The Quarterly Journal of Mathematics Advance Access published on January 18, 2006
Quart. J. Math. 57 (2006), 105–132; doi:10.1093/qmath/hah063

TWISTORIAL HARMONIC MORPHISMS WITH ONE-DIMENSIONAL FIBRES ON SELF-DUAL FOUR-MANIFOLDS

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[Received 22 January 2004; in final form 2 April 2005]

Abstract

We introduce a general notion of twistorial map and classify twistorial harmonic morphisms with one-dimensional fibres from self-dual four-manifolds. Such maps can be characterized as those that pull back Abelian monopoles to self-dual connections. In fact, the constructions involve solving a generalized monopole equation, and also the Beltrami fields equation of hydrodynamics, and lead to constructions of self-dual metrics.

0. Introduction

Harmonic morphisms between Riemannian manifolds are smooth maps which preserve Laplace's equation. By the basic characterization theorem [**12**, **19**], they are harmonic maps that are horizontally weakly conformal (see below).

Classification results for harmonic morphisms with one-dimensional fibres can be found in [**6**, **8**, **30–32**]. In [**32**] it is proved that, from an Einstein manifold of dimension at least five, there are just two types [**2**, **8**] of harmonic morphism with one-dimensional fibres. In dimension four, the situation is different: from an Einstein four-manifold there are precisely three types of harmonic morphism with one-dimensional fibres [**30**, **31**] (see also [**33**]), where the first two types are as before. It is significant that all these three types of harmonic morphism are twistorial maps in the sense of Definition 3.3 below. Moreover, by a result of [**37**], submersive harmonic morphisms from Einstein four-manifolds to Riemann surfaces are twistorial maps.

We shall see that a submersion with (nowhere degenerate) one-dimensional fibres from a four-dimensional (complex-)Riemannian manifold is twistorial if and only if it is self-dual in the sense of [**9**].

In this paper we classify twistorial harmonic morphisms with one-dimensional fibres from real-analytic Riemannian four-manifolds, finding precisely one more type, which is related to a metric construction of [**9**].

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In section 1, we review some basic facts on harmonic morphisms with one-dimensional fibres. In section 2, we introduce a notion of (almost) twistorial structure. Then, we recall [**1**, **18**, **25**, **34**] some basic examples from twistor theory and show how they fit into our framework. In section 3, we introduce the notion of twistorial map and then, in sections 4 and 5, we show that many examples and facts from twistor theory appear as natural properties of such twistorial maps. To make our definition of twistorial, we must complexify our manifolds, so that we shall often work with complex-analytic maps between complex-Riemannian manifolds (see [**25**]).

The main result of section 5 is a reformulation (Theorem 5.3) of a result of [**9**] which gives a nice characterization for twistorial maps with nowhere degenerate fibres from a four-dimensional oriented conformal manifold. We prove that the induced Weyl connection on the codomain coincides with the one hinted at in [**9**]. Such twistorial maps pull back (Abelian) monopoles to self-dual connections (cf. [**9**, **10**]); by using Theorem 5.3, we show that this property is equivalent to the property of being twistorial. Then we show that twistorial harmonic morphisms $\varphi : (M^4, g) \to (N^3, h, D)$ are characterized by the property that there exist non-trivial monopoles on (N^3, h, D) which are pulled back to flat connections.

In section 6 we prove (Theorem 6.3) that, from a real-analytic four-dimensional Riemannian manifold, there are, up to conformal changes with basic factor, just four types of twistorial harmonic morphism with one-dimensional fibres, where the first three types are as above with a slight extension of type 3, and type 4 is new. The proof involves solving the monopole equation (5.15) (cf. [**33**]). Recall [**30**, **31**, **33**] that harmonic morphisms of type 3 are determined by the Beltrami fields equation (see [**21**]). Here, this equation appears once more: for a harmonic morphism $\varphi : (M^4, g) \to (N^3, h)$ of type 4, the *Lee form* α with respect to h of the Weyl connection on N^3, with respect to which φ is twistorial, satisfies the Beltrami fields equation $d\alpha = \pm * \alpha$ outside the zero set of a function, up to a conformal change of h (Proposition 6.2).

Theorem 6.3 together with a result of [**9**] (see Theorem 5.3, below), gives the classification of twistorial harmonic morphisms with one-dimensional fibres from a self-dual four-dimensional manifold with real-analytic metric (Corollary 6.4).

In [**33**], we gave a new construction of Ricci-flat self-dual metrics based on harmonic morphisms of type 3. In section 7, we show that harmonic morphisms are related to constructions of Einstein and self-dual metrics in [**9**, **20**].

1. Some facts on harmonic morphisms with one-dimensional fibres

In this section we present some basic facts on harmonic morphisms with one-dimensional fibres. See [**6**, **29**] for general accounts and [**16**] for a list of papers on harmonic morphisms.

DEFINITION 1.1 A *harmonic morphism* is a smooth map $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds which pulls back (locally defined) harmonic functions to harmonic functions, that is, if $f : V \to \mathbb{R}$ is a harmonic function on an open subset of N with $\varphi^{-1}(V)$ non-empty, then $f \circ \varphi$ is a harmonic function on $\varphi^{-1}(V)$.

DEFINITION 1.2 A smooth map $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds is *horizontally* (*weakly*) *conformal* if, at each point $x \in M$, either $d\varphi_x = 0$, in which case x is called a *critical point* of φ, or $d\varphi_x \colon T_x M \to T_{\varphi(x)} N$ is surjective and its restriction to the horizontal space $\mathscr{H}_x = (\ker d\varphi_x)^{\perp}$ is a conformal (linear) map $(\mathscr{H}_x, g_x|_{\mathscr{H}_x}) \to (T_{\varphi(x)}N, h_{\varphi(x)})$, in which case x is called a *regular point* of φ. Denote the conformality factor by $\lambda(x)$. The resulting function λ is called the *dilation* of φ. The dilation is smooth outside the set of critical points and can be extended to a continuous function on M^m, with λ^2 smooth, by setting it equal to zero on the set of critical points.

A smooth map is called *horizontally homothetic* if it is horizontally conformal with dilation constant along horizontal curves.

A *homothetic foliation* is a foliation which is locally defined by horizontally homothetic submersions [**29**, **30**].

REMARK 1.3 A map $\varphi : (M, g) \to (N, h)$ is horizontally weakly conformal if and only if, for each $x \in M$, the adjoint $d\varphi_x^* : (T_{\varphi(x)}N, h_{\varphi(x)}) \to (T_x M, g_x)$ is a weakly conformal linear map (with image $\mathscr{H}_x$). This formulation shows that the condition of horizontal weak conformality is dual to that of weak conformality (see also [**6**, §2.4]).

The basic characterization result for harmonic morphisms is the following.

THEOREM 1.4 [**12**, **19**] *A smooth map between Riemannian manifolds is a harmonic morphism if and only if it is a harmonic map which is horizontally weakly conformal.*

It follows that the set of regular points of a non-constant harmonic morphism is an open dense subset of the domain [**12**].

The following two propositions give two of the four types of harmonic morphism with one-dimensional fibres that we shall meet (see section 3 below).

PROPOSITION 1.5 [**2**] *Let $\varphi : (M^{n+1}, g) \to (N^n, h)$ be a non-constant horizontally weakly conformal map between Riemannian manifolds of dimensions $n+1$ and n, respectively $(n \geq 1)$. If $n = 2$, then φ is a harmonic morphism if and only if its fibres are geodesic at regular points. If $n \neq 2$, then any two of the following assertions imply the third:*

(i) *φ is a harmonic morphism;*
(ii) *the fibres of φ are geodesic at regular points;*
(iii) *φ is horizontally homothetic.*

Proof. Let $\varphi : (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion. Then, at regular points we have the following fundamental equation (see, for example, [**6**, §4.5]) for the tension field $\tau(\varphi) \in \Gamma(\varphi^*(TN))$ of φ:

$$\tau(\varphi) + d\varphi(\mathrm{trace}(B^{\mathscr{V}})) = -(n-2)d\varphi(\mathrm{grad}(\log \lambda)), \tag{1.1}$$

where $\mathrm{trace}(B^{\mathscr{V}})$ is the trace of the second fundamental form of $\mathscr{V} = \ker \mathrm{d}\varphi$ and λ is the dilation of φ.

The proposition is an immediate consequence of (1.1).

REMARK 1.6 Proposition 1.5 is true for maps with higher-dimensional fibres after replacing ‘geodesic’ with ‘minimal’.

Recall that a foliation is called *Riemannian* if it is locally defined by the fibres of Riemannian submersions; then we have the following result (see also [**6**, §12.3]; see [**22**] for the definition of Killing vector fields).

PROPOSITION 1.7 [**8**] *Let $\varphi : (M^{n+1}, g) \to N^n$ be a submersion with connected one-dimensional fibres from a Riemannian manifold of dimension $n+1$ to a smooth manifold of dimension n $(n \geq 3)$. Suppose that the fibres of φ form an orientable Riemannian foliation. Then the following assertions are equivalent.*

(i) *There exists a Riemannian metric h on N^n with respect to which $\varphi : (M^{n+1}, g) \to (N^n, h)$ is a harmonic morphism;*
(ii) *there exists a nowhere-zero Killing vector field on (M^{n+1}, g) tangent to the fibres of φ.*

In general, harmonic morphisms with one-dimensional fibres have a local normal form as follows.

THEOREM 1.8 [**8**] *Let (M^{n+1}, N^n, S^1) be a principal bundle with projection $\varphi : M^{n+1} \to N^n$, endowed with a principal connection $\mathscr{H} \subseteq TM$. Let h be a Riemannian metric on N^n and λ a smooth positive function on M^{n+1}.*

Define a Riemannian metric on M^{n+1} by

$$g = \lambda^{-2}\,\varphi^*(h) + \lambda^{2n-4}\,\theta^2, \tag{1.2}$$

where θ is the connection form of $\mathscr{H}$. Then $\varphi : (M^{n+1}, g) \to (N^n, h)$ is a harmonic morphism.

Conversely, any submersive harmonic morphism with one-dimensional fibres is locally *of this form, up to isometries.*

See [**6**, §12.2; **29**; **30**] (and [**5**] for the case $n = 2$) for a proof of Theorem 1.8 and a more explicit version of the converse.

The vector field V on M^{n+1} which is the vertical dual of θ (that is, $d\varphi(V) = 0$ and $\theta(V) = 1$) is the infinitesimal generator of the local S^1-action; it is called *the fundamental* (*vertical*) *vector field*; up to sign, it is characterized by the property that it is vertical and $g(V, V) = \lambda^{2n-4}$.

REMARK 1.9

(1) By Proposition 1.5, any horizontally homothetic submersion with geodesic fibres is a harmonic morphism. In the context of Theorem 1.8 this corresponds to the case when λ is constant along horizontal curves; then if $d\lambda$ is nowhere zero, the horizontal distribution is integrable and totally umbilical.
(2) The Killing vector field of Proposition 1.7 is equal to (a multiple of) the fundamental vector field. In fact, from Proposition 1.7 and Theorem 1.8 (see [**6**; **8**, §12.3; **29**]) we deduce the following. Let V be a Killing vector field on (M^{n+1}, g) $(n \neq 2)$ whose integral curves are the fibres of the submersion $\varphi : (M^{n+1}, g) \to N^n$; let $\check{g}$ denote the unique metric on N^n with respect to which $\varphi : (M^{n+1}, g) \to (N^n, \check{g})$ is a Riemannian submersion and $\check{\lambda}$ the positive smooth function on N^n such that $\varphi^*(\check{\lambda}^{2n-4}) = g(V, V)$; then $\varphi : (M^{n+1}, g) \to (N^n, \check{\lambda}^{-2}\check{g})$ is a harmonic morphism. In the context of Theorem 1.8 this corresponds to the case when λ is constant along the fibres of φ.

We end this section by recalling the following.

DEFINITION 1.10 ([**30**], cf. [**36**]) Let $\mathscr{V}$ be (the tangent bundle of) a foliation on the Riemannian manifold (M^m, g).

We say that $\mathscr{V}$ *produces harmonic morphisms* if it can be locally defined by submersive harmonic morphisms.

2. Twistorial structures

In this section we shall work in the complex-analytic category. Thus, all manifolds will be complex and all maps will be complex-analytic. Examples of such manifolds and maps can be obtained by complexifying real-analytic maps between real-analytic manifolds. In particular, by a *Riemannian manifold* we shall mean a complex-Riemannian manifold in the sense of [**25**].

We define horizontally (weakly) conformal maps similarly to Definition 1.2, but λ may then be a complex number defined only up to sign with the *square dilation* λ^2 globally defined.

In what follows, it is convenient to work with the following definition.

DEFINITION 2.1 Let M be a (complex) manifold. By an *almost twistorial structure* (*on the manifold* M) we shall mean a quadruple $(P, M, \pi, \mathscr{F})$, where $\pi : P \to M$ is a proper surjective (complex-analytic) submersion and $\mathscr{F} \subseteq TP$ is a distribution on P such that $(\ker d\pi) \cap \mathscr{F} = \{0\}$; we call $\mathscr{F}$ the *twistor distribution* of $(P, M, \pi, \mathscr{F})$.

We shall call the almost twistorial structure $(P, M, \pi, \mathscr{F})$ *integrable* if its twistor distribution $\mathscr{F}$ is integrable. An integrable almost twistorial structure will be called a *twistorial structure*. If $(P, M, \pi, \mathscr{F})$ is a twistorial structure then the leaf space of $\mathscr{F}$ is called the *twistor space* of $(P, M, \pi, \mathscr{F})$.

