Introduction to harmonic morphisms between Weyl spaces and twistorial maps

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This book is dedicated to our families

Contents

Preface

The aim of this book is to give a self-contained introduction to the study of the interplay between harmonic morphisms and twistorial maps. It is primarily intended for graduate students but, also, for undergraduate students who are interested in Differential Geometry, and for mathematicians willing to be initiated in harmonic morphisms, Twistor Theory or Weyl Geometry. To understand it, some familiarity with principal connections and Riemannian manifolds would be desirable.

The importance of harmonic maps and minimal submanifolds is well known (see, for example, [12], [11], [5]).

Harmonic morphisms are maps which pull back (local) harmonic functions to harmonic functions. In Differential Geometry, a natural setting for harmonic morphisms was provided by Riemannian manifolds (see [5] and, also, [26] for a fairly up to date account of the subject). However, in lower dimensions, for example, the classification results are involving, at least implicitly, twistorial methods. Consequently, maps having natural twistorial interpretations (like the horizontally conformal submersions with geodesic fibres on three-dimensional Einstein–Weyl spaces) suggested the necessity to broaden the geometric setting for the study of harmonic morphisms.

On the other hand, the Twistor Theory itself was lacking a systematization which takes into account the similarities between its numerous interesting constructions.

The resolution of these two general problems was initiated in [28] and [21] , and then continued and developed in [24] , [6] , [22] , [17] , [23] .

Given the above stated aim, the book is organized as follows.

In Chapter 1 , we collect a few definitions and results (most notably, Proposition 1.2.1 and Theorem 1.3.1), subsequently invoked several times throughout the book.

The first properties of harmonic morphisms between Weyl spaces are given in Chapter 2 . Besides the necessary generalizations to Weyl spaces of basic facts on harmonic morphisms between Riemannian manifolds (such as, Theorem 2.4.3 and Proposition 2.5.3), there we include two natural constructions of harmonic morphisms between Weyl spaces (Proposition 2.3.2 and Corollary 2.5.5).

Chapter 3 is an introduction to Twistor Theory. There we discuss, in detail, the twistorial structures of three-dimensional Einstein–Weyl spaces and of anti-self-dual manifolds, and the corresponding twistorial maps between them.

In Chapter 4 , we focus on the interplay between harmonic morphisms and twistorial maps. From there, we mention here only the basic Examples 4.2.3 and 4.2.4 of twistorial harmonic morphisms, based on the monopole and the Beltrami fields equations, respectively.

Finally, the Appendix presents the decomposition of the curvature form of a Weyl space (and two consequences of it: Propositions A.1.9 and A.1.11), results of which are used in Chapters 3 and 4 .

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CHAPTER 1

Preliminaries

In this chapter, we recall the definition and the basic properties of (co-)CR structures and we prove two results (Proposition 1.2.1 and Theorem 1.3.1) which will be useful later on.

1.1. CR and co-CR structures

Unless otherwise stated, all the manifolds are assumed smooth and connected, and all the maps are assumed smooth.

Let J be an almost complex structure on a manifold M; that is, J is a $(1,1)$-tensor field on M such that $J^2 = -\mathrm{Id}_{TM}$. Then J is determined by its eigenbundle $\mathcal{C}$ corresponding to $-\mathrm{i}$; also, $T^{\mathbb{C}}M = \mathcal{C} \oplus \overline{\mathcal{C}}$. Moreover, J is integrable (that is, J is given by local systems of complex coordinates on M) if and only if the space of sections of $\mathcal{C}$ is closed under the (Lie) bracket (the Newlander-Nirenberg Theorem; see [20] and the references therein).

More generally, we have the following definitions (see [3], [23]).

Definition 1.1.1. Let $\mathcal{C}$ be a complex vector subbundle of $T^{\mathbb{C}}M$.

We say that

(1) $\mathcal{C}$ is an *almost CR structure* if $\mathcal{C} \cap \overline{\mathcal{C}} = \{0\}$.

(2) $\mathcal{C}$ is an *almost co-CR structure* if $\mathcal{C} + \overline{\mathcal{C}} = T^{\mathbb{C}}M$.

