'$ *is nonintegrable. Furthermore, the following assertions are equivalent*:

(i) $\mathscr{G}$ *is integrable.*
(ii) $T(\Lambda^2 p) \subseteq p$, $R_1(\Lambda^2 p)(p) \subseteq p$, $R_2(\Lambda^2 p)(q) \subseteq q$, *for all* $(p,q) \in Q$.

Proof. Let $G = G_1 \times G_2$ and let $\mathfrak{g}$ and $\mathfrak{g}_j$ be the Lie algebras of G and G_j, respectively, $(j = 1,2)$. Let $P = \iota^*(P_1 \times P_2)$ and denote, also, by π and $\mathscr{H}$ the projection $\pi : P \to M$ and the connection $\mathscr{H} \subseteq TP$ induced by ∇_1 and ∇_2 on P. Note that, $(\ker d\pi)^{\mathbb{C}} = P \times (\mathfrak{g} \oplus \bar{\mathfrak{g}})$.

Let H_1 and H_2 be the isotropy groups of G_1 and G_2 at some points p_0 and q_0 of F_1 and F_2, respectively. Let $H = H_1 \times H_2$ and denote by $\mathfrak{h}$ its Lie algebra.

Let $\mathscr{G}_{0,P} \subseteq \mathscr{H}^{\mathbb{C}}$ be characterized by $(d\pi)_{(u,v)}(\mathscr{G}_{0,P}) = u(p_0)$, for all $(u,v) \in P$. Clearly, $d\psi(\mathscr{G}_{0,P}) = \mathscr{G}_0$ and $(d\psi)^{-1}(\mathscr{G}_0) = \mathscr{G}_{0,P} \oplus P \times \mathfrak{h}^{\mathbb{C}}$ where $\psi : P \to Q$ is the projection.

Define

$$\begin{aligned} \mathscr{G}_P &= \mathscr{G}_{0,P} \oplus P \times \mathfrak{h} \oplus P \times \bar{\mathfrak{g}}, \\ \mathscr{G}'_P &= \mathscr{G}_{0,P} \oplus P \times \bar{\mathfrak{h}} \oplus P \times \mathfrak{g}. \end{aligned} \tag{1.1}$$

Obviously, $d\psi(\mathscr{G}_P) = \mathscr{G}$, $d\psi(\mathscr{G}'_P) = \mathscr{G}'$, $(d\psi)^{-1}(\mathscr{G}) = \mathscr{G}_P$, $(d\psi)^{-1}(\mathscr{G}') = \mathscr{G}'_P$. Therefore, $\mathscr{G}$ is integrable if and only if $\mathscr{G}_P$ is integrable. Similarly, $\mathscr{G}'$ is nonintegrable if and only if $\mathscr{G}'_P$ is nonintegrable.

For each $\xi \in \mathbb{C}^{m_1}$ let $B(\xi)$ be the horizontal (complex) vector field on P characterized by $(d\pi)_{(u,v)}(B(\xi)) = u(\xi)$ for any $(u,v) \in P$. Obviously, the map $P \times p_0 \to \mathscr{G}_{0,P}$, $((u,v),\xi) \mapsto B(\xi)_{(u,v)}$, for $(u,v) \in P$ and $\xi \in p_0$, is an isomorphism of vector bundles. Also, if $A \in \mathfrak{g}_1$ and $\xi \in \mathbb{C}^{m_1}$ then $[A, B(\xi)] = B(A\xi)$ and $[\overline{A}, B(\xi)] = 0$ (cf. [7, Chapter III]).

If $\dim_{\mathbb{C}} F_1 \geqslant 1$ then for any $A \in \mathfrak{g}_1 \setminus \mathfrak{h}_1$ there exists $\xi \in p_0$ such that $A\xi \notin p_0$. Hence, $[A, B(\xi)] = B(A\xi)$ is nowhere tangent to $\mathscr{G}_{0,P}$. Therefore $\mathscr{G}'_P$ and $\mathscr{G}'$ are nonintegrable.

The equivalence (i) ⇔ (ii) follows straightforwardly from Cartan's structural equations (cf. [15]). □

Let p be a section of Q_1; we shall denote by the same letter p the corresponding complex vector subbundle of $T^{\mathbb{C}}M$. Then the map $Q_2 \hookrightarrow Q$, $q \mapsto (p_{\pi_2(q)}, q)$, for any $q \in Q_2$, is an embedding, where $\pi_2 : Q_2 \to M$ is the projection. Denote by the same symbol Q_2 the image of this embedding. Then $\mathscr{G}^p = \mathscr{G} \cap TQ_2$ is a subbundle of $T^{\mathbb{C}}Q_2$.

Note that, $\mathscr{G}^p$ does not depend of ∇_1. In fact, we could define $\mathscr{G}^p$ as follows. Firstly, let $\mathscr{G}^p_0$ be the subbundle of $T^{\mathbb{C}}Q_2$ which is horizontal, with respect to the connection induced by ∇_2 on Q_2, and such that $d\pi_2(\mathscr{G}^p_0) = p$. Then we have $\mathscr{G}^p = \mathscr{G}^p_0 \oplus (\ker d\pi_2)^{0,1}$.

From Theorem 1.1 we easily obtain the following result.

Corollary 1.2. *(Cf. [8].) The following assertions are equivalent*:

(i) $\mathscr{G}^p$ *is integrable.*
(ii) p *is integrable and* $R_2(\Lambda^2_x p)(q) \subseteq q$ *for any* $x \in M$ *and* $q \in \pi_2^{-1}(x)$.

Remark 1.3. Theorem 1.1 and Corollary 1.2 can be easily generalized to the case when Q_2 is a fibre bundle for which the typical fibre is a complex manifold and the structural group is a complex Lie group whose action on the typical fibre is transitive and holomorphic.