REMARK 2.2

(1) Let $(P, M, \pi, \mathscr{F})$ be a twistorial structure. Assume that the foliation $\mathscr{F}$ is simple (that is, its leaf space Z is a manifold and the projection $\pi_Z : P \to Z$ is a submersion) and that any of its leaves intersect each fibre of π at most once. Then $\{\pi_Z(\pi^{-1}(x))\}_{x\in M}$ is an analytic family [**23**] of compact submanifolds of Z, which we shall call *twistor submanifolds*, or *twistor lines* when they are projective lines.

(2) For all of the almost twistorial structures $(P, M, \pi, \mathscr{F})$ ($\dim \mathscr{F} = k$) which will appear in this paper, the map $P \to G_k(TM)$, $p \mapsto d\pi(\mathscr{F}_p)$ is an embedding of P into the Grassmann bundle of k-dimensional planes on TM. Such almost twistorial structures can be obtained as follows.

By a *linear partial connection on a vector bundle* $E \to M$*, over a distribution* $\mathscr{H}$ *on* M [**7**], we mean a morphism ∇ from the sheaf of sections of E to the sheaf of sections of $E \otimes \mathscr{H}^*$ such that, if U is an open set of M, then $\nabla : \Gamma(U, E) \to \Gamma(U, E \otimes \mathscr{H}^*)$ is a $\mathbb{C}$-linear map that satisfies

$$\nabla(sf) = (\nabla s)f + s \otimes \mathrm{d}f|_{\mathscr{H}}$$

for any function $f : U \to \mathbb{C}$ and section $s \in \Gamma(U, E)$; in a similar way to the case of usual connections, any linear partial connection over $\mathscr{H}$ corresponds to a principal partial connection over $\mathscr{H}$ on the frame bundle of E, where by a *principal partial connection, over* $\mathscr{H}$*, on the principal bundle* (P, M, G)*, with projection* $\pi : P \to M$, we mean a G-invariant distribution $\mathscr{K}$ on P such that $\mathscr{K} \cap \ker d\pi = \{0\}$ and $d\pi(\mathscr{K}) = \mathscr{H}$.

Now let $\mathscr{H} \subseteq TM$ be an n-dimensional distribution endowed with a linear partial connection D, over itself. Suppose that D corresponds to a principal partial connection, over $\mathscr{H}$, on a principal subbundle (Q, M, G), $(G \subseteq GL(n, \mathbb{C}))$, of the bundle of (complex) frames of $\mathscr{H}$. Let $F \subseteq G_k(\mathbb{C}^n)$ be a submanifold which is invariant under the action of G and let $P = Q \times_G F$ be the associated bundle. Clearly, $P \subseteq G_k(\mathscr{H})$. Also, D induces a connection $\mathcal{D} \subseteq TP$ on $\pi : P \to M$. Then, for each $p \in P$ we define $\mathscr{F}_p \subseteq T_pP$ to be the horizontal lift, with respect to $\mathcal{D}$, at $p \in P$ of $p \subseteq \mathscr{H}_{\pi(p)}$.

Usually, $G \subseteq CO(n, \mathbb{C})$ where $CO(n, \mathbb{C})$ is the complex-conformal group in dimension n, so that $\mathscr{H}$ is endowed with a conformal structure. Then, if D corresponds to a principal partial connection on the corresponding frame bundle $(Q, M, CO(n, \mathbb{C}))$, it is called a *conformal partial connection*.

(3) Let $(P, M, \pi, \mathscr{F})$ ($\dim M = m$, $\dim \mathscr{F} = k$) be a twistorial structure where, as above, P is a subbundle of the Grassmann bundle $G_k(TM)$ and $\mathscr{F}$ is induced by some connection D on M which preserves P. Suppose that D is torsion-free and, for any $p \in P$, there exists a totally geodesic submanifold of M, of dimension $(m - k)$, which passes through $\pi(p)$ and which is transversal to p. Then each point of M has an open neighbourhood U such that $\mathscr{F}|_{\pi^{-1}(U)}$ is simple.

Next, we give the basic examples of almost twistorial structures with which we shall work. Recall that we are working in the complex-analytic category. First, we consider structures over two- and three-dimensional manifolds.

EXAMPLE 2.3 (LeBrun [**25**]) Let $M = M^2$ be a two-dimensional Riemannian manifold and let $\pi : P \to M$ be the bundle of null directions on M^2. Clearly, $P = \det(O(M))$ and hence $\pi : P \to M$ is a $\mathbb{Z}_2$-covering. Furthermore, there exists a canonical one-dimensional foliation $\mathscr{F}$ on P such that π locally maps leaves of $\mathscr{F}$ to (local) null geodesics on M^2. Hence any two-dimensional Riemannian (or conformal, if one prefers) manifold M^2 is canonically endowed with the twistorial structure $(P, M, \pi, \mathscr{F})$.

Conversely, any (almost) twistorial structure $(P, M, \pi, \mathscr{F})$ with $\dim M = 2$, $\dim \mathscr{F} = 1$ and $\pi : P \to M$ a $\mathbb{Z}_2$-covering, such that the map $P \to P(TM)$, $p \mapsto d\pi(\mathscr{F}_p)$ is an embedding, is induced by a conformal structure as above.

If the Riemannian manifold M^2 is orientable (equivalently, if the $\mathbb{Z}_2$-covering $\pi : P \to M$ is trivial) then $P = M_+ \sqcup M_-$, where M_+ and M_- are copies of M. Hence, there exist two foliations by null geodesics $\mathscr{F}_+$ and $\mathscr{F}_-$, on M, which are the projections of $\mathscr{F}$ restricted to M_+ and M_-, respectively. Therefore, the twistor space $Z = Z(M)$ of the canonical twistorial structure of an oriented two-dimensional Riemannian manifold M^2 (more generally, of a two-dimensional manifold endowed with an oriented conformal structure) is the *space of null geodesics* [**25**] of M^2. Then, locally, Z is the disjoint union of two curves $Z = C_+ \sqcup C_-$ and $M = C_+ \times C_-$. Also, note that the (complex-analytic) almost Hermitian structures $J_\pm$ on M defined by $J_\pm(X) = \pm \mathrm{i}X$ for $X \in \mathscr{F}_\pm$ are integrable.

Let (M, g) be a Riemannian manifold, $\dim M = m$. A *degenerate hyperplane* $H \subseteq TM$ is a subspace of codimension one such that $g|_H$ is degenerate (equivalently, H is the orthogonal complement of a null vector which is thus contained in H) (cf. [**25**]); if $\dim M = 3$ we shall say *degenerate plane*. For fixed $x \in M$, the space of degenerate hyperplanes in T_xM can be identified with the non-singular quadric Q_{m-2} in $P(T_xM)$.

EXAMPLE 2.4 (Hitchin [**18**]) Let M^3 be a three-dimensional Riemannian manifold. Let D be a Weyl connection (that is, a torsion-free conformal connection) on M. Let $\pi : P \to M$ be the bundle of degenerate planes on M^3. Then $P = CO(M) \times_\rho \mathbb{C}P^1$, where $CO(M)$ is the bundle of conformal frames of M^3 and ρ is the action of $CO(3, \mathbb{C})$ on $\mathbb{C}P^1 = \{p \mid p \subseteq \mathbb{C}^3 \text{ degenerate plane}\}$ induced by the canonical action of $CO(3, \mathbb{C})$ on $\mathbb{C}^3$. Thus D induces a connection on $\pi : P \to M$ and, for each $p \in P$, we define $\mathscr{F}_p \subseteq T_pP$ to be the horizontal lift at $p \in P$ of $p \subseteq T_{\pi(p)}M$. Obviously, the almost twistorial structure $(P, M, \pi, \mathscr{F})$ depends only on D and on the conformal class c of the metric of M^3.

THEOREM 2.5 [**18**] *The twistor distribution $\mathscr{F}$ is integrable if and only if D is Einstein–Weyl.*

Furthermore, if D is Einstein–Weyl then (locally) the leaf space Z of $\mathscr{F}$ contains a locally complete analytic family of projective lines each of which has normal bundle $\mathcal{O}(2)$. Conversely, any surface Z containing a projective line with normal bundle $\mathcal{O}(2)$ is (locally) the twistor space of a three-dimensional Riemannian manifold M^3 endowed with an Einstein–Weyl connection D. The conformal structure of M^3 and the Einstein–Weyl connection D are uniquely determined.

Let $(P, M, \pi, \mathscr{F})$ be the twistorial structure corresponding, as above, to (M^3, D), where D is an Einstein–Weyl connection on M^3. Then, as just explained, $\mathscr{F}$ is integrable and the locally complete analytic family [**23**] of projective lines on Z of Theorem 2.5 appears as in Remark 2.2(1) (see, also, Remark 2.2(3)): each projective line represents all the (local) degenerate surfaces through a given point which are totally geodesic with respect to D. The fact that the normal bundle of any twistor line in Z is $\mathcal{O}(2)$ can be proved as follows. Let $t = \mathbb{C}P^1$ be a fibre of π. Then the normal bundle of $\pi_Z(t) = \mathbb{C}P^1$ in Z is (isomorphic to) $(t \times \mathbb{C}^3)\big/(\mathscr{F}|_t)$. Now, $\mathscr{F}|_t$ is the restriction to $t = Q_1 \subseteq \mathbb{C}P^2 = G_2(\mathbb{C}^3)$ of the tautological plane bundle E over $G_2(\mathbb{C}^3)$, where Q_1 is the one-dimensional non-singular quadric and $G_2(\mathbb{C}^3)$ is the Grassmann manifold of planes in $\mathbb{C}^3$. As $(G_2(\mathbb{C}^3) \times \mathbb{C}^3)\big/E = H$, where $H \to \mathbb{C}P^2$ is the hyperplane bundle and the embedding $t = \mathbb{C}P^1 = Q_1 \hookrightarrow \mathbb{C}P^2$ has degree two, we obtain $(t \times \mathbb{C}^3)\big/(\mathscr{F}|_t) = \mathcal{O}(2)$.

Note that, locally, any leaf of $\mathscr{F}$ is mapped by π to a degenerate surface in M^3 which is totally geodesic with respect to D, and so Z is, locally, the space of degenerate surfaces on M^3 which are totally geodesic with respect to D.

Finally, we discuss an important example of almost twistorial structures over a four-dimensional manifold.

EXAMPLE 2.6 (Penrose [**34**], Atiyah *et al.* [**1**])

(i) Let M^4 be a four-dimensional Riemannian manifold. A plane $H \subseteq TM$ on (M^4, g) is called *null* if $g|_H = 0$. Let $\pi : P \to M$ be the bundle of null planes on M^4. Then the Levi-Civita connection of M^4 induces a connection on $\pi : P \to M$ and, for each $p \in P$, we define $\mathscr{F}_p \subseteq T_pP$ to be the horizontal lift at $p \in P$ of $p \subseteq T_{\pi(p)}M$. As $\mathscr{F}$ is conformally invariant, the almost twistorial structure $(P, M, \pi, \mathscr{F})$ canonically associated to M^4 is conformally invariant. It is well known that $\mathscr{F}$ is integrable if and only if M^4 is conformally flat (a consequence of Theorem 2.7, below).

(ii) We have that (M^4, g) is orientable if and only if $P = P_+ \sqcup P_-$ is the disjoint union of two $\mathbb{C}P^1$-bundles over M^4. In this case, with respect to a choice of orientation on M^4, $P_+ \to M$ is the bundle of self-dual planes on M^4 and $P_- \to M$ is the bundle of anti-self-dual planes on M^4. (Self-dual and anti-self-dual planes are sometimes called *α-planes* and *β-planes*; see [**27**, **35**].) Then, with $\pi_\pm = \pi|_{P_\pm}$ and $\mathscr{F}_\pm = \mathscr{F}|_{P_\pm}$, $(P_\pm, M, \pi_\pm, \mathscr{F}_\pm)$ are almost twistorial structures on M^4. These almost twistorial structures are conformally invariant and so are well defined on any four-dimensional manifold endowed with an oriented conformal structure.

THEOREM 2.7 [**1**, **34**] *The twistor distribution $\mathscr{F}_-$ is integrable if and only if M^4 is self-dual.* (*Similarly, $\mathscr{F}_+$ is integrable if and only if M^4 is anti-self-dual.*)

Furthermore, if M^4 is self-dual then (*locally*) *the leaf space Z of $\mathscr{F}_-$ contains a locally complete analytic family of projective lines each of which has normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$. Conversely, any three-dimensional manifold Z containing a projective line with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$ is* (*locally*) *the twistor space of a self-dual Riemannian four-manifold which is uniquely determined, up to a conformal change of the metric.*

As the oriented $\mathbb{Z}_2$-covering of a *non*-orientable Riemannian manifold is canonically endowed with an orientation-reversing isometry, the oriented $\mathbb{Z}_2$-covering of a non-orientable Riemannian four-manifold M^4 is (anti-)self-dual if and only if M^4 is conformally flat.

Let $(P_-, M, \pi_-, \mathscr{F}_-)$ be the twistorial structure corresponding, as above, to the four-dimensional self-dual Riemannnian manifold M^4. Then, by Theorem 2.7, $\mathscr{F}_-$ is integrable and the locally complete analytic family of projective lines [**23**] on Z appears again as in Remark 2.2(1) (see, also, Remark 2.2(3)): each projective line represents all the (local) anti-self-dual surfaces through a given point. The fact that the normal bundle of any twistor line in Z is $\mathcal{O}(1) \oplus \mathcal{O}(1)$ can be proved as follows. Let $t = \mathbb{C}P^1$ be a fibre of π_-. Then the normal bundle of $\pi_Z(t) = \mathbb{C}P^1$ in Z is (isomorphic to) $(t \times \mathbb{C}^4)\big/(\mathscr{F}_-|_t)$. Now, $\mathscr{F}_-|_t$ is the restriction to $t = \mathbb{C}P^1 \subseteq G_2(\mathbb{C}^4)$ of the tautological plane bundle E over $G_2(\mathbb{C}^4)$. As any anti-self-dual p-plane on $\mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2$ is given by $p = \{ v \otimes v' \mid v \in \mathbb{C}^2 \}$ for some fixed $v' \in \mathbb{C}^2$ we have that $E|_t = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ and hence the normal bundle of $\pi_Z(t)$ in Z is $\mathcal{O}(1) \oplus \mathcal{O}(1)$.