Let $\mathcal{C}$ be an almost (co-)CR structure on M. Then $\mathcal{C}$ is *integrable* if its space of sections is closed under the bracket. A *(co-)CR structure* is an integrable almost (co-)CR structure. A *(co-)CR manifold* is a manifold endowed with a (co-)CR structure.

A map $\varphi : (M, \mathcal{C}) \to (N, \mathcal{D})$ between almost (co-)CR manifolds is *holomorphic* if $\mathrm{d}\varphi(\mathcal{C}) \subseteq \mathcal{D}$.

The 'CR' stands both for 'Cauchy-Riemann' and 'complex-real'.

In this book, we are mainly interested in co-CR structures.

Let N be a submanifold of an (almost) co-CR manifold $(M, \mathcal{C})$. If $\mathcal{C}|_N \cap T^{\mathbb{C}}N$ is an (almost) co-CR structure on N, we say that $\mathcal{C}$ *induces an (almost) co-CR structure* on N.

Example 1.1.2. Let $\varphi : M \to (N, J)$ be a submersion onto a complex manifold. Then

$$\mathcal{C} = (\mathrm{d}\varphi)^{-1}(T^{0,1}N)$$

is a co-CR structure on M; if, further, φ has connected fibres then $\mathcal{C}$ is called a *simple* co-CR structure.

Note that, $\varphi : (M, \mathcal{C}) \to (N, J)$ is holomorphic.

Next, we prove that any co-CR structure is, locally, given as in Example 1.1.2 .

Proposition 1.1.3. *Let $(M, \mathcal{C})$ be a co-CR manifold. Then any point of M has an open neighbourhood U for which there exists a submersion $\varphi : U \to (N, J)$ onto a complex manifold such that $\mathcal{C}|_U = (\mathrm{d}\varphi)^{-1}(T^{0,1}N)$. Moreover, any such submersion is unique, up to the composition with a holomorphic diffeomorphism from its codomain onto some complex manifold.*

Before proving Proposition 1.1.3 , note that we could have replaced the condition

$$\mathcal{C}|_U = (\mathrm{d}\varphi)^{-1}(T^{0,1}N)$$

with the condition φ is holomorphic and $(\ker \mathrm{d}\varphi)^{\mathbb{C}} = (\mathcal{C} \cap \overline{\mathcal{C}})|_U$.

Further, recall that if $\varphi : M \to N$ is a submersion then the vectors tangent to $\ker \mathrm{d}\varphi$ are called *vertical vectors*. Also, a vector field X on M is *projectable* if there exists a vector field Y on N such that $\mathrm{d}\varphi(X_x) = Y_{\varphi(x)}$, for any $x \in M$. Obviously, any vertical vector field is projectable.

We also need the following lemma.

Lemma 1.1.4. *Let $\varphi : M \to N$ be a surjective submersion (with connected fibres) and let X be a vector field on M.*

Then X is projectable if and only if $[V, X]$ is vertical, for any vertical vector field V.

Proof. If f is a function on N then $\mathrm{d}\varphi(X_x)(f) = X_x(f \circ \varphi)$, for any $x \in M$. Therefore X is projectable if and only if $\mathrm{d}\varphi(X)$ is constant along the fibres of φ.

As the fibres of φ are connected, the latter condition is equivalent to the fact that, for any vertical vector field V, we have that $\mathrm{d}\varphi(\mathrm{d}\psi_t(X))$ doesn't depend of t, where ψ_t is the local flow of V. By [20, Corollary 1.10], this holds if and only if

$$\mathrm{d}\varphi\big(\mathrm{d}\psi_s\big([V, X]\big)\big) = 0 ,$$

for any s; equivalently, $\mathrm{d}\varphi\big([V, X]\big) = 0$.

The proof is complete. $\square$

Proof of Proposition 1.1.3. If $\mathcal{C}$ is integrable then $\mathcal{C} \cap \overline{\mathcal{C}}$ is (the tangent bundle of) a foliation on M. Hence, each point of M has an open neighbourhood U for which there exists a surjective submersion $\varphi : U \to N$ such that $(\ker \mathrm{d}\varphi)^{\mathbb{C}} =$

$(\mathcal{C}\cap\overline{\mathcal{C}})|_U$.