2. Almost twistorial structures and twistorial maps

An *almost CR-structure* on a (smooth connected) manifold M is a section J of $\mathrm{End}(\mathscr{H})$ such that $J^2 = -\mathrm{Id}_{\mathscr{H}}$, for some distribution $\mathscr{H}$ on M; if $\mathscr{H} = TM$ then J is an *almost complex structure* on M. Let $\mathscr{F}$ be the eigenbundle of (the complexification of) J corresponding to $-\mathrm{i}$; we say that $\mathscr{F}$ is *the complex distribution associated to* J. Then J is *integrable* if $\mathscr{F}$ is integrable (that is, for any $X, Y \in \Gamma(\mathscr{F})$ we have $[X,Y] \in \Gamma(\mathscr{F})$). A *CR-structure* is an integrable almost CR-structure; a *complex structure* is an integrable almost complex structure (see [13, §2]).

An *almost f-structure* on M is a section F of $\mathrm{End}(TM)$ such that $F^3 + F = 0$. Let $\mathscr{F} = T^0M \oplus T^{0,1}M$ where T^0M and $T^{0,1}M$ are the eigenbundles of F corresponding to 0 and $-\mathrm{i}$, respectively; we say that $\mathscr{F}$ is *the complex distribution associated to* F. Then F is *integrable* if $\mathscr{F}$ is integrable. An *f-structure* is an integrable almost f-structure (see [13, §2]).

If $\mathscr{F}$ is the complex distribution associated to a CR-structure or an f-structure on a manifold M then, obviously, $\mathscr{F} \cap \overline{\mathscr{F}}$ is (the tangent bundle of) a foliation on M.

Let F be an almost f-structure on M and let $T^{1,0}M$ and $T^{0,1}M$ be its eigenbundles corresponding to i and $-\mathrm{i}$, respectively. Then $J = F|_{T^{1,0}M \oplus T^{0,1}M}$ is an almost CR-structure on M; we shall call J *the almost CR-structure induced by* F. Note that, F is not determined by J; also, if F is integrable then J is not necessarily integrable.

An (*almost*) *CR-structure on a conformal manifold* (M, c) is an (almost) CR-structure J on M such that $J^* + J = 0$; obviously, this holds if and only if the complex distribution associated to J is isotropic.

An (*almost*) *f-structure on a conformal manifold* (M, c) is an (almost) f-structure F on M such that $F^* + F = 0$; obviously, this holds if and only if $T^{0,1}M$ is isotropic and $T^0M = (T^{1,0}M \oplus T^{0,1}M)^\perp$. Therefore an almost f-structure on a conformal manifold is determined by its eigenbundle corresponding to i (or $-\mathrm{i}$). Equivalently, if we denote by J the almost CR-structure whose eigenbundle corresponding to i is $T^{1,0}M$ then $F \leftrightarrow J$ establishes a bijective correspondence (which depends on c) between almost f-structures on (M, c) and almost CR-structures on (M, c).

Definition 2.1. A map $\varphi : (M, F^M) \to (N, F^N)$, between manifolds endowed with almost f-structures (or, almost CR-structures), is *holomorphic* if $d\varphi(\mathscr{F}^M) \subseteq \mathscr{F}^N$, where $\mathscr{F}^M$ and $\mathscr{F}^N$ are the complex distributions associated to F^M and F^N, respectively.

Remark 2.2. An almost f-structure F on M is integrable if and only if for any $x \in M$ there exists an open neighborhood $U \ni x$ and a holomorphic submersion φ from $(U, F|_U)$ onto some complex manifold (N, J) such that $\ker d\varphi = T^0M$ [15]; we say that the f-structure $F|_U$ is *defined by* φ. A *simple f-structure* is an f-structure (globally) defined by a holomorphic submersion with connected fibres.

We end this section with the definitions of almost twistorial structure and twistorial map suitable for the purpose of this paper; more general definitions are given in [13] (cf. [17]).

Definition 2.3. An *almost twistorial structure* on a manifold M is a quadruple $\tau = (Q, M, \pi, \mathcal{J})$, where $\pi : Q \to M$ is a locally trivial fibre space and $\mathcal{J}$ is an almost CR-structure or an almost f-structure on Q which induces almost complex structures on each fibre of π. We say that τ is *integrable* if $\mathcal{J}$ is integrable; a *twistorial structure* is an integrable almost twistorial structure. Suppose that τ is a twistorial structure such that there exists a surjective submersion $\varphi : Q \to Z$ whose fibres are the leaves of $\mathscr{F} \cap \overline{\mathscr{F}}$, where $\mathscr{F}$ is the complex distribution associated to $\mathcal{J}$. Then Z, endowed with the CR-structure $d\varphi(\mathscr{F})$, is the *twistor space* of τ.

Definition 2.4. Let $\varphi : M \to N$ be a map between manifolds endowed with the almost twistorial structures $\tau_M = (Q_M, M, \pi_M, \mathcal{J}^M)$ and $\tau_N = (Q_N, N, \pi_N, \mathcal{J}^N)$. Suppose that there exists a section p of Q_M and a map $\Phi : p(M) \to Q_N$ such that $\pi_N \circ \Phi = \varphi \circ \pi_M|_{p(M)}$ and the tangent bundle of $p(M)$ is preserved by $\mathcal{J}^M$; denote by $\mathcal{J}^p$ the restriction of $\mathcal{J}^M$ to the tangent bundle of $p(M)$. We shall say that $\varphi : (M, \tau_M) \to (N, \tau_N)$ is a *twistorial map* (*with respect to* Φ), if $\Phi : (p(M), \mathcal{J}^p) \to (Q_N, \mathcal{J}^N)$ is holomorphic; that is, $d\Phi(\mathscr{F}^p) \subseteq \mathscr{F}^N$ where $\mathscr{F}^p$ and $\mathscr{F}^N$ are the complex distributions associated to $\mathcal{J}^p$ and $\mathcal{J}^N$, respectively.

3. Twistorial immersions between Weyl spaces

We start this section with two related examples of almost twistorial structures.