Note that, locally, any leaf of $\mathscr{F}_-$ is mapped by π_- to an anti-self-dual surface in M^4 so Z is, locally, the space of anti-self-dual surfaces in M^4.

REMARK 2.8 The twistorial structures of Example 2.4 are *reductions* of the twistorial structures of Example 2.6 [**18**] (see [**9**]).

3. Twistorial maps

In this section, unless otherwise stated, we again work in the complex-analytic category; thus all the manifolds are complex and all the maps are complex-analytic.

In what follows, it is convenient to work with the following definitions. We start with an important special case.

DEFINITION 3.1 Let $\sigma = (P, M, \pi_P, \mathscr{F})$ and $\tau = (Q, N, \pi_Q, \mathscr{G})$ be almost twistorial structures over M and N, respectively. Let $\varphi : M \to N$ be a map. Suppose that there exists a map $\Phi : P \to Q$ which covers φ (that is, $\pi_Q \circ \Phi = \varphi \circ \pi_P$).

Then $\varphi : (M, \sigma) \to (N, \tau)$ will be called a *twistorial map* (*with respect to* Φ) if $d\Phi(\mathscr{F}_p) \subseteq \mathscr{G}_{\Phi(p)}$ for all $p \in P$.

The following lemma allows us to construct 'substructures' of an almost twistorial structure; its proof is omitted.

LEMMA 3.2 *Let $\sigma = (P, M, \pi, \mathscr{F})$ be an almost twistorial structure and let $P' \subseteq P$ be a closed submanifold such that $\pi(P') = M$. Suppose that $\mathscr{F}_p \subseteq T_pP'$ for all $p \in P'$ and* $\dim(T_pP' \cap \ker d\pi_p)$ *does not depend on $p \in P'$.*

Then $\sigma' = (P', M, \pi|_{P'}, \mathscr{F}|_{P'})$ is an almost twistorial structure.

Next, we generalize the definition of twistorial map by allowing Φ to be a map between such substructures.

DEFINITION 3.3 Let $\sigma = (P, M, \pi_P, \mathscr{F})$ and $\tau = (Q, N, \pi_Q, \mathscr{G})$ be almost twistorial structures over M and N, respectively, and let $\varphi : M \to N$ be a map. Let $P' \subseteq P$ be as in Lemma 3.2 and similarly for $Q' \subseteq Q$ so that $\sigma' = (P', M, \pi_P|_{P'}, TP' \cap \mathscr{F})$ and $\tau' = (Q', N, \pi_Q|_{Q'}, TQ' \cap \mathscr{G})$ are almost twistorial structures. Suppose that φ is covered by a map $\Phi : P' \to Q'$. The map $\varphi : (M, \sigma) \to (N, \tau)$ will be called a *twistorial map* (*with respect to* Φ) if $d\Phi(\mathscr{F}_p) \subseteq \mathscr{G}_{\Phi(p)}$ for all $p \in P'$.

Often, our choice of P' and Q' will depend on the map φ.

If the distributions $\mathscr{F}|_{P'}$ and $\mathscr{G}|_{Q'}$ are integrable with leaf spaces $Z(M, \varphi)$ and $Z(N, \varphi)$, respectively, then we have an induced local map $Z(\varphi) : Z(M, \varphi) \to Z(N, \varphi)$ which we shall call the *twistorial representation* of φ.

REMARK 3.4

(1) Any real-analytic (Riemannian) manifold has a germ-unique complexification to a complex (-Riemannian) manifold [**25**]. We can *complexify* any real-analytic map between real-analytic manifolds, that is, extend it to (the germ of) a complex-analytic map between complexifications of those manifolds. A map between real-analytic manifolds will be called twistorial if its complexification is twistorial.
(2) Note that Definition 3.1 is the case when $P' = P$ and $Q' = Q$.
(3) In all our examples, P is embedded as a subbundle of some Grassmann bundle $G_k(TM)$, as in Remark 2.2(2). Also, Φ will be naturally induced by $d\varphi$ so that we shall write $\Phi = d\varphi|_{P'}$.
(4) In Definition 3.3, if the distributions $\mathscr{F}$ and $\mathscr{G}$ are integrable and $Z(M)$, $Z(M, \varphi)$, $Z(N)$, $Z(N, \varphi)$ are the leaf spaces of $\mathscr{F}$, $\mathscr{F}|_{P'}$, $\mathscr{G}$, $\mathscr{G}|_{Q'}$, respectively, then we have induced local maps $Z(M, \varphi) \to Z(M)$, $Z(N, \varphi) \to Z(N)$.
(5) Most of the twistorial maps which will appear in this paper are submersive maps. We could also consider *twistorial foliations*, that is, foliations that are locally defined by submersive twistorial maps; most of the facts that follow can be easily reformulated in terms of such foliations.
(6) (*Compositions*) If $M_j (j = 1, 2, 3)$ are endowed with almost twistorial structures $\tau_j = (P_j, M_j, \pi_j, \mathscr{F}_j)$ $(j = 1, 2, 3)$ and $\varphi_1 : (M_1, \tau_1) \to (M_2, \tau_2)$, $\varphi_2 : (M_2, \tau_2) \to (M_3, \tau_3)$ are twistorial maps with respect to $\Phi_1 : P_1' \to Q_1'$ and $\Phi_2 : P_2' \to Q_2'$, such that $\Phi_1(P_1') \subseteq P_2'$, then $\varphi_2 \circ \varphi_1 : (M, \tau_1) \to (M, \tau_3)$ is a twistorial map with respect to $\Phi_2 \circ \Phi_1$.

The following simple proposition will be useful later on.

PROPOSITION 3.5 *Let M be a manifold endowed with a linear connection D and let $\mathscr{H} \subseteq T(G_k(TM))$ be the connection induced by D on the Grassmann bundle $\pi : G_k(TM) \to M$ of k-dimensional planes on M ($k \leq \dim M$). Let $\mathscr{F}$ be the k-dimensional distribution on $G_k(TM)$ defined by setting $\mathscr{F}_p$ equal to the horizontal lift at $p \in G_k(TM)$ of $p \subseteq T_{\pi(p)}M$.*

Then, for a distribution $s : M \to G_k(TM)$ on M the following assertions are equivalent:

(i) *$\mathscr{F}_{s(x)} \subseteq T_{s(x)}(s(M))$ for all $x \in M$;*
(ii) *for any curve c tangent to the distribution s on M we have that $s \circ c$ is a parallel section of $G_k(TM)$;*
(iii) *for any vector fields $X, Y \in \Gamma(s)$ we have $D_X Y \in \Gamma(s)$.*

Furthermore, if D is torsion free, then the following assertion can be added to this list:

(iv) *the distribution s is integrable and its leaves are totally geodesic with respect to D.*

Proof. Assertion (i) is equivalent to the fact that for any $X \in s(x)$ we have $ds(X) \in \mathscr{F}_{s(x)}$; this is clearly equivalent to assertion (ii).

Assertion (ii) is equivalent to the fact that the distribution s is invariant under parallel transport along curves tangent to s. Then, the equivalence of (ii) and (iii) follows from the fact that, if $U \subseteq \mathbb{C}$ is a domain in $\mathbb{C}$, H is a Lie subgroup of the Lie group G and $(a_z)_{z\in U}$ is a curve in G such that $a_{z_0} \in H$ for some $z_0 \in U$ and $\theta(\mathrm{d}a_z/\mathrm{d}z) \in L(H)$ for all $z \in U$, where $\theta \in \Gamma(L(G) \otimes T^*G)$ is the canonical form of G, then $a_z \in H$ for all $z \in U$. (Here '$L(G)$' and '$L(H)$' denote the Lie algebras of 'G' and 'H', respectively.)

If D is torsion free, the equivalence (iii) $\Longleftrightarrow$ (iv) is trivial.

EXAMPLE 3.6 We interpret the twistor lift of Eells and Salamon [**11**] in our framework. Let M^2 and N^4 be manifolds of dimensions two and four, respectively, endowed with oriented conformal structures and let $\varphi : M^2 \to N^4$ be an injective conformal immersion.

Endow M^2 and N^4 with the almost twistorial structures $(P, M, \pi, \mathscr{F})$ and $(P_-, N, \pi_-, \mathscr{F}_-)$ of Examples 2.3 and 2.6, respectively.

As φ maps null directions on M^2 to null directions on N^4, we can define $\Phi_- : P \to P_-$ by $\Phi_-(p)$ is the anti-self-dual plane containing $d\varphi(p)$, for each null direction p on M^2. Then, φ is twistorial with respect to Φ_- if and only if, for any null geodesic γ on M^2, $\Phi_- \circ \gamma$ is parallel along $\varphi \circ \gamma$. If N^4 is self-dual then φ is twistorial with respect to Φ_- if and only if, for any null geodesic γ on M^2 there exists a (necessarily unique) anti-self-dual surface $S_\gamma \subseteq N^4$ such that $\varphi(\gamma) \subseteq S_\gamma$. Then, the map $\gamma \mapsto S_\gamma$ is the twistorial representation of φ whose image is a pair of (local) curves in $Z(N)$.

A nowhere degenerate surface in N^4 will be called a $(-)$*twistorial surface* if the corresponding inclusion map is twistorial, in the above sense.

Similarly, we define $(+)$*twistorial surfaces* by using instead the almost twistorial structure $(P_+, N, \pi_+, \mathscr{F}_+)$ of Example 2.6(ii). *A nowhere degenerate surface on an oriented four-dimensional conformal manifold is totally umbilical if and only if it is both $(+)$twistorial and $(-)$twistorial* [**11**].

4. Twistorial maps to surfaces

We shall now describe the twistorial maps which we shall need, namely those with nowhere degenerate fibres.

EXAMPLE 4.1 (cf. [**3, 4, 10**]) Let $\varphi : M^3 \to N^2$ be a surjective submersion whose fibres are nowhere null from a Riemannian manifold onto an oriented Riemannian manifold. Let D be a Weyl connection on M^3. We consider M^3 to be endowed with the almost twistorial structure $(P, M, \pi, \mathscr{F})$ of Example 2.4, and N^2 with the twistorial structure of Example 2.3 with $Q \to N$ the bundle of null directions on N^2.

At each $x \in M$ there are precisely two horizontal null directions $h_+(x)$ and $h_-(x)$. Thus to φ correspond the two disjoint embeddings $M_\pm = h_\pm(M) \hookrightarrow P$ given by $x \mapsto h_\pm(x)^\perp = \mathrm{Span}(h_\pm(x), \ker \mathrm{d}\varphi_x)$.

As $\dim P = 4$, $\dim \mathscr{F} = 2$ and $\dim M_\pm = 3$, at each degenerate plane $p \in M_\pm$, $d_\pm(p) = \dim(\mathscr{F}_p \cap T_p M_\pm) \in \{1, 2\}$.

By Proposition 3.5, $d_{\pm} = 2$ on $M_{\pm}$ if and only if the fibres of $\varphi : M^3 \to N^2$ form a conformal foliation which is geodesic with respect to D. *Therefore, φ is a twistorial map with $P' = M_+ \sqcup M_-$ (and $\Phi = d\varphi|_{P'} : P' \to Q$) if and only if it is a horizontally conformal submersion with geodesic fibres (with respect to D).* If D is Einstein–Weyl, then $Z(M, \varphi) = C_+ \sqcup C_-$ is the leaf space of the foliation induced by $\mathscr{F}$ on $M_+ \sqcup M_-$ and the map $Z(M, \varphi) \to Z(M)$ is simply the inclusion $C_+ \sqcup C_- \hookrightarrow Z(M)$. Obviously C_+ and C_- are transversal to the twistor lines in $Z(M)$. Also, $Z(N, \varphi) = Z(N)$ is the space of null geodesics of N^2, which, as a consequence of the horizontal conformality of φ can be canonically identified with $C_+ \sqcup C_-$; then the twistorial representation of φ is simply the identification map $C_+ \sqcup C_- \to Z(N)$. To retrieve φ from its twistorial representation, let $x \in M^3$; this corresponds to the twistor line $t_x \subseteq Z(M)$. Then, locally, t_x meets C_+ and C_- in two points which, under the identification $C_+ \sqcup C_- = Z(N)$ correspond to two null geodesics on N^2 whose intersection is precisely $\varphi(x)$.

If $d_{\pm} = 1$ on $M_{\pm}$ then $\mathscr{F}$ induces a one-dimensional foliation on $M_+ \cup M_-$. For example, if $\varphi : M^3 \to N^2$ is a horizontally conformal submersion (with nowhere null fibres) whose fibres are nowhere geodesics, then $d_{\pm} = 1$ on $M_{\pm}$ and the leaves of the induced foliation on $M_+ \sqcup M_-$ project onto the horizontal null geodesics. For such a horizontally conformal submersion, if D is Einstein–Weyl, then, at least locally, any totally geodesic (with respect to D) degenerate surface contains precisely one horizontal null geodesic and so we have two *local* sections $Z_{\pm} \hookrightarrow \mathcal{N}(M)$ of the canonical projection [**18**] $\mathcal{N}(M) \to Z(M)$, where $\mathcal{N}(M)$ is the space of null geodesics of M^3 so that $Z_+ \sqcup Z_-$ is the space of null geodesics of M^3 which are horizontal with respect to φ. If φ is twistorial (with respect to D), then the fibres of $\mathcal{N}(M) \to Z(M)$ induce a one-dimensional foliation on $Z_+ \sqcup Z_-$ whose leaf space is $C_+ \sqcup C_-$.