Now, Lemma 1.1.4 implies that $\mathrm{d}\varphi(\mathcal{C})$ defines a complex structure on N, whilst the unicity of φ (with given U) is obvious. $\square$

Note that, for CR structures the result corresponding to Proposition 1.1.3 does not hold. Nevertheless, the following result is obvious.

Proposition 1.1.5. *Let $(M,\mathcal{C})$ be an almost CR manifold and let $\mathscr{H}$ be the distribution on M such that $\mathscr{H}^{\mathbb{C}}=\mathcal{C}\oplus\overline{\mathcal{C}}$.*

If $\mathscr{H}$ is integrable then the following assertions are equivalent:

(i) *$\mathcal{C}$ is integrable.*

(ii) *Each leaf N of $\mathscr{H}$, endowed with $\mathcal{C}|_N$, is a complex manifold.*

1.2. Other useful facts

A *complex vector bundle* is a (real) vector bundle E endowed with a section J of its endomorphism bundle such that $J^2=-\mathrm{Id}_E$. Therefore the fibres and the space of sections $\Gamma(E)$ of E are complex vector spaces. Obviously, also, $\mathrm{Hom}(TM,E)$ is a complex vector bundle, where M is the base of E. Moreover, there exists a canonical isomorphism of complex vector bundles $\mathrm{Hom}(TM,E)=\mathrm{Hom}_{\mathbb{C}}(T^{\mathbb{C}}M,E)$.

Let $(P, M, \mathrm{GL}(r, \mathbb{C}))$ be the (principal) bundle of complex frames on E, where $\mathrm{rank}_{\mathbb{C}} E = r$. We shall denote by $\pi : P \to M$ the projection and by $\mathfrak{gl}(r, \mathbb{C})$ the Lie algebra of $\mathrm{GL}(r, \mathbb{C})$ (that is, $\mathfrak{gl}(r, \mathbb{C})$ is the Lie algebra determined of $r \times r$ complex matrices).

Recall that, there exists a canonical isomorphism of (complex) vector bundles $\ker \mathrm{d}\pi = P \times \mathfrak{gl}(r, \mathbb{C})$, under which the constant sections of $P \times \mathfrak{gl}(r, \mathbb{C})$ correspond to the fundamental vector fields on P (see [20]).

A *complex connection* on a complex vector bundle E is a connection ∇ on E such that $\nabla J = 0$, where J is the linear complex structure of E; equivalently, $\nabla : \Gamma(E) \to \Gamma(\mathrm{Hom}(TM, E))$ is complex linear. Furthermore, this holds if and only if for any complex local frame $u = (u_j)_{j=1,\dots,r}$ on E, over some open set U of M, there exists a $\mathfrak{gl}(r, \mathbb{C})$-valued one-form A on U characterised by $\nabla u_j = A^k_j \otimes u_k$, for any $j = 1, \dots, r$, where we have used the Einstein summation convention. The one-form A is the *local connection form* of ∇, with respect to u.

It follows that ∇ corresponds to a principal connection $\mathscr{H}$ on P. Recall that, $\mathscr{H}$ is a $\mathrm{GL}(r, \mathbb{C})$-invariant distribution on P which is complementary to $\ker \mathrm{d}\pi$. Also, by composing the 'vertical' projection from TP onto $\ker \mathrm{d}\pi$ with the projection $\ker \mathrm{d}\pi \to \mathfrak{gl}(r, \mathbb{C})$ we obtain the *connection form* ω of $\mathscr{H}$. Then, under the isomorphism of principal bundles $P|_U = U \times \mathrm{GL}(r, \mathbb{C})$, corresponding to a complex local frame u, we have

$$\omega = a^{-1}\,\mathrm{d}a + a^{-1}Aa$$

and $u^*(\omega) = A$, where

$$a : U \times \mathrm{GL}(r, \mathbb{C}) \to \mathrm{GL}(r, \mathbb{C})$$

is the projection and we have denoted by the same letter the local connection form A and its pull back to $U \times \mathrm{GL}(r, \mathbb{C})$ (see [20]).

Another characteristic condition for a connection ∇ on E to be complex is that, with respect to the isomorphism $E^{\mathbb{C}} = E \oplus \overline{E}$, we have $\nabla^{\mathbb{C}} = \nabla \oplus \overline{\nabla}$.