Example 3.1. Let (M, c, D) be an oriented, even-dimensional Weyl space and let $\pi : Q \to M$ be the bundle of positive maximal isotropic spaces on (M, c) (the positive maximal isotropic spaces on (M, c) are the eigenspaces, corresponding to $-\mathrm{i}$, of the positive orthogonal complex structures on (M, c)). As $\ker d\pi$ is a complex vector bundle, we have an isomorphism of complex vector bundles $(\ker d\pi)^{\mathbb{C}} = (\ker d\pi)^{1,0} \oplus (\ker d\pi)^{0,1}$. Let $\mathscr{H} \subseteq TQ$ be the connection induced by D on Q. Let $\mathscr{G}^0 \subseteq \mathscr{H}^{\mathbb{C}}$ be the complex vector subbundle characterized by $d\pi(\mathscr{G}^0_q) = q$, for any $q \in Q$, and define

$$\mathscr{G} = \mathscr{G}_0 \oplus (\ker d\pi)^{0,1},$$
$$\mathscr{G}' = \mathscr{G}_0 \oplus (\ker d\pi)^{1,0}.$$

Let $\mathcal{J}$ and $\mathcal{J}'$ be the almost complex structures whose eigenbundles corresponding to $-\mathrm{i}$ are $\mathscr{G}$ and $\mathscr{G}'$, respectively.

Obviously, if $\dim M = 2$ then $Q = M$ and $\mathcal{J} = \mathcal{J}'$ is the positive Hermitian structure of (M^2, c).

Note that, $\mathcal{J}$ does not depend of D whilst if $\dim M \geqslant 4$ then $\mathcal{J}'$ determines D (that is, if D_1 is another Weyl connection on (M, c) which induces $\mathcal{J}'$ then $D = D_1$; this follows from [13, Proposition 2.6]).

If $\dim M \geqslant 4$ then $\mathcal{J}'$ is nonintegrable (that is, always not integrable) whilst if $\dim M = 4$ then $\mathcal{J}$ is integrable if and only if (M^4, c) is anti-self-dual and if $\dim M \geqslant 6$ then $\mathcal{J}$ is integrable if and only if (M, c) is flat; these well-known results (see [6, §4], [14, §5], [15, §3]) follow from Theorem 1.1.

Obviously, $(Q, M, \pi, \mathcal{J})$ and $(Q, M, \pi, \mathcal{J}')$ are almost twistorial structures on M; we shall call $(Q, M, \pi, \mathcal{J}')$ the *nonintegrable almost twistorial structure associated to* (M, c, D).

Let (M, c_M, D^M) and (N, c_N, D^N) be even-dimensional oriented Weyl spaces and let $\tau'_M = (Q_M, M, \pi_M, \mathcal{J}'_M)$ and $\tau'_N = (Q_N, N, \pi_N, \mathcal{J}'_N)$ be the associated nonintegrable almost twistorial structures.

Suppose that $\varphi : M \hookrightarrow N$ is an injective immersion. Then orient $(TM)^\perp$ such that the isomorphism $TN|_M = TM \oplus (TM)^\perp$ be orientation preserving and let $\pi : Q \to M$ be the bundle of positive maximal isotropic spaces on $((TM)^\perp, c_N|_{(TM)^\perp})$.

If p is a (local) section of Q_M then we shall denote by J^p the almost Hermitian structure on (M, c_M) such that p is the eigenbundle of J^p corresponding to $-\mathrm{i}$; similarly, for Q_N. Standard arguments show that the following assertions are equivalent:

(i) $p : (M, J^p) \to (Q_M, \mathcal{J}'_M)$ is holomorphic.
(ii) $D^M_X Y$ is a section of p for any sections X and Y of p.
(iii) $D^M_{J^p X} J^p = -J^p D^M_X J^p$, for any $X \in TM$.
(iv) $(d^{D^M} \omega_M)^{(1,2)} = 0$, where ω_M is the Kähler form of (M, c_M, J^p) (defined by $\omega_M(X, Y) = c_M(J^p X, Y)$, for any $X, Y \in TM$).

Furthermore, if assertion (i), (ii), (iii) or (iv) holds then D^M is the Weyl connection of (M, c_M, J^p) (see [12, Remark 3.3]).

We shall denote by $\mathcal{J}^p$ the almost complex structure on Q whose eigenbundle corresponding to i is constructed, similarly to $\mathscr{G}^p$ of Corollary 1.2, by using the connection induced by $\Pi \circ D^N$ on Q and the complex vector subbundle $\overline{p}$ of $T^{\mathbb{C}} M$, where $\Pi : TN|_M \to (TM)^\perp$ is the orthogonal projection.

Let L be the line bundle of (N, c_N). We define a section A of the bundle $(L|_M)^2 \otimes \Lambda^2 T^* M \otimes \Lambda^2((TM)^\perp)^*$ by

$$A(X, Y, U, V) = \sum_a c_N(D^N_X Z_a, U) c_N(D^N_Y Z_a, V) - c_N(D^N_Y Z_a, U) c_N(D^N_X Z_a, V),$$

for any $x \in M$ and $X, Y \in T_x M$, $U, V \in (T_x M)^\perp$, where $\{Z_a\}$ is any conformal local frame on (M, c_M) defined on some open neighborhood of x. It is easy to see that A does not depend of D^N. Furthermore, if M is an umbilical submanifold of (N, c_N) then $A = 0$.

Corollary 3.2. *(Cf. [19].) The almost complex structure $\mathcal{J}^p$ does not depend of the Weyl connection D^N. Moreover, the following assertions are equivalent*:

(i) *$\mathcal{J}^p$ is integrable.*
(ii) *J^p is integrable and $(W + A)(\Lambda^2_x p, \Lambda^2 q) = 0$ for any $x \in M$ and $q \in Q_x$, where W is the Weyl tensor of (N, c_N).*

Proof. A straightforward calculation gives the following relation, essentially due to Ricci (see [2, 1.72(e)]),

$$c_N\big(R^N(X, Y)U, V\big) = c_N\big(R^\Pi(X, Y)U, V\big) + A(X, Y, U, V), \tag{3.1}$$

for any $X, Y \in TM$ and $U, V \in (TM)^{\perp}$, where R^N and R^{Π} are the curvature forms of D^N and $\Pi \circ D^N$, respectively.