The following result (cf. Theorem 4.5 and Theorem 5.11, below), which is a consequence of Proposition 1.5 and Example 4.1, is a rephrasing of the starting point of the classification results of [**3**, **4**] (see also [**6**, Chapters 1 and 6]), there given for the smooth category.

PROPOSITION 4.2 *Let $\varphi : M^3 \to N^2$ be the complexification of a real-analytic submersive map between Riemannian manifolds. Equip M^3 with the Levi-Civita connection and endow it with the almost twistorial structure of Example* 2.4*; equip N^2 with the twistorial structure of Example* 2.3.

Then, φ is a twistorial map, in the sense of Example 4.1, *if and only if it is the complexification of a harmonic morphism.*

DEFINITION 4.3 Let M^4 be an oriented four-dimensional Riemannian manifold considered with the almost twistorial structure $\tau_- = (P_-, M, \pi_-, \mathscr{F}_-)$ of Example 2.6(ii), with $\pi_- : P_- \to M$ the bundle of anti-self-dual planes. Let $\varphi : M^4 \to N^n$ be a surjective submersive map with n $(= \dim N) = 2$ or 3. Let $\tau = (Q, N, \pi, \mathscr{G})$ be the almost twistorial structure of Example 2.3 (if $n = 2$), or of Example 2.4 for some Weyl connection on N (if $n = 3$).

We shall say that φ is $(-)$*twistorial* (*with respect to* Φ) if $\varphi : (M, \tau_-) \to (N, \tau)$ is a twistorial map with respect to $\Phi = d\varphi|_{P'} : P' \to Q$ for some $P' \subseteq P_-$ as in Lemma 3.2.

Similarly, we define what is meant by φ is $(+)$*twistorial* by using instead the almost twistorial structure $(P_+, N, \pi_+, \mathscr{F}_+)$ of Example 2.6(ii). We shall write $(\pm)$*twistorial* to mean $(+)$*twistorial or* $(-)$*twistorial.*

EXAMPLE 4.4 (cf. [**37**]) Let $\varphi : M^4 \to N^2$ be a surjective submersion between oriented Riemannian manifolds, whose fibres are nowhere degenerate. We consider M^4 endowed with the almost twistorial structure $(P_-, M, \pi_-, \mathscr{F}_-)$ of Examples 2.6(ii) with $P = P_-$ the bundle of anti-self-dual planes on M^4, and N^2 endowed with the twistorial structure $(Q, N, \pi, \mathscr{G})$ of Examples 2.3, with $Q \to N$ the bundle of null directions on N^2.

At each point $x \in M^4$ there are precisely two horizontal null directions $h_+(x)$, $h_-(x)$ and two vertical null directions $v_+(x)$, $v_-(x)$. Let $p_+(x)$ (respectively, $p_-(x)$) be the (null) plane spanned by $h_+(x)$ and $v_+(x)$ (respectively, $h_-(x)$, and $v_-(x)$). Then, from the fact that any self-dual plane intersects any anti-self-dual plane along a null line, it easily follows that either both $p_+(x)$ and $p_-(x)$ are self-dual or both are anti-self-dual. We assume that we have chosen the orientations such that both $p_+(x)$ and $p_-(x)$ are anti-self-dual planes. These give two disjoint embeddings $M_\pm = p_\pm(M) \hookrightarrow P_-$.

As $\dim P_- = 5$, $\dim \mathscr{F} = 2$ and $\dim M_\pm = 4$, at each null plane $p \in M_\pm$ we have $d_\pm(p) = \dim(\mathscr{F}_p \cap T_p M_\pm) \in \{1, 2\}$.

By Proposition 3.5, $d_\pm = 2$ on $M_\pm$ if and only if the two distributions p_+ and p_- are integrable and totally geodesic. This is equivalent to the fact that the almost Hermitian structures $J_\pm$ defined by $J_\pm(X) = \pm\, \mathrm{i}X$ for $X \in p_\pm$ are integrable. Then, on setting $P' = M_+ \sqcup M_-$ and $\Phi = d\varphi|_{P'} \colon P' \to Q$, *$\varphi$ is a $(-)$twistorial map with respect to Φ if and only if J_- $(= -J_+)$ is integrable and φ is horizontally conformal.*

If φ is $(-)$twistorial, then the anti-self-dual surfaces on M^4 which are leaves of the distributions p_+ or p_- are foliated by horizontal null geodesics (this follows from the fact that φ is horizontally conformal and, by definition, at each point $x \in M^4$, the space $p_\pm(x)$ intersects the horizontal space of φ at x along a null direction). If, further, M^4 is self-dual, then $Z(M, \varphi) = S_+ \sqcup S_-$ is the leaf space of the foliation induced by $\mathscr{F}$ on $M_+ \sqcup M_-$, and the map $Z(M, \varphi) \to Z(M)$ is simply the inclusion $S_+ \sqcup S_- \hookrightarrow Z(M)$. Obviously S_+ and S_- are transversal to the twistor lines. Furthermore, as the preimage of a null geodesic on N^2 through φ is a hypersurface on M^4 foliated by anti-self-dual surfaces, the two 'surfaces' S_+ and S_- in $Z(M)$ are foliated by curves with leaf spaces C_+ and C_-. Also, $Z(N, \varphi) = Z(N)$ is the space of null geodesics of N^2 which, as a consequence of the horizontal conformality of φ, can be canonically identified with $C_+ \sqcup C_-$; then the twistorial representation of φ is simply the projection $S_+ \sqcup S_- \to Z(N)$. To retrieve φ from its twistorial representation, let $x \in M^4$; this corresponds to the twistor line $t_x \subseteq Z(M)$. Then, locally, t_x meets S_+ and S_- in two points which are projected by $S_+ \sqcup S_- \to Z(N)$ to two points: one in C_+ and the other in C_-. These correspond to two null geodesics on N^2 whose intersection is precisely $\varphi(x)$.

In a similar way to Example 4.1, if $\varphi : M^4 \to N^2$ is a horizontally conformal submersion for which J_- $(= -J_+)$ is nowhere integrable then $d_\pm = 1$ on $M_\pm$ and, if M^4 is self-dual then, locally, each anti-self-dual surface contains precisely one horizontal null geodesic and so we have two *local* sections $Z_\pm \hookrightarrow \mathcal{N}(M)$ of the canonical projection (see LeBrun's papers in [**28**], cf. [**18**]) $\mathcal{N}(M) \to Z(M)$, where $\mathcal{N}(M)$ is the space of null geodesics of M^4 [**25**]. Note that if φ is $(-)$twistorial then the fibres of $\mathcal{N}(M) \to Z(M)$ induce on $Z_+ \sqcup Z_-$ a one-dimensional foliation whose leaf space is $S_+ \sqcup S_-$.

A horizontally conformal submersion $\varphi \colon M^4 \to N^2$ with nowhere degenerate fibres is $(-)$twistorial if and only if its fibres are $(-)$twistorial in the sense of Example 3.6. In particular, *a horizontally conformal submersion $\varphi : M^4 \to N^2$ with nowhere degenerate fibres is both $(+)$twistorial and $(-)$twistorial if and only if it has totally umbilical fibres* (cf. Example 3.6).

The following theorem is a reformulation of a result of [**37**] (cf. Proposition 4.2 and Theorem 5.11, below).

THEOREM 4.5 *Let M^4 be an orientable four-dimensional Einstein manifold and let $\varphi : M^4 \to N^2$ be the complexification of a submersive harmonic morphism.*

Then, with a suitable choice of orientations, φ is $(-)$twistorial.

REMARK 4.6

(1) Let M be a manifold endowed with an almost twistorial structure $\tau = (P, M, \pi, \mathscr{F})$, $(\dim \mathscr{F} = k)$, such that the map $P \to G_k(TM)$ given by $p \mapsto \mathrm{d}\pi(\mathscr{F}_p)$ is an embedding of P in the Grassmann bundle of k-dimensional planes on M.

Then τ is integrable if, locally, there are sufficiently many twistorial maps from (M, τ); for example, if for each $p \in P$ there exists a (local) twistorial map φ from (M, τ) such that $\ker \mathrm{d}\varphi_{\pi(p)} = p$ then τ is integrable. In the above examples this condition is also necessary; moreover other similar necessary and sufficient conditions can easily be formulated. In Theorem 5.3 below we shall see that the existence of a *single* suitable twistorial map may be sufficient for a twistorial structure to be integrable.

(2) Twistorial maps as in Example 4.1, Example 4.4 and as in the next section appear, in a more or less explicit way, in [**26**].

5. Twistorial maps with one-dimensional fibres from four-dimensional Riemannian manifolds

In this section we continue to work in the complex-analytic category. The almost twistorial structures which may appear in this section will be those of Examples 2.4 and 2.6. As usual, the results can be applied to the real-analytic category by complexification (Remark 3.4).

Let $\varphi : M^4 \to N^3$ be a surjective submersive map with nowhere degenerate fibres from an oriented four-dimensional Riemannian manifold to a three-dimensional Riemannian manifold. Suppose, for the moment, that M^4 is self-dual and is endowed with the twistorial structure of Example 2.6(ii), with $P = P_-$ the bundle of anti-self-dual planes of M^4, and that N^3 is endowed with the twistorial structure of Example 2.4, which corresponds to an Einstein–Weyl connection D on N^3, with Q the bundle of degenerate planes on N^3. Then, it is clear that φ is $(-)$twistorial for a suitable choice of $P' \subseteq P$ if and only if it maps some, if not all, of the anti-self-dual surfaces on M^4 to totally geodesic degenerate surfaces on (N^3, D) (cf. [**18**, **20**]). But, unless we introduce some extra structure, there is no reason to ignore any of the anti-self-dual surfaces on M^4. Moreover, if φ maps anti-self-dual surfaces on M^4 to totally geodesic degenerate surfaces on (N^3, D) then, in particular, $d\varphi$ maps anti-self-dual planes on M^4 to degenerate planes on N^3, which is equivalent to the condition that φ be horizontally conformal (indeed, the differential $d\varphi_x$, $(x \in M)$, maps the orthogonal complement, in $\mathscr{H}_x$, of a null direction $l \subseteq \mathscr{H}_x$ onto the orthogonal complement of a null direction $k \subseteq T_{\varphi(x)}N$ if and only if the adjoint of $d\varphi_x$ maps k onto l; thus, as the horizontal projection of any anti-self-dual plane is a degenerate horizontal plane and any degenerate horizontal plane is obtained in this way, $d\varphi$ maps anti-self-dual planes on M^4 to degenerate planes on N^3 if and only if the adjoint of $d\varphi$ is conformal at each point).

Therefore, given an oriented Riemannian manifold M^4 and a horizontally conformal submersion $\varphi : M^4 \to N^3$ with nowhere degenerate fibres onto a Riemannian manifold, we shall look for necessary and sufficient conditions under which there exists a Weyl connection D on N^3 with respect to which $\varphi : M^4 \to (N^3, D)$ is a $(-)$twistorial map with $P' = P_-$ the bundle of anti-self-dual planes on M^4.

To do this, we first place the discussion in a slightly more general context. Let M^4 be an oriented Riemannian manifold and let $\mathscr{H}$ be a nowhere degenerate three-dimensional distribution. Denote, as usual, $\mathscr{V} = \mathscr{H}^\perp$ and assume that $(\mathscr{V}, g|_\mathscr{V})$ and $(\mathscr{H}, g|_\mathscr{H})$ are oriented so that the isomorphism $(TM, g) = (\mathscr{V}, g|_\mathscr{V}) \oplus (\mathscr{H}, g|_\mathscr{H})$ is orientation preserving. Let $I^\mathscr{H}$ be the integrability 2-form of $\mathscr{H}$ defined by $I^\mathscr{H}(X, Y) = -g(U, [X, Y])$ for all local sections X, Y of $\mathscr{H}$, where U is the positive unit section of $\mathscr{V}$. Also, let $B^\mathscr{H}$ be the second fundamental form of $\mathscr{H}$ defined by $B^\mathscr{H}(X, Y) = \frac{1}{2}\mathscr{V}(\nabla_X Y + \nabla_Y X)$ for any local horizontal vector fields X and Y, where ∇ is the Levi-Civita connection of (M^4, g). Then we define a section $\mathcal{B}^\mathscr{H}$ of $\mathscr{H}^* \otimes \mathscr{H}^*$ by $\mathcal{B}^\mathscr{H}(X, Y) = g(U, B^\mathscr{H}(X, Y))$ for any horizontal X and Y; we shall denote by $\mathcal{B}_0^\mathscr{H}$ the trace-free part of $\mathcal{B}^\mathscr{H}$. Let $*_{\mathscr{H},g}$ be the Hodge star operator on $(\mathscr{H}, g|_\mathscr{H})$.

Recall (see [**13**]) that if D is a conformal connection on a conformal manifold (M, c) and g is a local representative of c on some open set U, then the *Lee form* of D *with respect to* g is the one-form $\alpha \in \Gamma(T^*U)$ characterized by $Dg = -2\alpha \otimes g$. The Lee form of a conformal *partial* connection can be defined in a similar way. Also, if D is a partial connection on $\mathscr{H}$, over $\mathscr{H}$, then its *torsion* T, with respect to $\mathscr{V}$, is defined by $T(X, Y) = D_X Y - D_Y X - \mathscr{H}[X, Y]$ for any local sections X and Y of $\mathscr{H}$. We now introduce a conformal partial connection which encodes the condition of twistoriality.