Let H be the closed complex Lie subgroup of $\mathrm{GL}(r, \mathbb{C})$ formed by those $a \in \mathrm{GL}(r, \mathbb{C})$ preserving the space spanned by the first k vectors of the canonical basis of $\mathbb{C}^r$. Then P/H is the Grassmann bundle of complex vector subspaces of complex dimension k of E; in particular, the typical fibre of P/H is the complex Grassmannian $\mathrm{Gr}_k(r, \mathbb{C})$ of k-dimensional complex vector subspaces of $\mathbb{C}^r$.

Let ∇ be a complex connection on E. We shall denote by $\mathscr{K}\ \big(\subseteq T(P/H)\big)$ the connection induced by ∇ on P/H. Note that, as P/H is a bundle whose typical fibre is a complex manifold on which the structural group acts by holomorphic diffeomorphisms, $\ker \mathrm{d}\rho$ is a complex vector bundle. Therefore

$$(\ker \mathrm{d}\rho)^{\mathbb{C}} = (\ker \mathrm{d}\rho)^{1,0} \oplus (\ker \mathrm{d}\rho)^{0,1} \ ,$$

where $\rho : P/H \to M$ is the projection.

Proposition 1.2.1. *Let F be a complex vector subbundle of E, $\operatorname{rank}_{\mathbb{C}} F = k$, and denote by q the corresponding section of P/H. Let $X \in T^{\mathbb{C}}_{x_0}M$, for some $x_0 \in M$, and denote $q_0 = q(x_0)$.*

The following assertions are equivalent:

(i) $(\mathrm{d}q)^{\mathbb{C}}(X) \in \mathscr{K}^{\mathbb{C}}_{q_0} \oplus (\ker \mathrm{d}\rho)^{0,1}_{q_0}$.

(ii) $\nabla_X v \in F_{x_0}$ *for any section v of F.*

Proof. Let $\sigma : P \to P/H$ be the projection. Then $\sigma^{-1}(q(M))$ is the reduction of P to H formed of the complex frames on E whose first k vectors are complex frames on F.

Let u be a local section of $\sigma^{-1}(q(M))$ defined on some open neighbourhood U of x_0. We, obviously, have $q|_U = \sigma \circ u$.

Therefore assertion (i) is equivalent to

$$\omega^{\mathbb{C}}(\mathrm{d}u(X)) \in \mathfrak{h} \oplus \overline{\mathfrak{gl}(r,\mathbb{C})} \ ,$$

where $\mathfrak{h}$ is the Lie algebra of H, and we have used the isomorphism of Lie algebras

$$\mathfrak{gl}(r,\mathbb{C})^{\mathbb{C}} = \mathfrak{gl}(r,\mathbb{C}) \oplus \overline{\mathfrak{gl}(r,\mathbb{C})} \, .$$

As, with respect to this isomorphism, we have $\omega^{\mathbb{C}} = \omega \oplus \overline{\omega}$, assertion (i) is, further, equivalent to

$$\omega(\mathrm{d}u(X)) \in \mathfrak{h} \, .$$

By using the fact that $u^*(\omega)$ is the local connection form of ∇, with respect to u, we obtain that assertion (i) holds if and only if $\nabla_X u_j \in F_{x_0}$, for any $j = 1, \dots, k$. $\square$

1.3. An integrability result

Let G be a complex Lie subgroup of $\mathrm{GL}(m,\mathbb{C})$ which acts transitively on a complex submanifold F of the complex Grassmannian $\mathrm{Gr}_k(m,\mathbb{C})$, $k \leq m$; in particular, $F = G/H$, where H is the isotropy group of G at some $q_0 \in F$.

Let M be a manifold, $\dim M = m$, for which the bundle of complex frames on $T^{\mathbb{C}}M$ is endowed with a reduction P to G. Suppose, further, that $T^{\mathbb{C}}M$ is endowed with a complex connection, compatible with G; that is, ∇ corresponds to a principal connection on P.

Note that, the usual formula for the torsion of a connection on a manifold can be applied to obtain the *torsion* T of ∇, which is a section of $T^{\mathbb{C}}M \otimes \Lambda^2(T^*M)$. Alternatively, T is the exterior covariant derivative of the $T^{\mathbb{C}}M$-valued one-form on M given by the inclusion $TM \to T^{\mathbb{C}}M$.