Also, we have (see [4])

$$c_N\big(R^N(X,Y)U, V\big) = -W(X,Y,U,V) + F^N(X,Y)c_N(U,V), \tag{3.2}$$

for any $X, Y \in TM$ and $U, V \in (TM)^{\perp}$, where W is the Weyl tensor of (N, c_N) and F^N is the curvature form of the connection induced by D^N on L.

The proof now follows quickly from Corollary 1.2. □

Remark 3.3. In Corollary 3.2, if $\dim M = 2$ then assertion (ii) is automatically satisfied whilst if $\operatorname{codim} M = 2$ then the second part of assertion (ii) is automatically satisfied.

Let p be a section of Q_M which is isotropic with respect to c_N. Then for any map $\Phi : p(M) \to Q_N$ such that $\pi_N \circ \Phi = \varphi \circ \pi_M|_{p(M)}$ there exists a unique section q of Q such that $\Phi \circ p = p \oplus q$.

The following result reduces to [6, Theorem 5.3], when $\dim M = 2$, $\dim N = 4$ (see [13, Proposition 5.2]).

Proposition 3.4. *(Cf. [19].) Let Φ be given by the sections p and q of Q_M and Q, respectively, with p isotropic with respect to c_N. Then the following assertions are equivalent:*

(i) *$\varphi : (M, \tau'_M) \to (N, \tau'_N)$ is twistorial, with respect to Φ.*
(ii) *φ is $(1,1)$-geodesic with respect to J^p and, $p : (M, J^p) \to (P_M, \mathcal{J}'_M)$ and $q : (M, J^p) \to (Q, \mathcal{J}^p)$ are holomorphic.*

Proof. Assertion (i) holds if and only if $p(M)$ is an almost complex submanifold of $(Q_M, \mathcal{J}'_M)$ and $\Phi : (p(M), \mathcal{J}'_M|_{p(M)}) \to (Q_N, \mathcal{J}'_N)$ is holomorphic. It is clear that $p(M)$ is an almost complex submanifold of $(Q_M, \mathcal{J}'_M)$ if and only if $p : (M, J^p) \to (P_M, \mathcal{J}'_M)$ is holomorphic. Then $\Phi : (p(M), \mathcal{J}'_M|_{p(M)}) \to (Q_N, \mathcal{J}'_N)$ is holomorphic if and only if $\Phi \circ p : (M, J^p) \to (Q_N, \mathcal{J}'_N)$ is holomorphic. From [13, Proposition 2.6] it follows quickly that, assertion (i) is equivalent to (a) $D^M_{\overline{X}} Y \in \Gamma(p)$, for any $X, Y \in \Gamma(p)$, (b) $D^N_{\overline{X}} Y \in \Gamma(p \oplus q)$, for any $X, Y \in \Gamma(p)$, and (c) $D^N_{\overline{X}} U \in \Gamma(p \oplus q)$, for any $X \in \Gamma(p)$, $U \in \Gamma(q)$.

Note that, if (b) holds, condition (c) is equivalent to $\Pi(D^N_{\overline{X}} U) \in \Gamma(q)$, for any $X \in \Gamma(p)$, $U \in \Gamma(q)$. Thus, if (b) holds, condition (c) is equivalent to $q : (M, J^p) \to (Q, \mathcal{J}^p)$ be holomorphic.

Also, if (a) holds, condition (b) is equivalent to $(Dd\varphi)(\overline{X}, Y) \in \Gamma(p \oplus q)$, for any $X, Y \in \Gamma(p)$. As D^M and D^N are torsion free, $Dd\varphi$ is symmetric. It follows quickly that, if (a) holds, then (b) is equivalent to $(Dd\varphi)^{(1,1)} = 0$.

The proposition is proved. □

Remark 3.5. 1) If assertion (i) or (ii) of Proposition 3.4 holds then D^M is the Weyl connection of (M, c_M, J^p); if, further, φ is conformal then D^M is equal to the connection induced by D^N on M.

2) A result similar to (but more complicated than) Proposition 3.4 can be given for twistorial submersions between Weyl spaces endowed with the nonintegrable almost twistorial structures. It follows again that such maps are $(1,1)$-geodesic (in particular, harmonic) and, if the codomain is of dimension two, harmonic morphisms.

Let $\varphi : (M, c_M) \hookrightarrow (N, c_N)$ be a conformal injective immersion. Denote by $Q_M + Q$ the pull-back by ι of $Q_M \times Q$, where $\iota : M \to M \times M$ is defined by $\iota(x) = (x, x)$, for any $x \in M$. Let $\mathcal{J}$ and $\mathcal{J}'$ be the almost complex structures on $Q_M + Q$ whose eigenbundles corresponding to $-\mathrm{i}$ are constructed, similarly to $\mathcal{G}$ and $\mathcal{G}'$, respectively, of Theorem 1.1, by using the connections induced by D^M and $\Pi \circ D^N$ on Q_M and Q, respectively.

Proposition 3.6. *Let $\Phi : Q_M + Q \to Q_N$ be defined by $\Phi(p, q) = p \oplus q$, for any $(p, q) \in Q_M + Q$.*

(i) *The following assertions are equivalent:*
   (i1) *$\Phi : (Q_M + Q, \mathcal{J}) \to (Q_N, \mathcal{J}_N)$ is holomorphic.*
   (i2) *M is an umbilical submanifold of (N, c_N).*
(ii) *If $\dim M \geqslant 4$ then the following assertions are equivalent:*

(ii1) $\Phi : (Q_M + Q, \mathcal{J}') \to (Q_N, \mathcal{J}'_N)$ *is holomorphic.*
(ii2) φ *is geodesic.*

Proof. Let $x_0 \in M$ and let p_0, q_0 be positive maximal isotropic spaces which are tangent and normal, respectively, to M at x_0. Let $S \subseteq M$ be a surface such that $x_0 \in S$ and one of the two isotropic directions tangent to S at x_0 are contained in p_0; denote by X_0 a nonzero element of $T^{\mathbb{C}}_{x_0} S \cap p_0$ (obviously, X is well-defined, up to some complex factor).

We may suppose that there exist two sections p and q of Q_M and Q, respectively, over S which are horizontal at x_0 and such that $p_{x_0} = p_0$ and $q_{x_0} = q_0$.