DEFINITION 5.1 (cf. [**9**]) The *conformal partial connections $D_\pm$ induced by g on $\mathscr{H}$* are the unique conformal partial connections on $(\mathscr{H}, g|_\mathscr{H})$, over $\mathscr{H}$, whose Lee forms $\alpha_\pm$ and torsion tensors $T_\pm$, with respect to $\mathscr{V}$, are given by

$$\begin{aligned} \alpha_\pm &= \operatorname{trace}(B^\mathscr{V})^b \pm *_{\mathscr{H},g} I^\mathscr{H}, \\ *_{\mathscr{H},g} T_\pm &= \mp 2\mathcal{B}_0^\mathscr{H}, \end{aligned} \tag{5.1}$$

where we have identified $\mathscr{H} \otimes \mathscr{H}^*$ and $\mathscr{H}^* \otimes \mathscr{H}^*$ by using $g|_\mathscr{H}$.

If $\mathscr{H}$ is totally umbilical then D_- is the horizontal part of a Weyl connection of [**9**, §4].

Let $\pi^\mathscr{H}\colon P^\mathscr{H} \to M$ be the bundle of degenerate planes of $\mathscr{H}$. As in Example 2.4, D_- induces a twistor distribution $\mathscr{F}_-^\mathscr{H}$ defined by setting $\mathscr{F}_-^\mathscr{H}(p) \subseteq T_p P^\mathscr{H}$ equal to the horizontal lift at $p \in P^\mathscr{H}$, with respect to D_-, of $p \subseteq \mathscr{H}_{\pi^\mathscr{H}(p)}$. Assume, for simplicity, that $\mathscr{V} = \ker d\varphi$, where $\varphi\colon M^4 \to N^3$ is a surjective submersion.

PROPOSITION 5.2 (a) *Let γ be a null geodesic on* (M^4, g).

(1) *If γ is horizontal then it is a geodesic of D_-.*
(2) *Suppose that γ is not horizontal at $x_0 = \gamma(0)$; let $p_{0-} \subseteq T_{x_0}M$ be the anti-self-dual plane containing $(d\gamma/dz)(0)$ and let $p_0 = \mathscr{H}(p_{0-})$. Denote by c the (local) horizontal curve such that $c(0) = x_0$ and $\varphi \circ c = \varphi \circ \gamma$.*

If p is the field of horizontal degenerate planes along c such that $p(0) = p_0$ then $(dp/dz)(0) \in \mathscr{F}_-^\mathscr{H}$.

(b) *Moreover, if D is a conformal partial connection on $(\mathscr{H}, g|_{\mathscr{H}})$ which has this property for all non-horizontal null geodesics on (M^4, g), and whose torsion T^D, with respect to $\mathscr{V}$, is such that $*T^D$ is self-adjoint and trace-free, then $D = D_-$. In particular, D_- is conformally invariant.*

Proof. (1) Suppose that γ is horizontal and let Y be its velocity vector field. Then $\mathcal{B}_0^{\mathscr{H}}(Y, Y) = 0$ and hence also $(*_{\mathscr{H},g} T_-)(Y) = 0$; equivalently, T_- restricted to $Y^\perp$ is zero, and thus $g(Y, T_-(X, Y)) = 0$ for any $X \in Y^\perp$.

Because γ is a null curve and D_- is conformal, we have that $D_{-\,Y} Y \in Y^\perp$ at each point of γ. Thus, to prove that γ is a geodesic of D_-, it is sufficient to prove that $g(D_{-\,Y} Y, X) = 0$ for any $X \in Y^\perp$. The proof follows from the fact that $D_{-\,Y} Y = \nabla_Y Y + 2\alpha_-(Y)Y$, where ∇ is the Levi-Civita connection of g.

(2) If γ is not horizontal then we can write $d\gamma/dz = X + \mathrm{i}\,U$, where X is horizontal, U is vertical and $g(X, X) = g(U, U) = a^2 \neq 0$. Extend X and U to local sections of $\mathscr{H}$ and $\mathscr{V}$, respectively. We can assume that X is a basic vector field and $a^{-1}U$ is positive on $(\mathscr{V}, g|_{\mathscr{V}})$. Then, along γ, we have

$$\nabla_{X+\mathrm{i}\,U}(X + \mathrm{i}\,U) = \nabla_X X - \nabla_U U + \mathrm{i}\,(\nabla_X U + \nabla_U X) = 0, \tag{5.2}$$

where ∇ is the Levi-Civita connection of (M^4, g).

We see from (5.2) that, for any horizontal vector field Y, we have

$$g(\nabla_X X, Y) - g(\nabla_U U, Y) + \mathrm{i}\,g(\nabla_X U, Y) + \mathrm{i}\,g(\nabla_U X, Y) = 0$$

along γ.

Now, assume that Y is null (and horizontal) and $g(X, Y) = 0$. Then, by using the fact that X is basic, the last relation implies that

$$g(\nabla_X X, Y) = a^2\big\{g(\mathrm{trace}(B^{\mathscr{V}}), Y) - \mathrm{i}\,I^{\mathscr{H}}(a^{-1}X, Y) + 2\mathrm{i}\mathcal{B}^{\mathscr{H}}(a^{-1}X, Y)\big\} \tag{5.3}$$

along γ, where we have assumed that $\{a^{-1}U\}$ is a positive local frame of $(\mathscr{V}, g|_{\mathscr{V}})$.

On the other hand, let D be a conformal partial connection on $(\mathscr{H}, g|_{\mathscr{H}})$, let T be its torsion with respect to $\mathscr{V}$, and let α be its Lee form with respect to $g|_{\mathscr{H}}$. Then we have

$$g(\nabla_X X, Y) = g(D_X X, Y) + \alpha(Y)g(X, X) + g(X, T(X, Y)). \tag{5.4}$$

Let $\{X_0, X_1, X_2, X_3\}$ be a positive local orthonormal frame on (M^4, g) such that $X_0 = a^{-1}U$ and $X_1 = a^{-1}X$. Then $\big\{X_1, X_2, X_3\big\}$ is a positive orthonormal local frame of $(\mathscr{H}, g|_{\mathscr{H}})$. Take $Y = X_2 + \mathrm{i}X_3$ and note that p_{0-} is spanned by $\{X_{x_0} + \mathrm{i}\,U_{x_0}, Y_{x_0}\}$ and p_0 by $\{X_{x_0}, Y_{x_0}\}$. Also, c is the integral curve of X through x_0 so that p_z is spanned by $\{X_{c(z)}, Y_{c(z)}\}$ at each z.

Then $(dp/dz)(0)$ is horizontal with respect to the connection induced by D on $\pi^{\mathscr{H}} : P^{\mathscr{H}} \to M$ if and only if $(D_X X)_{x_0} \in p_0$; equivalently $g\big((D_X X)_{x_0}, Y_{x_0}\big) = 0$. By (5.3) and (5.4), the last condition is equivalent to

$$\alpha(Y) - g(\operatorname{trace}(B^{\mathscr{V}}), Y) + \mathrm{i}\, I^{\mathscr{H}}(X_1, Y) = -g(X_1, T(X_1, Y)) + 2\mathrm{i}\mathcal{B}^{\mathscr{H}}(X_1, Y) \tag{5.5}$$

at x_0. Condition (5.5) is equivalent to

$$\big(\alpha - \operatorname{trace}(B^{\mathscr{V}})^{b} + *_{\mathscr{H},g} I^{\mathscr{H}}\big)(Y) = -g(X_1, T(X_1, Y)) + 2\mathrm{i}\mathcal{B}^{\mathscr{H}}(X_1, Y) \tag{5.6}$$

at x_0. It is easy to see that, if in (5.6) we replace α and T by α_- and T_-, respectively, then both sides are zero. Hence, $(dp/dz)(0) \in \mathscr{F}_-^{\mathscr{H}}$.

Suppose now that D satisfies condition (2) of the proposition. Then (5.6) must hold. Moreover, the relation obtained from (5.6) by replacing Y with $\widetilde{Y} = X_2 - \mathrm{i}X_3$ and X_1 by $-X_1$ (so that $\{X_0, -X_1, X_2, -X_3\}$ is positive) must also hold, namely

$$\big(\alpha - \operatorname{trace}(B^{\mathscr{V}})^{b} + *_{\mathscr{H},g} I^{\mathscr{H}}\big)(\widetilde{Y}) = -g(X_1, T(X_1, \widetilde{Y})) - 2\mathrm{i}\mathcal{B}^{\mathscr{H}}(X_1, \widetilde{Y}) \tag{5.7}$$

at x_0. By taking the sum of (5.6) and (5.7) we obtain

$$\big(\alpha - \operatorname{trace}(B^{\mathscr{V}})^{b} + *_{\mathscr{H},g} I^{\mathscr{H}}\big)(X_2) = -g(X_1, (*T)(X_3)) - 2\mathcal{B}^{\mathscr{H}}(X_1, X_3) \tag{5.8}$$

at x_0.

From the fact that the right-hand side of (5.8) does not depend on X_2 it follows that $*T + 2\mathcal{B}^{\mathscr{H}}$ is proportional to $g|_{\mathscr{H}}$. But $*T$ is self-adjoint and trace-free, hence $*T + 2\mathcal{B}_0^{\mathscr{H}} = 0$. Then, by (5.8), we also have $\alpha = \operatorname{trace}(B^{\mathscr{V}})^{b} - *_{\mathscr{H},g} I^{\mathscr{H}}$ and hence $D = D_-$.

The following theorem is a reformulation of results of [**9**] (see Definitions 4.3 and 5.1 for $(-)$twistorial map and D_-, respectively). It involves the dilation λ; recall that this is defined only up to sign, however, we use it only to define a one-form (5.9) which does not depend on that choice.

THEOREM 5.3 *Let (M^4, g) and (N^3, h) be orientable Riemannian manifolds and let $\varphi : (M^4, g) \to (N^3, h)$ be a surjective horizontally conformal submersion with connected nowhere degenerate fibres.*

Suppose that orientations on (M^4, g), $(\mathscr{V}, g|_{\mathscr{V}})$ and $(\mathscr{H}, g|_{\mathscr{H}})$ are chosen such that the isomorphism $(TM, g) = (\mathscr{V}, g|_{\mathscr{V}}) \oplus (\mathscr{H}, g|_{\mathscr{H}})$ is orientation preserving.

Then, the following assertions are equivalent:

(i) *there exists a Weyl connection D on N^3 with respect to which φ is $(-)$twistorial (with $P' = P_-$ the bundle of anti-self-dual planes on (M^4, g));*

(ii) *the one-form*

$$\operatorname{trace}(B^{\mathscr{V}})^{b} - \lambda^{-1}\mathrm{d}^{\mathscr{H}}\lambda - *_{\mathscr{H},g} I^{\mathscr{H}} \tag{5.9}$$

is basic, where λ *is the dilation of* φ, $\mathrm{d}^{\mathscr{H}} = \mathscr{H} \circ \mathrm{d}$ *and* $*_{\mathscr{H},g}$ *is the Hodge star-operator on* $(\mathscr{H}, g|_{\mathscr{H}})$*;*

(iii) *the two-form* $\mathrm{d}\big(\operatorname{trace}(B^{\mathscr{V}})^{b} + \frac{1}{3}\operatorname{trace}(B^{\mathscr{H}})^{b}\big)$ *is self-dual.*

Moreover, if any of the assertions (i), (ii) *or* (iii) *holds then*

(a) $\varphi^{*}(D)^{\mathscr{H}} = D_{-}$*; in particular, D is the* unique *Weyl connection that satisfies* (i)*;*

(b) M^4 *is self-dual if and only if D is Einstein–Weyl.*

If M^4 *is self-dual then, the following assertion can be added to the assertions* (i) *to* (iii).

(iv) *there exists a* (*unique*) *Einstein–Weyl connection D on* N^3 *such that, for any local map* $\psi : N^3 \to P^2$ *which is twistorial in the sense of Example* 4.1, *the map* $\psi \circ \varphi$: $M^4 \to P^2$ *is* $(-)$*twistorial* (*in the sense of Example* 4.4).

Proof. Let $\pi_{-} : P_{-} \to M$ be the bundle of anti-self-dual planes on M^4 and let $\Phi = d\varphi|_{P_{-}}$ be the map induced by φ from P_{-} to the bundle of degenerate planes on N^3. Assertion (i) is equivalent to the existence of a Weyl connection D on N^3 such that for any $x \in M^4$ and any null geodesic γ on M^4 with $\gamma(0) = x$, if we denote by p the field of anti-self-dual planes along γ such that $(d\gamma/dz)(z) \in p(z)$ for all z, then $d\Phi\big(dp/dz)(0)\big) \in \mathscr{F}$, where $\mathscr{F}$ is the twistor distribution induced by D on the bundle of degenerate planes on N^3 (Example 2.4). Thus, by Proposition 5.2, assertion (i) is equivalent to the existence of a Weyl connection D on N^3 such that $\varphi^{*}(D)^{\mathscr{H}} = D_{-}$. Now, applying the conformal change $\varphi^{*}(h)|_{\mathscr{H}} = \lambda^2\, g|_{\mathscr{H}}$ to (5.1) introduces the term $-\lambda^{-1}\mathrm{d}^{\mathscr{H}}\lambda$ so that the Lee form of D_{-} with respect to $\varphi^{*}(h)|_{\mathscr{H}}$ is equal to the one-form (5.9). Hence we have the equivalence of (i) and (ii). Furthermore, if (i) holds then assertion (a) also holds.