Now, note that, P/H is a subbundle of the Grassmannian bundle of k-dimensional complex vector spaces on M, and ∇ induces a connection on it.

Let $\mathcal{B}$ be the complex subbundle of $T^{\mathbb{C}}(P/H)$ whose fibre at each $q \in P/H$ is the horizontal lift of $q \subseteq T^{\mathbb{C}}_{\rho(q)}M$, with respect to ∇, where $\rho : P/H \to M$ is the projection. Define $\mathcal{C} = \mathcal{B} \oplus (\ker \mathrm{d}\rho)^{0,1}$.

Theorem 1.3.1. *The following assertions are equivalent:*

(i) $\mathcal{C}$ *is integrable.*

(ii) $R(\Lambda^2 q)q \subseteq q$ *and* $T(\Lambda^2 q) \subseteq q$, *for any* $q \in P/H$, *where* R *and* T *are the curvature and torsion of* ∇, *respectively.*

Proof. Let $\mathcal{C}_0 = (\mathrm{d}\sigma)^{-1}(\mathcal{C})$, where $\sigma : P \to P/H$ is the projection, defined by $\sigma(u) = u(q_0)$, for any $u \in P$. As, also, $(\mathrm{d}\sigma)(\mathcal{C}_0) = \mathcal{C}$, we have that $\mathcal{C}$ is integrable if and only if $\mathcal{C}_0$ is integrable.

Denote by $\mathcal{B}_0$ the complex vector subbundle of $T^{\mathbb{C}}P$ whose fibre at each u is the horizontal lift, with respect to ∇, of $u(q_0) \subseteq T^{\mathbb{C}}_{\pi(u)}M$, where π :

$P \to M$ is the projection. We, obviously, have $(\mathrm{d}\sigma)(\mathcal{B}_0) = \mathcal{B}$.

Note that, $\ker \mathrm{d}\pi = P \times \mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G (see [20]). Also, we have an isomorphism of Lie algebras $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$, with respect to which, as usual, the decomposition of any $A \in \mathfrak{g}$ is given by

$$A \otimes 1 = \tfrac{1}{2}(A \otimes 1 - \mathrm{i}A \otimes \mathrm{i}) + \tfrac{1}{2}(A \otimes 1 + \mathrm{i}A \otimes \mathrm{i}) \, .$$

Then $\mathcal{C}_0 = \mathcal{B}_0 \oplus P \times \mathfrak{h} \oplus P \times \overline{\mathfrak{g}}$, where $\mathfrak{h}$ is the Lie algebra of H.

We shall denote by the same letters the elements of $\mathfrak{g}^{\mathbb{C}}$ and the corresponding fundamental (complex) vector fields on P,

For each $\xi \in \mathbb{C}^m$ let $B(\xi)$ be the (complex) vector field on P which, at each $u \in P$, is the horizontal lift, with respect to ∇, of $u(\xi)$ (cf. [20]). Thus, $\mathcal{B}_0$ is generated by $\{B(\xi) \,|\, \xi \in q_0\}$. Furthermore, similarly to [20, Proposition III.2.3] , we have $[A \otimes 1, B(\xi)] = B(A\xi)$, for any $A \in \mathfrak{g}$ and $\xi \in \mathbb{C}^m$.

Consequently,

$$[\mathrm{i}A \otimes \mathrm{i}, B(\xi)] = \mathrm{i}\,[\mathrm{i}A \otimes 1, B(\xi)] = \mathrm{i}B(\mathrm{i}A\xi) = -B(A\xi) \, ,$$

and, hence, under $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \oplus \overline{\mathfrak{g}}$, we have $[A, B(\xi)] = B(A\xi)$ and $[\,\overline{A}, B(\xi)] = 0$, for any $A \in \mathfrak{g}$ and $\xi \in \mathbb{C}^m$.

Also, by using the fact that H is embedded in G, we obtain

$$\mathfrak{h} = \{ A \in \mathfrak{g} \,|\, A(q_0) \subseteq q_0 \} \,. \tag{1.3.1}$$

Therefore $\mathcal{C}_0$ is integrable if and only if $[B(\xi), B(\eta)]$ is a section of $\mathcal{C}_0$, for any $\xi, \eta \in q_0$.