From [13, Proposition 2.6] it follows quickly that $\Phi : (Q_M + Q, \mathcal{J}) \to (Q_N, \mathcal{J}_N)$ is holomorphic if and only if, for any $x_0 \in M$ and any such sections p and q, we have $D^N_{X_0} Y \in p_0 \oplus q_0$ and $D^N_{X_0} U \in p_0 \oplus q_0$, for any local sections Y of p and U of q; equivalently, $c_N(D^N_{X_0} Y, U) = 0$ for any local section Y of p and U of q. The proof of (i) follows quickly.

Similarly, $\Phi : (Q_M + Q, \mathcal{J}') \to (Q_N, \mathcal{J}'_N)$ is holomorphic if and only if, for any $x_0 \in M$ and any such sections p and q, we have $D^N_{\overline{X_0}} Y \in p_0 \oplus q_0$ and $D^N_{\overline{X_0}} U \in p_0 \oplus q_0$, for any local sections Y of p and U of q. It follows that (ii1) is equivalent to the fact that the Weyl connection induced by D^N on M is equal to D^M and, for any $x_0 \in M$ and any such sections p and q, we have $c_N(D^N_{\overline{X_0}} Y, U) = 0$ for any local sections Y of p and U of q. The proof of (ii) follows quickly. □

Similarly to the proof of Proposition 3.6(ii), we obtain the following:

Remark 3.7. (Cf. [6].) If $\dim M = 2$ then the equivalence (ii1) ⇔ (ii2), of Proposition 3.6, remains true if we replace (ii2) with the following assertion:

(ii2) M^2 is a minimal surface in (N, c_N, D^N).

4. On (1, 1)-geodesic submanifolds

Let (N, c_N, D^N) be a Weyl space. For $1 \leqslant r < \frac{1}{2} \dim N$, let $\pi_{N,r} : Q_{N,r} \to N$ be the bundle of isotropic spaces on (N, c_N) of complex dimension r. Denote by $\mathcal{J}_{N,r}$ and $\mathcal{J}'_{N,r}$ the almost CR-structures on $Q_{N,r}$ whose eigenbundles corresponding to $-\mathrm{i}$ are constructed, similarly to $\mathscr{G}$ and $\mathscr{G}'$, respectively, of Theorem 1.1, by using the connection induced by D^N on $Q_{N,r}$, and by taking $Q = N$ the trivial bundle over N.

Note that, $\mathcal{J}_{N,r}$ does not depend of D^N whilst $\mathcal{J}'_{N,r}$ determines D^N. Furthermore, by Theorem 1.1, the almost CR-structure $\mathcal{J}'_{N,r}$ is nonintegrable whilst, if $r = 1$ then $\mathcal{J}_{N,1}$ is integrable [10]. We shall prove the following result.

Theorem 4.1. *The following assertions are equivalent, if* $r \geqslant 2$:

(i) $\mathcal{J}_{N,r}$ *is integrable.*
(ii) (N, c_N) *is flat.*

Proof. Assume $r \geqslant 2$ and let R and W be the curvature form of D^N and the Weyl tensor of (N, c_N), respectively. We shall prove that the following assertions are equivalent:

(a) $R(\Lambda^2 p)(p) \subseteq p$ for any $p \in Q_{N,r}$.
(b) $c_N(R(X, Y)X, Y) = 0$ for any $X, Y \in T^{\mathbb{C}} N$ spanning an isotropic space.
(c) $W = 0$.

Indeed, as any two-dimensional isotropic space on (N, c_N) is contained in some $p \in Q_{N,r}$, we obviously have (a) ⇒ (b). Also, (b) ⟺ (c) (see [16]) and, as $R(\Lambda^2 p)(p) = W(\Lambda^2 p)(p)$, for any isotropic space p on (N, c_N), we have (c) ⇒ (a).

By Theorem 1.1, we have (i) ⇔ (a), and, by the Weyl theorem on flat conformal manifolds, (ii) ⇔ (c). The theorem is proved. □

Let $M \subseteq N$ be a submanifold, $\dim M = 2r$. Let $c_M = c_N|_M$ and let D^M be the Weyl connection on (M, c_M) induced by D^N. Also, let $\tau'_M = (Q_M, M, \pi_M, \mathcal{J}'_M)$ be the nonintegrable almost twistorial structure associated to (M, c_M, D^M).

Suppose that there exists a section p of $Q_{N,r}$ which is tangent to M. As before, denote by J^p the almost Hermitian structure on (M, c_M) whose eigenbundle corresponding to $-\mathrm{i}$ is p.

Similarly to Proposition 3.4, we obtain the following result (cf. [18]).

Proposition 4.2. *The following assertions are equivalent.*

(i) $p : (M, J^p) \to (Q_{N,r}, \mathcal{J}'_{N,r})$ *is holomorphic.*
(ii) (M, J^p) *is a* $(1,1)$*-geodesic submanifold of* (N, c_N, D^N) *and the map* $p : (M, J^p) \to (Q_M, \mathcal{J}'_M)$ *is holomorphic.*

Remark 4.3. 1) Proposition 4.2 can be easily formulated in similar vein to Proposition 3.4.

2) With the same notations as in Proposition 4.2, $p : (M, J^p) \to (Q_{N,r}, \mathcal{J}_{N,r})$ is holomorphic if and only if J^p is integrable and (M, J^p) is a $(2,0)$-geodesic submanifold of (N, c_N, D^N).

3) In Proposition 4.2, assume that (N, c_N, D^N) is the Euclidean space $\mathbb{R}^n$ with its canonical conformal structure and flat connection. Then $Q_{N,r} = \mathbb{R}^n \times Q_{n,r}$ where $Q_{n,r} \subseteq \mathrm{Gr}_r(n, \mathbb{C})$ is the manifold of isotropic r-dimensional subspaces of $\mathbb{C}^n$.

Let $\widetilde{p} = \pi_2 \circ p : M \to F$ where $\pi_2 : \mathbb{R}^n \times Q_{n,r} \to Q_{n,r}$ is the projection. Then $p : (M, J^p) \to (Q_{N,r}, \mathcal{J}'_{N,r})$ is holomorphic if and only if $\widetilde{p} : (M, J^p) \to Q_{n,r}$ is holomorphic.