The equivalence (ii) $\Longleftrightarrow$ (iii) can be found in the proof of [**9**, Proposition 4.4].

The fact that, if M^4 is self-dual, then (i) $\Longleftrightarrow$ (iv) is obvious. Also, if M^4 is self-dual and φ is $(-)$twistorial, then from Theorem 2.5 it follows that D is Einstein–Weyl. Conversely, if φ is $(-)$twistorial and D is Einstein–Weyl, then by composing φ with local twistorial maps $(N^3, D) \to P^2$ we obtain sufficiently many local twistorial maps $M^4 \to P^2$ to show that any anti-self-dual plane on M^4 is tangent to some anti-self-dual surface in M^4. Thus, by Theorem 2.7, M^4 is self-dual.

REMARK 5.4

(1) Calderbank [**9**] calls a horizontally conformal submersion, between *real*-analytic Riemannian manifolds, satisfying condition (iii) of Theorem 5.3 *self-dual*. We have thus interpreted self-dual submersions as those which are $(-)$twistorial.

(2) By using the null-tetrad formalism (see [**27**]), the equivalence (i) $\Longleftrightarrow$ (ii) of Theorem 5.3 can also be obtained after a straightforward but tedious computation.

In the following, let $\varphi : (M^4, c_M) \to (N^3, c_N)$ be a surjective horizontally conformal submersion with nowhere degenerate fibres from a four-dimensional oriented conformal manifold to a three-dimensional conformal manifold. Let L be the line bundle over N^3 associated with the bundle of conformal frames on N^3 through the morphism of Lie groups $\rho : CO(3, \mathbb{K}) \to \mathbb{K} \setminus \{0\}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), characterized by $a \in CO(3, \mathbb{K})$ if and only if $a^t\, a = \rho(a)^2 I$ (in the smooth category, L can be defined as the line bundle associated to the frame bundle of N^3 through the morphism of Lie groups $GL(3, \mathbb{R}) \to (0, \infty)\mathbb{R} \setminus \{0\}$, $a \mapsto (\det(a))^{1/3}$; cf. [**9**]). Note that L can be defined on any odd-dimensional conformal manifold (N, c_N) and that its local sections correspond to oriented local representatives of c_N. Let E be some line bundle over N and let $\widetilde{E} = \varphi^{*}(E)$.

To any pair (s, ∇), where s is a section of L^* over some open subset U of N and ∇ is a connection on $E|_U$, there can be associated a connection $\widetilde{\nabla}$ of $\widetilde{E}|_{\varphi^{-1}(U)}$ as follows: assume, initially, that s is nowhere zero, let h be the oriented local representative of c_N over U corresponding to it and let g be the oriented local representative of c_M over $\varphi^{-1}(U)$ such that $\varphi : (\varphi^{-1}(U), g) \to (U, h)$ is a Riemannian submersion. Denote, as usual, $\mathscr{V} = \ker d\varphi$, $\mathscr{H} = \mathscr{V}^\perp$, and let $\omega \in \Gamma(\varphi^{-1}(U), \mathscr{V}^*)$ be the induced orientation on $\mathscr{V}|_{\varphi^{-1}(U)}$ such that the isomorphism $\big(T(\varphi^{-1}(U)), g\big) = \big(\mathscr{V}|_{\varphi^{-1}(U)}, g|_{\mathscr{V}}\big) \oplus \big(\mathscr{H}|_{\varphi^{-1}(U)}, \varphi^*(h)\big)$ is orientation preserving. If A is the local connection form of ∇ with respect to some nowhere zero local section $\sigma \in \Gamma(U, E)$, then we set the local connection form of $\widetilde{\nabla}$ corresponding to $\varphi^*(\sigma)$ equal to

$$\widetilde{A} = -\omega + \varphi^*(A).$$

We shall call $\widetilde{\nabla}$ the *pull-back, by φ, of the pair* (s, ∇). Note that, if u is a nowhere zero function locally defined on N, the pull-back by φ of (us, ∇) is given by $-u\,\omega + \varphi^*(A)$. Hence, we can extend the definition of the pull-back by φ of any pair (s, ∇) where the local section s of L^* may have zeros.

The monopole equation of [**20**] can be written as follows:

$$(d - \alpha)u = *F, \tag{5.10}$$

where u is a function on a three-dimensional oriented Riemannian manifold (N^3, h), α is a one-form on N^3, F is a two-form on N^3 and $*$ is the Hodge star-operator of (N^3, h). By interpreting α as the Lee form with respect to h of a Weyl connection on $(N^3, [h])$, and F as the curvature form of some connection on some line bundle over (N^3, h), the equation (5.10) can be written as follows.

DEFINITION 5.5 [**10**, **20**] Let D be a Weyl connection on (N^3, c_N). A *monopole on* (N^3, c, D) is a pair (s, ∇), where s is a section of L^* and ∇ is a connection on a line bundle E over N^3 such that

$$*_N Ds = F. \tag{5.11}$$

Here F is the curvature form of ∇. A monopole is called *non-trivial* if $s \neq 0$.

REMARK 5.6 Let (N^3, c) be a three-dimensional conformal manifold endowed with a Weyl connection D.

(1) It is well known (see [**10**]) that if (N^3, c, D) is Einstein–Weyl then, at least locally, there exist non-trivial monopoles on (N^3, c, D).

Conversely, let (N^3, c) be a three-dimensional conformal manifold and let E be a line bundle over N endowed with a connection ∇. Also, let s be a nowhere zero (local) section of L^* and let h be the oriented (local) representative of c corresponding to s. Define a one-form α such that $- *_h \alpha = \varphi^*(F)$, where F is the curvature form of ∇. Then (s, ∇) is a monopole on (N^3, c, D), where D is the Weyl connection on (N^3, c) whose Lee form with respect to h is α.

(2) Let h be a local representative of c over some open subset U of N and let α be the Lee form of D with respect to h. Then, it is well known (and easy to prove) that, on small enough open subsets U, monopoles on (N^3, c, D) correspond to solutions of the equation $\Delta u = \mathrm{d}^*(u\alpha)$, where Δ is the Laplacian and d^* is the codifferential on (U, h). Hence, if (N^3, c, D) is (the complexification of) a real-analytic conformal manifold endowed with a real-analytic Weyl connection then, at least locally, there exist non-trivial monopoles on (N^3, c, D).

Assertion (i) of the following proposition is essentially in [**9**, **10**].

COROLLARY 5.7 *Let $\varphi : (M^4, c_M) \to (N^3, c_N)$ be a surjective horizontally conformal submersion with nowhere degenerate fibres and let D be a Weyl connection on (N^3, c_N).*

(i) *If $\varphi : (M^4, c_M) \to (N^3, c_N, D)$ is a $(-)$twistorial map, then it pulls back monopoles on (N^3, c_N, D) to self-dual connections on (M^4, c_M).*
(ii) *Conversely, suppose that there exists a non-trivial monopole on (N^3, c_N, D) which is pulled back by φ to a self-dual connection on (M^4, c_M). Then, $\varphi : (M^4, c_M) \to (N^3, c_N, D)$ is $(-)$twistorial.*

Proof. Let (s, ∇) be a non-trivial monopole. We may assume that s is nowhere zero so that it corresponds to an oriented local representative h of c_N. Let α be the Lee form of D with respect to h. Then (5.11) is equivalent to

$$-*_h \alpha = F. \tag{5.12}$$

If A is a local connection of ∇, then the corresponding local connection form of $\widetilde{\nabla}$ is $\widetilde{A} = -\omega + \varphi^*(A)$, where $\omega \in \Gamma(\mathscr{V}^*)$ is as above. Now, by using the fact that

$$d\omega = -\mathrm{trace}(B^{\mathscr{V}})^b \wedge \omega + I^{\mathscr{H}} \tag{5.13}$$

it is easy to prove that $d\widetilde{A}$ is self-dual if and only if

$$-*_h \alpha_- = \varphi^*(dA), \tag{5.14}$$

where $\alpha_- = \mathrm{trace}(B^{\mathscr{V}})^b - *_{\mathscr{H},g} I^{\mathscr{H}}$ is the Lee form of D_- with respect to h.

From (5.12) and (5.14) it follows that $\alpha_- = \alpha$ is basic. The result follows from Theorem 5.3.

Let M^m be a manifold of dimension $m \geq 3$ endowed with a three-dimensional distribution $\mathscr{H}$. Suppose that $\mathscr{H}$ is endowed with a conformal structure c and a conformal partial connection D. In a similar fashion to the above, we consider a line bundle L over M whose nowhere zero local sections correspond to oriented local representatives of c.

Suppose that there exists a foliation $\mathscr{V}$ which is complementary to $\mathscr{H}$ and assume, for simplicity, that there exists a submersion $\varphi : M^m \to N^3$ whose fibres are leaves of $\mathscr{V}$.

Then Definition 5.5 can be easily generalized by defining a *monopole on* $(\mathscr{H}, c, D)$ (*with respect to* φ) to be a pair (s, ∇), where s is a section of L^* and ∇ is a connection on a line bundle E over N^3 such that $*_{\mathscr{H}} Ds = \varphi^*(F)$, and where F is the curvature form of ∇. (One could also consider line bundles on M endowed with partial connections over $\mathscr{H}$ but then the resulting equation would depend on a local section of E.)

Furthermore, if (M^4, c) is a four-dimensional oriented conformal manifold then a construction similar to the above associates to any pair (s, ∇) a connection $\widetilde{\nabla}$ on $\varphi^*(E)$. We then have the following simple fact (which may be known) whose proof is similar to the proof of Corollary 5.7.

PROPOSITION 5.8 *Let $\varphi : (M^4, c) \to N^3$ be a surjective submersion with nowhere degenerate fibres from an oriented four-dimensional conformal manifold. Suppose that D is a conformal partial connection on $(\mathscr{H}, c|_{\mathscr{H}})$ and let E be a line bundle over N^3. Let s be a section of L^* and ∇ a connection on E.*

Then any two of the following assertions imply the third:

(i) (s, ∇) *is a non-trivial monopole on* $(\mathscr{H}, c|_{\mathscr{H}}, D)$*;*

(ii) $\widetilde{\nabla}$ *is self-dual;*
(iii) $D = D_-$.

Next, we show how harmonic morphisms fit into the above discussion.

PROPOSITION 5.9 *Let* $\mathscr{V}$ *be (the complexification) of a one-dimensional (nowhere degenerate) conformal foliation on a four-dimensional oriented conformal manifold* (M^4, c)*; let* $\mathscr{H} = \mathscr{V}^\perp$.
Then the following assertions are equivalent:

(i) *there exist local representatives g of c with respect to which* $\mathscr{V}$ *is locally defined by harmonic morphisms;*
(ii) *there exist non-trivial monopoles* (s, ∇) *on* $(\mathscr{H}, c|_{\mathscr{H}}, D_-)$ *for which* $\widetilde{\nabla}$ *is flat.*

Proof. By Theorem 1.8, assertion (i) is equivalent to the following:
(i′) locally, there exist representatives g of c with respect to which $\mathscr{V}$ is geodesic.
Also, from (5.13), we easily obtain that a pair (s, ∇) is as in assertion (ii) if and only if the following hold:

(a) $\mathscr{V}$ has geodesic fibres with respect to the local representative g of c whose horizontal component corresponds to s;
(b) if θ is the local volume form induced by g on $\mathscr{V}$ then $d\theta$ is equal to the pull-back by φ of the curvature form of ∇.

We have thus shown that (ii) $\Rightarrow$ (i).
Conversely, suppose that (i) holds and let s be a nowhere zero local section of L^* such that, with respect to the corresponding oriented local representative g of c, the foliation $\mathscr{V}$ is geodesic. Then $I^{\mathscr{H}} = d\theta$, where θ is the local volume form induced by g on $\mathscr{V}$. Moreover, because $\mathscr{V}$ is geodesic with respect to g, we have, locally, $d\theta = \varphi^*(dA)$ for some one-form A on N^3. Furthermore, by Definition 5.1, we have $- *_{\mathscr{H},g}\, \alpha_- = I^{\mathscr{H}}$. Hence $- *_{\mathscr{H},g}\, \alpha_- = \varphi^*(dA)$; equivalently, the pair formed by s and the connection determined by A, on the trivial line bundle over N^3, is a monopole on $(\mathscr{H}, c|_{\mathscr{H}}, D_-)$. The proposition is proved.

The following result gives another characterization for twistorial harmonic morphisms between orientable Riemannian manifolds of dimension four and three, respectively.

COROLLARY 5.10 *Let* (M^4, g) *and* (N^3, h) *be orientable Riemannian manifolds. Let* $\varphi : (M^4, g) \to (N^3, h)$ *be a surjective submersive harmonic morphism with connected nowhere degenerate fibres. Suppose that orientations on* (M^4, g), (N^3, h), $(\mathscr{V}, g|_{\mathscr{V}})$ *and* $(\mathscr{H}, g|_{\mathscr{H}})$ *are chosen such that the isomorphisms* $(TM, g) = (\mathscr{V}, g|_{\mathscr{V}}) \oplus (\mathscr{H}|_{\mathscr{H}})$ *and* $(\mathscr{H}, g|_{\mathscr{H}}) = (\varphi^*(TN), \varphi^*(h))$ *are orientation preserving.*
Then, the following assertions are equivalent.