Let ω be the connection form of (the principal connection corresponding to) ∇ and let θ be the $\mathbb{C}^m$-valued one-form on P given by $\theta(X) = u^{-1}\big(\mathrm{d}\pi(X)\big)$, for any $X \in TP$. Alternatively, θ is the tensorial form corresponding to the $T^{\mathbb{C}}M$-valued one-form on M given by the inclusion $TM \to T^{\mathbb{C}}M$ (cf. [20]).

Let Ω and Θ be the tensorial forms on P corresponding to R and T, respectively; that is,

$$\begin{aligned} &\Omega(X,Y) = u^{-1} \circ R(\mathrm{d}\pi(X), \mathrm{d}\pi(Y)) \circ u \,, \\ &\Theta(X,Y) = u^{-1}\big(T(\mathrm{d}\pi(X), \mathrm{d}\pi(Y))\big) \,, \end{aligned} \tag{1.3.2}$$

for any $u \in P$ and $X, Y \in T_uP$.

By the (complex) Cartan's structural equations, we have

$$\begin{aligned} \Omega &= \mathrm{d}\omega + \omega \wedge \omega \,, \\ \Theta &= \mathrm{d}\theta + \omega \wedge \theta \,. \end{aligned} \tag{1.3.3}$$

Let $\xi, \eta \in q_0$. The vertical and horizontal components of $[B(\xi), B(\eta)]$ are given by $\omega^{\mathbb{C}}\big([B(\xi), B(\eta)]\big)$

and $\theta\big([B(\xi), B(\eta)]\big)$, respectively. Hence, by using (1.3.2), (1.3.3) and the fact that $\omega^{\mathbb{C}} = \omega \oplus \overline{\omega}$, we obtain that, at each $u \in P$, we have $[B(\xi), B(\eta)]_u \in \mathcal{C}_0$ if and only if the following two relations hold: $u^{-1} \circ R(u\xi, u\eta) \circ u \in \mathfrak{h}$ and $u^{-1}(T(u\xi, u\eta)) \in q_0$.

Finally, from (1.3.1) it follows that $\mathcal{C}$ is integrable if and only if, for any $\xi, \eta \in q_0$ and $u \in P$, we have $R(u\xi, u\eta)(u(q_0)) \subseteq u(q_0)$ and $T(u\xi, u\eta) \in u(q_0)$. The proof is complete. $\square$

From the proof of Theorem 1.3.1 we obtain the following fact which will be used later on.

Remark 1.3.2. If ∇ is torsion free then the bracket of any two sections of $\mathcal{C}$ is a section of $\mathcal{C} \oplus (\ker\rho)^{1,0}$.

CHAPTER 2

Harmonic morphisms between Weyl spaces

In this chapter, we present the basic properties of Weyl spaces (see [10] , [8]) and of the harmonic morphisms between them [21] (cf. [5]).

2.1. Conformal manifolds

Let M^m be a (smooth connected) manifold, with $\dim M = m$. For simplicity we shall assume that M^m is oriented. Then there exists an oriented line bundle L such that $L^m = \Lambda^m TM$, where $L^m = \underbrace{L \otimes \cdots \otimes L}_{m \text{ factors}}$. We call L *the line bundle of* M^m.

Two Riemannian metrics g and h on M^m are *conformally equivalent* if there exists a positive function f on M^m such that $h = f^2 g$. A *conformal structure* on M^m is an equivalence class c of conformally equivalent Riemannian metrics on M^m ; if g is a representative of c we write $c = [g]$. A *conformal manifold* is a manifold endowed with a conformal structure.

There are other ways, equally useful, to describe a conformal structure. Two of these are based on the following simple observation.

Proposition 2.1.1. *Let (M^m, c) be a conformal manifold. Then there exists a bijective correspondence which to any positive section s of L associates the representative g_s of c such that s^{-m} is the volume form of g_s, where $s^{-m} = \underbrace{s^* \otimes \cdots \otimes s^*}_{m \text{ factors}}$, with s^* the section of L^* dual to s. Moreover, the section $s^2 \otimes g_s$ of $L^2 \otimes \big(\odot^2 T^*M\big)$ depends only of c, where $\odot^2$ denotes the second symmetric power.*

Proof. Let s be a positive section of L and let ω be the volume form of some representative g of c. Then there exists a positive function f on M such that $s^{-m} = f^m\omega$. Consequently, the volume form of $f^2 g$ is s^{-m}; equivalently, $g_s = f^2 g$.