Thus, by Proposition 4.2, (M, J^p) *is a* $(1,1)$*-geodesic submanifold of* (N, c_N, D^N) *and* $p : (M, J^p) \to (Q_M, \mathcal{J}'_M)$ *is holomorphic if and only if* $\widetilde{p} : (M, J^p) \to Q_{n,r}$ *is holomorphic*. In the particular case $\dim M = 2$, this gives M^2 minimal in $\mathbb{R}^n$ if and only if $\widetilde{p}$ holomorphic which leads to the Weierstrass representation of minimal surfaces in Euclidean space.

4) A result similar to Proposition 3.6 can be easily written by working with the inclusion map $Q_M \hookrightarrow Q_{N,r}$.

5. f-structures and pseudo horizontally conformal submersions

We start this section by recalling the following definition.

Definition 5.1. (See [1,3].) A map $\varphi : (M, c) \to (N, J)$ from a conformal manifold to an almost complex manifold is *pseudo horizontally weakly conformal* if it pulls back $(1,0)$-forms on N to isotropic 1-forms on (M, c). A map is *pseudo horizontally conformal* if it is submersive and pseudo horizontally weakly conformal.

Remark 5.2. 1) A submersion $\varphi : (M, c) \to (N, J)$ from a conformal manifold to an almost complex manifold is pseudo horizontally conformal if and only if there exists an almost f-structure F on (M, c) such that $T^0 M = \ker d\varphi$ and $\varphi : (M, F) \to (N, J)$ is holomorphic (cf. [11]).

2) Let (M, c) be a conformal manifold and let F be an almost f-structure on M. Then F is an f-structure on (M, c) if and only if it is locally defined by pseudo horizontally conformal submersions onto complex manifolds.

Let (M, c, D) be a Weyl space, $\dim M = m$. For $1 \leqslant r < \frac{1}{2}m$, let $\pi_{M,r} : Q_{M,r} \to M$ be the bundle of isotropic spaces on (M, c) of complex dimension r. For $p \in Q_{M,r}$ let F^p be the skew-adjoint f-structure on $(T_{\pi_{M,r}(p)} M, c_{\pi_{M,r}(p)})$ whose eigenspace corresponding to $-\mathrm{i}$ is p. Thus, $Q_{M,r}$ is also the bundle of skew-adjoint f-structures on (M, c) with kernel of dimension $m - 2r$.

Let $\mathscr{H}$ be the connection induced by D on $Q_{M,r}$ and let $T^0 Q_{M,r} \subseteq \mathscr{H}$ be the subbundle characterized by $d\pi_{M,r}(T^0_p Q_{M,r}) = \ker F^p$, for all $p \in Q_{M,r}$. Also, let $\mathscr{G}_0 \subseteq \mathscr{H}^{\mathbb{C}}$ be the subbundle such that $d\pi_{M,r}((\mathscr{G}_0)_p)$ is the eigenspace of F^p corresponding to $-\mathrm{i}$, for all $p \in Q_{M,r}$. Denote $T^{0,1} Q_{M,r} = \mathscr{G}_0 \oplus (\ker d\pi_{M,r})^{0,1}$ and let $\mathcal{F}_{M,r}$ be the almost f-structure on $Q_{M,r}$ whose eigenbundles corresponding to 0 and $-\mathrm{i}$ are $T^0 Q_{M,r}$ and $T^{0,1} Q_{M,r}$, respectively. Also, let $\mathcal{F}'_{M,r}$ be the almost f-structure on $Q_{M,r}$ whose eigenbundles corresponding to 0 and $-\mathrm{i}$ are $T^0 Q_{M,r}$ and $\mathscr{G}_0 \oplus (\ker d\pi_{M,r})^{1,0}$, respectively.

Remark 5.3. 1) Each of the almost f-structures $\mathcal{F}_{M,r}$ and $\mathcal{F}'_{M,r}$ determines D.

2) With the same notations as in Section 4, the almost CR-structures induced by $\mathcal{F}_{M,r}$ and $\mathcal{F}'_{M,r}$ are $\mathcal{J}_{M,r}$ and $\mathcal{J}'_{M,r}$, respectively.

It is well-known (see [15, Theorem 3.5]) that if $m=3$ then $\mathcal{F}_{M,1}$ is integrable if and only if (M,c,D) is Einstein–Weyl. Also, from Theorem 1.1 it easily follows that $\mathcal{F}'_{M,r}$ is nonintegrable. We shall prove the following:

Theorem 5.4. *If $m \geqslant 4$ then the following assertions are equivalent*:

(i) *$\mathcal{F}_{M,r}$ is integrable.*
(ii) *D is, locally, the Levi-Civita connection of a constant curvature representative of c.*

Proof. Let R be the curvature form of the connection induced by D on $L^* \otimes TM$, where L is the line bundle of M. We claim that the following assertions are equivalent:

(a) $R(\Lambda^2(p^\perp))(p^\perp) \subseteq p^\perp$ for any $p \in Q_{M,r}$.
(b) $c(R(X,Y)X,Y)=0$ for any $X,Y \in T^{\mathbb{C}}M$ spanning a degenerate space.
(c) (M,c,D) is flat and Einstein–Weyl.

Indeed, as any two-dimensional degenerate space on (M,c) is contained in $p^\perp$ for some $p \in Q_{M,r}$, we obviously have (a) $\Rightarrow$ (b). Also, by [16], assertion (b) implies that (M,c) is flat; it follows quickly that (b) $\Rightarrow$ (c). By a result of M.G. Eastwood and K.P. Tod ([5, Theorem 1]; see [4, Theorem 5.2]), (c) $\Leftrightarrow$ (ii). Clearly, (ii) $\Rightarrow$ (a) and the proof follows from Theorem 1.1. □

Remark 5.5. By Theorems 4.1 and 5.4, if $\mathcal{F}_{M,r}$ is integrable then $\mathcal{J}_{M,r}$ is integrable.