(i) *There exists a Weyl connection D on* N^3 *with respect to which* φ *is* $(-)$*twistorial.*
(ii) *There exists a basic one-form* α *on* M^4 *such that*

$$(d^{\mathscr{H}} - \alpha)(\lambda^{-2}) = *_{\mathscr{H}}\Omega, \tag{5.15}$$

where λ *is a dilation of* φ, $*_{\mathscr{H}}$ *is the Hodge star operator on* $(\mathscr{H}, \varphi^*(h))$ *and* Ω *is the curvature form of the horizontal distribution (that is, in the notation of Theorem* 1.8, $\Omega = d\theta$).

Moreover, if (i) *or* (ii) *holds then the Weyl connection D of assertion* (i) *is unique and the one-form α of assertion* (ii) *is the pull-back of the Lee form of D with respect to h.*

Proof. By comparing the fundamental equation (1.1) with (5.9) and by using the fact that $I^{\mathscr{H}} = \lambda\,\Omega$, it follows that a one-form α on M^4 satisfies (5.15) if and only if it is the Lee form of D_- with respect to $\varphi^*(h)$. The proof follows from Theorem 5.3.

From [**31**, Corollary 1.9] we obtain the following result on maps from Einstein manifolds (cf. Proposition 4.2 and Theorem 4.5).

THEOREM 5.11 *Let M^4 be an orientable four-dimensional Einstein manifold and let φ: $M^4 \to N^3$ be the complexification of a submersive harmonic morphism.*

Then, there exists a Weyl connection on N^3 with respect to which, with a suitable choice of orientations, φ is $(-)$twistorial.

Another consequence of Theorem 5.3 is that a one-dimensional conformal foliation with nowhere degenerate leaves on an oriented four-dimensional Riemannnian manifold is both $(+)$twistorial and $(-)$twistorial (Remark 3.4(5)) if and only if it is locally generated by conformal vector fields [**9**] (the 'if' part follows also from [**20**]). In particular, (the complexification of) any one-dimensional homothetic foliation locally defined by harmonic morphisms on an oriented four-dimensional Riemannian manifold is both $(+)$twistorial and $(-)$twistorial.

With the same notation as above and in section 1, we have the following consequences of Theorem 5.3 (in which we do not claim that (i) $\Longleftrightarrow$ (iii) is new).

COROLLARY 5.12 *Let $\mathscr{V}$ be a one-dimensional conformal foliation with integrable orthogonal complement on an oriented four-dimensional conformal manifold (M^4, c).*

Then the following assertions are equivalent:

(i) *$\mathscr{V}$ is $(\pm)$twistorial;*
(ii) *$\mathscr{V}$ produces harmonic morphisms with respect to any local representative of c for which $\mathscr{V}$ has geodesic leaves;*
(iii) *locally, there exist representatives of c with respect to which $\mathscr{V}$ is defined by totally geodesic Riemannian submersions.*

We end this section with a result on $(\pm)$twistorial maps from conformally flat four-manifolds. Recall (see [**35**]) that, any conformally flat four-manifold can be locally embedded in the four-dimensional non-singular quadric Q_4.

PROPOSITION 5.13 *Let M^4 be an oriented conformally flat four-manifold and let $\varphi : M^4 \to N^3$ be a horizontally conformal submersion with nowhere degenerate fibres. Then, the following assertions are equivalent:*

(i) *φ is $(-)$twistorial*;
(ii) *φ is $(+)$twistorial.*

Furthermore, if φ is $(\pm)$twistorial and M^4 is locally embedded in Q_4, then the twistor space $Z(N)$ of the induced Einstein–Weyl connection on N^3 is, locally, a surface in Q_4 such that the space of horizontal null geodesics of φ is equal to the space of null geodesics on Q_4 which pass through $Z(N) \subseteq Q_4$.

6. The classification of twistorial harmonic morphisms with one-dimensional fibres from self-dual four-manifolds

In the last two sections we shall work in the real-analytic category.

First, we list the four types of (−)twistorial harmonic morphisms with one-dimensional fibres which can be defined on a four-dimensional Riemannian manifold. Later on we shall see how they come from equation (5.15).

Type 1 (*'Killing type'* [**8**]). Harmonic morphisms $\varphi : M^4 \to N^3$ whose fibres are locally generated by nowhere zero Killing vector fields (see section 1).

Type 2 (*'warped-product type'* [**2**]). Horizontally homothetic submersions with geodesic fibres orthogonal to an umbilical foliation by hypersurfaces (see Remark 1.9(1)).

Type 3 (*cf.* [**30**, **31**, **33**]). $\varphi : (M^4, g) \to (N^3, h)$ is a real-analytic map which is, locally, the canonical projection $\mathbb{R} \times N^3 \to N^3$, $(\rho, x) \mapsto x$, and

$$g = \rho\, h + \rho^{-1}(d\rho + A)^2, \tag{6.1}$$

where A is a one-form on N^3 which satisfies the Beltrami fields equation $dA = - * A$.

Type 4 (*cf.* [**9**]). (N^3, h) is endowed with a Weyl connection D, φ: $(M^4, g) \to (N^3, h)$ is a real-analytic map which is, locally, the canonical projection $\mathbb{R} \times N^3 \to N^3$, $(\rho, x) \mapsto x$, and

$$g = (e^{\rho} + c)\, h + \frac{1}{e^{\rho} + c}\,(d\rho - \alpha)^2, \tag{6.2}$$

where the Lee form α of D with respect to h satisfies the equation

$$d\alpha - c * \alpha + *dc = 0$$

on N^3 with $c : N^3 \to \mathbb{R}$ a function (if $e^{\rho} + c < 0$, then we replace g with $-g$).

REMARK 6.1

(1) Note that maps of types 3 and 4 are always harmonic morphisms by Theorem 1.8.

(2) Let $\varphi : (M^{n+1}, g) \to (N^n, h)$ $(n \geq 1)$ be a submersive harmonic morphism with one-dimensional fibres. Then, the components of the fibres of φ form a homothetic foliation if and only if either φ is of type 1 or, up to a conformal change of (M^{n+1}, g) with factor constant along the fibres, φ is of type 2.

(3) If $\varphi : (M^4, g) \to (N^3, h)$ is a harmonic morphism of type 3 then ρ and $\varphi^*(A)$ are globally defined on M^4. Indeed, if we denote by λ the dilation of φ then $\rho = \lambda^{-2}$ and $\varphi^*(A) = -d^{\mathscr{H}}(\lambda^{-2})$.

(4) If φ is a harmonic morphism of type 4 with $d\alpha = 0$ then the horizontal distribution is integrable and, after a conformal change with basic factor, φ is of type 2. If $\alpha \neq 0$, then in general, ρ and $c \circ \varphi$ are globally defined on M^4 (this follows from the fact that, up to signs, we have $\theta = d\rho - \alpha$ and $\lambda^{-2} = e^{\rho} + c \circ \varphi$, where λ is the dilation of φ); in particular, $e^{\rho} + c \circ \varphi$ has constant sign on M^4.

PROPOSITION 6.2 *Let* $\varphi : (M^4, g) \to (N^3, h, D)$ *be a harmonic morphism of type* 4 *with* $c \neq 0$. *Then, outside the zero set of* c, *after a conformal change with factor constant along the fibres and a suitable* *change of coordinates we may assume that $c = \pm 1$—that is, φ is, locally, the canonical projection $\mathbb{R} \times N^3 \to N^3$, $(\rho, x) \mapsto x$, and*

$$g = (e^{\rho} \pm 1)\, h + \frac{1}{e^{\rho} \pm 1}\, (d\rho - \alpha)^2, \tag{6.3}$$

where the Lee form α of D with respect to h satisfies the Beltrami fields equation

$$d\alpha = \pm * \alpha \tag{6.4}$$

on N^3.

Proof. We shall show that $\widetilde{g} = |c|\, g$ is as in (6.3), (6.4) after the change of coordinates $\widetilde{\rho} = \rho - \log |c|$.

Assume for simplicity that $c > 0$ and let $\widetilde{h} = c^2 h$. Then the dilation $\widetilde{\lambda}$, fundamental vector field $\widetilde{V}$ and connection form $\widetilde{\theta}$ of the harmonic morphism $\varphi : (M^4, \widetilde{g}) \to (N^3, \widetilde{h})$ are as follows: $\widetilde{\lambda}^2 = c\lambda^2$, $\widetilde{V} = V$ and $\widetilde{\theta} = \theta$. Also the Lee form of D with respect to $\widetilde{h}$ is $\widetilde{\alpha} = \alpha - \mathrm{d}\log c$.

It is easy to see that

$$d\alpha - c * \alpha + *dc = 0$$

written with respect to $\widetilde{h}$ becomes $*d\widetilde{\alpha} = c\, \widetilde{\alpha}$. But, if we denote by $\widetilde{*}$ the Hodge-star operator on $(N^3, \widetilde{h})$, then $\widetilde{*}|_{\Lambda^2} = c^{-1} *|_{\Lambda^2}$ and hence

$$\widetilde{*} d\widetilde{\alpha} = \widetilde{\alpha}. \tag{6.5}$$

Now it is easy to see that, with respect to $\widetilde{\rho}$, $\widetilde{h}$ and $\widetilde{\alpha}$, the metric $\widetilde{g}$ has the required form.

Now we are able to state the main result of this section.

THEOREM 6.3 *Let M^4 and N^3 be real-analytic Riemannian manifolds of dimensions four and three, respectively. Suppose that N^3 is endowed with a Weyl connection D.*

Then $\varphi : M^4 \to (N^3, D)$ is a $(-)$twistorial harmonic morphism if and only if, up to conformal changes of the metric on the domain and codomain, φ is of type 1, 2, 3 *or* 4*; further, the conformal factor on M^4 can be taken to be constant along the fibres of φ. If φ is of type* 2*, then D is the Levi-Civita connection of some metric in the conformal class of N^3, whilst if φ is of type* 3*, then D is the Levi-Civita connection of N^3.*

Proof. By Corollary 5.10, equation (5.15) is satisfied. If $V(\lambda^{-2}) = 0$ then φ is of type 1. From [**32**, Corollary 1.5] it follows that $V(\lambda^{-2})$ is real-analytic, and from now on we shall assume that $V(\lambda^{-2})$ is nowhere zero on some open set. Moreover, by replacing, if necessary, V with $-V$, we can assume that $V(\lambda^{-2})$ is positive at each point.

Let X be a basic vector field. Note that the left-hand side of (5.15) must be basic and hence, using $[V, X] = 0$ we obtain $X(V(\lambda^{-2})) - V(\lambda^{-2})\alpha(X) = 0$. Thus

$$\alpha = d^{\mathscr{H}}(\log V(\lambda^{-2})) \ . \tag{6.6}$$

This implies that $X(\log V(\lambda^{-2}))$ is basic. Hence

$$X(V(\log V(\lambda^{-2}))) = V(X(\log V(\lambda^{-2}))) = 0.$$

It follows that, if $V(\log V(\lambda^{-2}))$ is non-constant then $\mathscr{H}$ is integrable; equivalently $\Omega = 0$ on an open set, and hence, by real-analyticity, on M^4. This, together with (5.15), gives $\alpha = -2d^{\mathscr{H}}(\log\lambda)$; hence $d^{\mathscr{H}}(\log\lambda)$ is basic, that is, $V(X(\log\lambda)) = 0$ for any basic vector field X. Because $\mathscr{H}$ is integrable this implies that $\mathscr{V}$ is homothetic, and hence after a conformal change with basic factor we get that φ is of type 2 (see Remark 6.1(2)). Moreover, as α is exact, D is the Levi-Civita connection of some Riemannian metric in the conformal class of N^3.

From now on we shall suppose that $V(\log V(\lambda^{-2})) = a$ for some constant $a \in \mathbb{R}$. If $a = 0$ this implies that $V(\lambda^{-2})$ is basic. We can write $V = \partial/\partial\rho$ for some function ρ which is zero on a chosen section of φ. Hence $\lambda^{-2} = b\rho + c$ for some basic functions b and c. After suitable conformal changes on N^3 and M^4 we have $\lambda^{-2} = \rho + c/b$, that is, $V(\lambda^{-2}) = 1$. By (6.6), this implies that $\alpha = 0$ and hence D is the Levi-Civita connection of N^3. Moreover, we can locally write $\theta = d(\lambda^{-2}) + \varphi^*(A)$ for some one-form A on N^3. Hence $d^{\mathscr{H}}(\lambda^{-2}) = -\varphi^*(A)$ and $\Omega = d\theta = \varphi^*(dA)$ which together with (5.15) gives $-A = *dA$. Thus we have proved that, if $a = 0$ then, up to a conformal change with basic factor, φ is of type 3.

It remains to consider the case when $V(\log V(\lambda^{-2})) = a$ for some non-zero constant a. Then, if we set $\rho = \log V(\lambda^{-2})$, we can assume that $V = a\,\partial/\partial\rho$. As $V(\lambda^{-2}) = e^{\rho}$ we have that $a\lambda^{-2} = e^{\rho} + c$ for some basic function c. Hence

$$\lambda^{-2} = \frac{1}{a}\,(e^{\rho} + c). \tag{6.7}$$

From $d^{\mathscr{V}}\rho = a\theta$ and (6.6) we get $d\rho = a\theta + \alpha$; equivalently

$$\theta = \frac{1}{a}\,\big(d\rho - \alpha\big) \ . \tag{6.8}$$

Now, we can write

$$\begin{aligned} a(d^{\mathscr{H}} - \alpha)(\lambda^{-2}) &= a\big(d^{\mathscr{H}} - d^{\mathscr{H}}\rho\big)\left(\frac{1}{a}\,(e^{\rho} + c)\right) \\ &= e^{\rho}d^{\mathscr{H}}\rho + d^{\mathscr{H}}c - e^{\rho}d^{\mathscr{H}}\rho - cd^{\mathscr{H}}\rho \\ &= d^{\mathscr{H}}c - cd^{\mathscr{H}}\rho = dc - cd^{\mathscr{H}}\rho = dc - c\alpha. \end{aligned} \tag{6.9}$$

On the other hand, from (6.8) it follows that $a\,\Omega = ad\theta = -d\alpha$ which, together with (5.15) and (6.9), gives $dc - c\alpha = -*d\alpha$. Thus we have proved that, if φ is not of type 1, 2 or 3 then

$$d\alpha - c*\alpha + *dc = 0 \tag{6.10}$$

and, locally, the metric of M^4 can be written in the form

$$g = \frac{1}{a}\left\{ (e^{\rho} + c)\, h + \frac{1}{e^{\rho} + c}\,(d\rho - \alpha)^2 \right\} \tag{6.11}$$

where, obviously, we can assume that $a = \pm 1$ and that c is non-zero; since, if $c = 0$, then after a suitable conformal change of basic factor, φ is of type 2.