Let (M,g) be a Riemannian manifold of constant curvature such that $\mathcal{F}_{M,r}$ is simple. Then there exists a holomorphic submersion from $(Q_{M,r},\mathcal{F}_{M,r})$ onto a complex manifold $Z_r(M)$ whose fibres are the leaves of $T^0Q_{M,r}$. Then $Z_r(M)$ is the twistor space of $(Q_{M,r},M,\pi_{M,r},\mathcal{F}_{M,r})$ (cf. [1, §6.8]).

Proposition 5.6. *(Cf. [13].) Let p be a section of $Q_{M,r}$ and let F^p be the corresponding almost f-structure on (M,c). The following assertions are equivalent*:

(i) *$p:(M,F^p) \to (Q_{M,r},\mathcal{F}_{M,r})$ is holomorphic.*
(ii) *F^p is integrable and locally defined by pseudo horizontally conformal submersions with geodesic fibres and for which the integrability tensor of the horizontal distribution is of degree $(1,1)$.*

Proof. Assertion (i) is equivalent to the fact that $D_XY \in \Gamma(p^\perp)$, for any $X,Y \in \Gamma(p^\perp)$; in particular, if (i) holds then F^p is integrable. Clearly, (i) is also equivalent to $D_XY \in \Gamma(\overline{p}^\perp)$, for any $X,Y \in \Gamma(\overline{p}^\perp)$. Therefore, if (i) holds then $p^\perp \cap \overline{p}^\perp$ $(=(p\oplus\overline{p})^\perp = \ker F^p)$ is geodesic.

Thus, if (i) holds then F^p is integrable and locally defined by pseudo horizontally conformal submersions with geodesic fibres; furthermore, if $X,Y \in \Gamma(p)$ and $U \in \Gamma((p\oplus\overline{p})^\perp)$ then, as F^p is integrable, we have $[U,X],[U,Y] \in \Gamma(p^\perp)$ and it follows that $c(U,[X,Y]) = -2c(D_UX,Y)=0$. This completes the proof of (i) $\Rightarrow$ (ii).

By definition, F^p integrable if and only if $p^\perp$ integrable. It follows that if F^p is integrable then $D_XY \in \Gamma(p^\perp)$, for any $X,Y \in \Gamma(p)$. Also, if $\ker F^p (=(p\oplus\overline{p})^\perp)$ is geodesic then $D_UV \in \Gamma(p^\perp)$, for any $U,V \in \Gamma((p\oplus\overline{p})^\perp)$. Furthermore, an argument as above shows that if F^p is integrable then the integrability tensor of $(p\oplus\overline{p})^\perp$ is of degree $(1,1)$ if and only if $D_UX \in \Gamma(p^\perp)$, for any $X \in \Gamma(p)$ and $U \in \Gamma((p\oplus\overline{p})^\perp)$. This completes the proof of (ii) $\Rightarrow$ (i). □

Remark 5.7. Let F be an f-structure on M. It is obvious that the almost CR-structure $T^{0,1}M$ is integrable if and only if the integrability tensor of $T^{1,0}M \oplus T^{0,1}M$ is of degree $(1,1)$.

From Proposition 5.6 we easily obtain the following result.

Corollary 5.8. *(Cf. [13].) Let p be a section of $Q_{M,1}$ and let F^p be the corresponding almost f-structure on (M, c). The following assertions are equivalent*:

(i) $p:(M, F^p) \to (Q_{M,1}, \mathcal{F}_{M,1})$ *is holomorphic.*
(ii) F^p *is integrable and locally defined by submersive harmonic morphisms with geodesic fibres* (*of codimension two*).

Let (M, g) be a real analytic Riemannian manifold, $\dim M = m$. Then (M, g) admits a (germ-unique) complexification $(M^{\mathbb{C}}, g^{\mathbb{C}})$. Let $\pi_{M^{\mathbb{C}},r}: Q_{M^{\mathbb{C}},r} \to M^{\mathbb{C}}$ be the bundle of r-dimensional isotropic spaces on $(M^{\mathbb{C}}, g^{\mathbb{C}})$. If $1 \leqslant r < \frac{1}{2}m$, the complex version of Theorem 5.4 says that the following assertions are equivalent (cf. [17, §2] and the references therein):

(i) For any $p \in Q_{M^{\mathbb{C}},r}$ there exists a coisotropic and geodesic complex submanifold S of $(M^{\mathbb{C}}, g^{\mathbb{C}})$ of (complex) rank $m - 2r$, with respect to $g^{\mathbb{C}}$, such that $T_{\pi_{M^{\mathbb{C}},r}(p)}S = p^{\perp}$.
(ii) $(M^{\mathbb{C}}, g^{\mathbb{C}})$ has constant (sectional) curvature.

Assume $(M^{\mathbb{C}}, g^{\mathbb{C}})$ (and, hence, also (M, g)) to be of constant curvature. Then, locally, the twistor space (in the sense of [17, Definition 2.1]) $Z_r(M^{\mathbb{C}})$ parametrizes the coisotropic geodesic (complex) submanifolds of $(M^{\mathbb{C}}, g^{\mathbb{C}})$ of rank $m - 2r$. It follows that, locally, we may assume $T^0 Q_{M,r}$ simple and such that each of its leaves intersects the fibres of $\pi_{M,r}$ at most once (apply [17, Remark 2.2(3)]). Then $Z_r(M)$ is an open submanifold of $Z_r(M^{\mathbb{C}})$; moreover, $Z_r(M)$ is endowed with a holomorphic m-dimensional family of submanifolds each of which is holomorphically diffeomorphic to the space of isotropic r-dimensional spaces on $\mathbb{C}^m$; the members of this family are called the *twistor submanifolds* of $Z_r(M)$ (see [17, Remark 2.2(1)]).

We shall say that two submanifolds S and S' of a manifold W are *transversal* if $T_xS \cap T_xS' = \{0\}$, at each $x \in S \cap S'$.

Corollary 5.9. *Let (M, g) be a Riemannian manifold of constant curvature and let $1 \leqslant r < \frac{1}{2}m$, where $m = \dim M$.*

Then any pseudo horizontally conformal submersion, locally defined on (M, g), with geodesic fibres of dimension $m - 2r$ and for which the integrability tensor of the horizontal distribution is of degree $(1, 1)$ corresponds, locally, to a complex submanifold, of dimension r, of $Z_r(M)$ which is transversal to the twistor submanifolds.

Proof. Any (local) pseudo horizontally conformal submersion φ on (M, g) with connected geodesic fibres of dimension $m - 2r$ and for which the integrability tensor of the horizontal distribution is of degree $(1, 1)$ defines an f-structure F^{φ} on (M, g). Moreover, by Proposition 5.6, F^{φ} corresponds to a holomorphic section $p^{\varphi}:(M, F^{\varphi}) \to (Q_{M,r}, \mathcal{F}_{M,r})$. Hence, $T^0 Q_{M,r}$ induces a foliation on $p^{\varphi}(M)$ whose leaves are mapped by $\pi_{M,r}$ onto the fibres of φ. Thus, locally, the projection $Q_{M,r} \to Z_r(M)$ maps $p^{\varphi}(M)$ onto a complex r-dimensional submanifold N^{φ} of $Z_r(M)$. Then $\varphi \longmapsto N^{\varphi}$ gives the claimed correspondence. □

Remark 5.10. Let (M, g) be a constant curvature Riemannian manifold and let $1 \leqslant r < \frac{1}{2}m$, where $m = \dim M$.

Then $Z_r(M)$ *parametrizes naturally the set of pairs (P, J) where P is a totally geodesic submanifold of (M, g), of codimension $2r$, and J is an orthogonal complex structure on the normal bundle of P which is parallel with respect to the normal connection*. (By (3.1) and (3.2), the normal connection on the normal bundle of any totally umbilical submanifold of a conformally-flat Riemannian manifold is flat.)

Let φ be a (local) pseudo horizontally conformal submersion on (M, g) with connected geodesic fibres of dimension $m - 2r$ and for which the integrability tensor of the horizontal distribution is of degree $(1, 1)$. Let N^{φ} be the codomain of φ and let J^{φ} be the orthogonal complex structure on $(\ker d\varphi)^{\perp}$ with respect to which $d\varphi|_{(\ker d\varphi)^{\perp}}$ is holomorphic at each point.

Then the correspondence of Corollary 5.9 is given by $\varphi \mapsto N^{\varphi}$ where the inclusion map $N^{\varphi} \hookrightarrow Z_r(M)$ is defined by $y \mapsto (\varphi^{-1}(y), J^{\varphi}|_{\varphi^{-1}(y)})$, $(y \in N^{\varphi})$.

From Corollary 5.9 we obtain the following result of P. Baird and J.C. Wood.

Corollary 5.11. *(See [1].) Let (M, g) be a Riemannian manifold of constant curvature. Then any submersive harmonic morphism, locally defined on (M, g), with geodesic fibres of codimension two corresponds, locally, to a complex one-dimensional submanifold of $Z_1(M)$ which is transversal to the twistor submanifolds.*

We end by describing the twistor spaces of the space forms $\mathbb{R}^m$, S^m and H^m (cf. [1, §6.8]). For this, we firstly describe the twistor spaces of the complex Euclidean space $\mathbb{C}^m$ and of the complex unit hypersphere $S^m(\mathbb{C})$.

Let $Q_{m,r} \subseteq \mathrm{Gr}_{m-r}(m, \mathbb{C})$ be the space of coisotropic subspaces of $\mathbb{C}^m$ of rank $m - 2r$. We shall denote by the same symbol $Q_{m,r}$ its image through the complex analytic diffeomorphism $\mathrm{Gr}_{m-r}(m, \mathbb{C}) \to \mathrm{Gr}_r(m, \mathbb{C})$ defined by $p \mapsto p^\perp$, for any $p \in \mathrm{Gr}_{m-r}(m, \mathbb{C})$. Thus, $Q_{m,r} \subseteq \mathrm{Gr}_r(m, \mathbb{C})$ is the space of isotropic subspaces of $\mathbb{C}^m$ of complex dimension r. Let $E_{m,r}$ and $F_{m,r}$ be the restrictions to $Q_{m,r}$ of the tautological vector bundles on $\mathrm{Gr}_{m-r}(m, \mathbb{C})$ and $\mathrm{Gr}_r(m, \mathbb{C})$, respectively. As $Z_r(\mathbb{C}^m)$ is the space of coisotropic planes in $\mathbb{C}^m$ of rank $m - 2r$, we have $Z_r(\mathbb{C}^m) = (Q_{m,r} \times \mathbb{C}^m)/E_{m,r} = F^*_{m,r}$.

Similarly, $Z_r(S^m(\mathbb{C}))$ is the space of (maximal) coisotropic geodesic submanifolds of $S^m(\mathbb{C})$, of rank $m - 2r$. As any such submanifold is the intersection of $S^m(\mathbb{C})$ with a coisotropic subspace, of rank $m - 2r + 1$, of $\mathbb{C}^{m+1}$, we have $Z_r(S^m(\mathbb{C})) = Q_{m+1,r}$.

It follows that $Z_r(\mathbb{R}^m) = F^*_{m,r}$, $Z_r(S^m) = Q_{m+1,r}$ and $Z_r(H^m) = Q_{m+1,r} \setminus \mathcal{C}_{m,r}$ for some closed set $\mathcal{C}_{m,r} \subseteq Q_{m+1,r}$. To describe $\mathcal{C}_{m,r}$, consider the complex Euclidean space $\mathbb{C}^{m+1}$ as the complexification of the Minkowski space $\mathbb{R}^{m+1}_1$ so that the complexification of $H^m \subseteq \mathbb{R}^{m+1}_1$ to be the complex hypersphere, of radius the imaginary unit. Then $\mathcal{C}_{m,r}$ is the set of coisotropic subspaces $p \subseteq \mathbb{C}^{m+1}$ of rank $m - 2r + 1$ such that $p^\perp \cap \mathbb{R}^{m+1}_1 \neq \{0\}$.

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