The following result is an immediate consequence of Theorem 5.3 and Theorem 6.3.

COROLLARY 6.4 *Let M^4 be a self-dual manifold with a real-analytic metric and let $\varphi : M^4 \to (N^3, D)$ be a $(-)$twistorial harmonic morphism.*

Then D is Einstein–Weyl and, up to conformal changes of the metrics, φ is of type 1, 2, 3 *or* 4*; further, the conformal factor on M^4 can be taken to be constant along the fibres of φ. If φ is of type* 2*, then both M^4 and N^3 are conformally flat, whilst if φ is of type* 3*, then N^3 has constant curvature.*

7. Constructions of self-dual metrics

In this section we show that harmonic morphisms are related to known constructions of self-dual metrics.

Firstly, we recall the following construction of Jones and Tod [**20**] (see [**26**], cf. [**15**, **17**]).

THEOREM 7.1 *Let $(N^3, [h], D)$ be an Einstein–Weyl three-manifold and let (M^4, N^3, S^1) be a (local) principal bundle endowed with a (local) principal connection $\mathscr{H} \subseteq TM$. Define a Riemannian metric g on M^4 by*

$$g = v\, \varphi^*(h) + v^{-1}\, \theta^2, \tag{7.1}$$

where $\varphi : M^4 \to N^3$ is the projection of the principal bundle (M^4, N^3, S^1), $v = u \circ \varphi$ for some positive smooth function u on N^3 and θ is the connection form of $\mathscr{H}$.

Then (M^4, g) is self-dual (respectively, anti-self-dual) if the following S^1-monopole equation holds on N^3:

$$(d - \alpha)\, u = *F \quad (\textit{respectively}, (d - \alpha)u = - * F\,), \tag{7.2}$$

where α is the Lee form of D with respect to h and $F \in \Gamma(\Lambda^2(T^*N))$ is the curvature form of $\mathscr{H}$.

REMARK 7.2 The construction of Theorem 7.1 clearly gives harmonic morphisms of type 1 and, if $\mathscr{H}$ is flat, of type 2, up to a conformal change with basic factor.

THEOREM 7.3 (Type 3, cf. [**9**, **33**]) *Let (N^3, h) be a constant curvature three-manifold and let A be a one-form on N^3. Define a Riemannian metric on $(0, \infty) \times N^3$ by*

$$g = \rho\, h + \rho^{-1}(d\rho + A)^2 \quad (\rho \in (0, \infty)\,). \tag{7.3}$$

Then g is self-dual (respectively, anti-self-dual) if the following Beltrami fields equation holds on N^3:

$$dA = - * A \quad (\textit{respectively}, dA = *A). \tag{7.4}$$

THEOREM 7.4 (Type 4 [**9**]) *Let* $(N^3, [h], D)$ *be an Einstein–Weyl three-manifold. Define a Riemannian metric on* $(0, \infty) \times N^3$ *by*

$$g = (e^\rho \pm 1)\, h + \frac{1}{e^\rho \pm 1}\,(d\rho - \alpha)^2 \quad (\rho \in (0, \infty)\,), \tag{7.5}$$

where α *is the Lee form of* D *with respect to* h.

Then g *is self-dual* (*respectively, anti-self-dual*) *if the following Beltrami fields equation holds on* N^3*:*

$$d\alpha = \pm * \alpha \quad (\textit{respectively, } d\alpha = \mp * \alpha\,)\,. \tag{7.6}$$

REMARK 7.5

(1) The construction of Theorem 7.1 gives Einstein metrics if $\alpha = 0$ and h is flat, in which case g is Ricci-flat self-dual [**15**, **17**] (see [**33**]).
(2) The construction of Theorem 7.3 gives Einstein metrics if and only if (N^3, h) has constant sectional curvature equal to $1/4$, in which case g is Ricci-flat self-dual. The construction of Theorem 7.4 gives Einstein metrics if and only if $\alpha = 0$ and so h has constant curvature, in which case g also has constant curvature. These facts follow from [**33**, Theorem 1.5] and are also true if we make a conformal change of g with factor constant along the fibres of φ.
(3) Let A be a solution of the Beltrami fields equation (7.4) on a three-dimensional Riemannian manifold (N^3, h) with constant curvature. Then the components of A with respect to an orthonormal basis of left invariant one-forms are eigenfunctions of the Laplace–Beltrami operator of (N^3, h) (cf. [**33**]; note that the corresponding eigenvalues are imaginary if (N^3, h) has negative sectional curvature).
(4) Let $(N^3, [h], D)$ be an Einstein–Weyl space for which there exists a surjective submersion $\pi : Z(N) \to \mathbb{C}P^1$ whose fibres are transversal to the twistor lines. Then, it is well known that, for any $x_1,\ x_2 \in \mathbb{C}P^1 = S^2$, the angle formed by any leaf of the foliation corresponding to $\pi^{-1}(x_1)$ and any leaf of the foliation corresponding to $\pi^{-1}(x_2)$ is equal to $\mathrm{dist}_{S^2}(x_1, x_2)$ (see [**10**, **14**]).

   For example, if D is the Levi-Civita connection of a Riemannian manifold (N^3, h) with constant non-negative sectional curvature then $(N^3, [h], D)$ has this property (see [**6**]). It follows that, if the codomain of a harmonic morphism of type 3 has non-negative constant sectional curvature then its domain is hyper-Hermitian.
(5) Calderbank [**9**] gives the type 4 construction with the extra condition on $(N^3, [h], D)$ that it is an Einstein–Weyl space for which there exists a surjective submersion $\pi : Z(N) \to \mathbb{C}P^1$ whose fibres are transversal to the twistor lines. Then the construction gives hyper-Hermitian metrics which, after a suitable conformal change, are Einstein with non-zero scalar curvature.

Acknowledgements

The authors gratefully acknowledge that this work was done under EPSRC grant number GR/N27897. We are grateful to D.M.J. Calderbank for useful comments on a preliminary version of this paper.

Note added in proof

For other treatments of twistorial structures and harmonic morphisms, see E. Loubeau, R. Pantilie, Harmonic morphisms between Weyl spaces and twistorial maps, http://maths2.univ-brest.fr/~loubeau/weyl.ps, preprint, Brest University, 2004, and R. Pantilie, Harmonic morphisms

between Weyl spaces, to appear in *Proceedings of the Seventh International Workshop on Differential Geomentry and Its Applications*, Deva, Romania, 5–11 September 2005.

References

1. M. F. Atiyah, N. J. Hitchin and I. M. Singer, Self-duality in four-dimensional Riemannian geometry, *Proc. R. Soc.* A **362** (1978) 425–461.
2. P. Baird and J. Eells, A conservation law for harmonic maps, *Geometry Symposium, Utrecht* 1980, Lecture Notes in Mathematics 894, Springer, Berlin, 1981, 1–25.
3. P. Baird and J. C. Wood, Bernstein theorems for harmonic morphisms from $\mathbf{R}^3$ and S^3, *Math. Ann.* **280** (1988) 579–603.
4. P. Baird and J. C. Wood, Harmonic morphisms and conformal foliations by geodesics of three-dimensional space forms, *J. Austral. Math. Soc.* (A) **51** (1991) 118–153.
5. P. Baird and J. C. Wood, Harmonic morphisms, Seifert fibre spaces and conformal foliations, *Proc. London Math. Soc.* **64** (1992) 170–196.
6. P. Baird and J. C. Wood, *Harmonic Morphisms Between Riemannian Manifolds*, London Mathematical Society Monographs (N.S.) 29, Oxford University Press, Oxford, 2003.
7. R. Bott, Lectures on characteristic classes and foliations. Notes by Lawrence Conlon, with two appendices by J. Stasheff, *Lectures on Algebraic and Differential Topology* (Second Latin American School in Math., Mexico City, 1971), Lecture Notes in Mathematics 279, Springer, Berlin, 1972, 1–94.
8. R. L. Bryant, Harmonic morphisms with fibers of dimension one, *Comm. Anal. Geom.* **8** (2000) 219–265.
9. D. M. J. Calderbank, *Selfdual Einstein metrics and conformal submersions*, preprint, Edinburgh University, 2000; ArXiv math DG/0001041.
10. D. M. J. Calderbank and H. Pedersen, Selfdual spaces with complex structures, Einstein–Weyl geometry and geodesics, *Ann. Inst. Fourier* (*Grenoble*) **50** (2000) 921–963.
11. J. Eells and S. Salamon, Twistorial construction of harmonic maps of surfaces into four-manifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **12** (1985) 589–640.
12. B. Fuglede, Harmonic morphisms between Riemannian manifolds, *Ann. Inst. Fourier* (*Grenoble*) **28** (1978) 107–144.
13. P. Gauduchon, Structures de Weyl–Einstein, espaces de twisteurs et variétés de type $S^1 \times S^3$, *J. Reine Angew. Math.* **469** (1995) 1–50.
14. P. Gauduchon and K. P. Tod, Hyper-Hermitian metrics with symmetry, *J. Geom. Phys.* **25** (1998) 291–304.
15. G. W. Gibbons and S. W. Hawking, Gravitational multi-instantons, *Phys. Lett.* B **78** (1978) 430–432.
16. S. Gudmundsson, *The bibliography of harmonic morphisms*, http://www.maths.lth.se/matematiklu/personal/sigma/harmonic/bibliography.html
17. S. W. Hawking, Gravitational instantons, *Phys. Lett.* A **60** (1977) 81–83.
18. N. J. Hitchin, *Complex Manifolds and Einstein's Equations, Twistor Geometry and Non-linear Systems* (Primorsko, 1980), Lecture Notes in Mathematics 970, Springer, Berlin, 1982, 73–99.
19. T. Ishihara, A mapping of Riemannian manifolds which preserves harmonic functions, *J. Math. Kyoto Univ.* **19** (1979) 215–229.

20. P. E. Jones and K. P. Tod, Minitwistor spaces and Einstein–Weyl spaces, *Classical Quantum Gravity* **2** (1985) 565–577.
21. P. C. Kendall and C. Plumpton, *Magnetohydrodynamics, With Hydrodynamics*, Pergamon Press, Oxford; Macmillan, 1964.
22. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, I, II, Wiley Classics Library (reprint of the 1963, 1969 original), Wiley-Interscience, New York, 1996.
23. K. Kodaira, A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds, *Ann. of Math.* **75** (1962) 146–162.
24. C. R. LeBrun, *Spaces of complex geodesics and related structures*, Ph.D. Thesis, Oxford, 1980.
25. C. R. LeBrun, Spaces of complex null geodesics in complex-Riemannian geometry, *Trans. Amer. Math. Soc.* **278** (1983) 209–231.
26. C. R. LeBrun, Explicit self-dual metrics on $\mathbb{C}P^2\#\cdots\#\mathbb{C}P^2$, *J. Differential Geom.* **34** (1991) 223–253.
27. L. J. Mason and N. M. J. Woodhouse, *Integrability, Self-Duality, and Twistor Theory*, London Mathematical Society Monographs (N.S.) 15, Oxford University Press, Oxford, 1996.
28. L. J. Mason, L. P. Hughston, P. Z. Kobak and K. Pulverer (editors), *Further Advances in Twistor Theory,* Vol. III*: Curved Twistor Spaces*, Chapman & Hall/CRC Research Notes in Mathematics 424, Chapman & Hall/CRC, Boca Raton, 2001.
29. R. Pantilie, Harmonic morphisms with one-dimensional fibres, *Internat. J. Math.* **10** (1999) 457–501.
30. R. Pantilie, *Submersive harmonic maps and morphisms*, Ph.D. Thesis, University of Leeds, 2000.
31. R. Pantilie, Harmonic morphisms with 1-dimensional fibres on 4-dimensional Einstein manifolds, *Comm. Anal. Geom.* **10** (2002) 779–814.
32. R. Pantilie and J. C. Wood, Harmonic morphisms with one-dimensional fibres on Einstein manifolds, *Trans. Amer. Math. Soc.* **354** (2002) 4229–4243.
33. R. Pantilie and J. C. Wood, A new construction of Einstein self-dual manifolds, *Asian J. Math.* **6** (2002) 337–348.
34. R. Penrose, Nonlinear gravitons and curved twistor theory, *Gen. Relativity Gravitation* **7** (1976) 31–52.
35. R. S. Ward and R. O. Wells, *Twistor Geometry and Field Theory*, Cambridge University Press, Cambridge, 1990.
36. J. C. Wood, Harmonic morphisms, foliations and Gauss maps, *Complex Differential Geometry and Nonlinear Differential Equations*, Contemporary Mathematics 49, American Mathematical, Society, Providence, 1986, 145–183.
37. J. C. Wood, Harmonic morphisms and Hermitian structures on Einstein 4-manifolds, *Internat. J. Math.* **3** (1992) 415–439.