From $(fs)^{-m} = f^{-m}s^{-m} = f^{-m}(f^m\omega) = \omega$ we obtain $g = g_{fs}$, which shows that the correspondence is bijective.

Finally, as $g_{fs} = g = f^{-2}g_s$, we have

$$(fs)^2 \otimes g_{fs} = s^2 \otimes g_s ,$$

which proves the last assertion. $\square$

Proposition 2.1.1 shows that a conformal structure c on M^m may be identified with $s^2 \otimes g_s$, where

s is any positive section of the line bundle L of M^m. In other words, we think of c as an L^2-valued Riemannian metric on M.

Furthermore, c also corresponds to a Riemannian metric on $L^* \otimes TM$, which we equally denote by c, induced by the canonical isomorphism of vector bundles $L^2 \otimes \left(\odot^2 T^*M\right) = \odot^2(L^* \otimes TM)^*$ (here, we use that $\operatorname{rank} L = 1$). Note that,

$$c(s^* \otimes X, s^* \otimes Y) = g_s(X,Y)\ ,$$

for any positive $s \in L$ and any $X, Y \in TM$.

Also, as c is a Riemannian metric on $L^* \otimes TM$, it corresponds to a musical isomorphism

$$(\cdot)^\sharp : L \otimes T^*M \to L^* \otimes TM\ .$$

Proposition 2.1.2. *Let s be a positive section of L. Then for any one-form α on M^m we have*

$$(s \otimes \alpha)^\sharp = s^* \otimes \alpha^{\sharp_s}, \tag{2.1.1}$$

where $(\cdot)^{\sharp_s} : T^*M \to TM$ is the musical isomorphism corresponding to g_s.

Proof. Let X be a vector field on M. Then $(s \otimes \alpha)(s^* \otimes X) = \alpha(X)$. On the other hand, we have $c(s^* \otimes \alpha^{\sharp_s}, s^* \otimes X) = g_s(\alpha^{\sharp_s}, X)$. Hence,

$$(s \otimes \alpha)(s^* \otimes X) = c(s^* \otimes \alpha^{\sharp_s}, s^* \otimes X)$$

and the proof is complete. $\square$

By tensorising $(\cdot)^\sharp$ with, for example, Id_{L^*} we obtain an isomorphism of vector bundles $T^*M \to (L^*)^2 \otimes TM$ which will, also, be denoted by $(\cdot)^\sharp$. If s is a positive section of L and α a one-form on M^m then, from (2.1.1), we obtain

$$\alpha^\sharp = s^{-2} \otimes \alpha^{\sharp_s}. \tag{2.1.2}$$

Similarly, we have, the isomorphism $(\cdot)^\flat : TM \to L^2 \otimes T^*M$, given by $X \mapsto c(X, \cdot)$, for any $X \in TM$.

We shall, also, use one more equivalent description for conformal structures. For this, say that a frame on a conformal manifold (M^m, c) is *conformal* if it is orthonormal with respect to some representative of c. Then the set of conformal frames on (M^m, c) is (the total space of) a principal bundle on M^m whose structural group is the (real) conformal group $\mathrm{CO}(m, \mathbb{R})$, formed of the $m \times m$ real matrices a for which there exists a positive number λ such that $a^T a = \lambda^2 I_m$, where I_m is the identity matrix; equivalently, $\mathrm{CO}(m, \mathbb{R})$ is formed of the conformal linear isomorphisms of $\mathbb{R}^m$, endowed with the conformal linear structure given by its canonical metric (cf. the paragraph before Definition 2.4.2, below).

Conversely, any reduction of the frame bundle of M^m to $\mathrm{CO}(m, \mathbb{R})$ is the bundle of conformal frames of a unique conformal structure on M^m.

2.2. Weyl spaces

We start this section with the following simple lemma.

Lemma 2.2.1. *Let* (M^m, c) *be a conformal manifold and let* D *be a connection on* M*; we shall, also, denote by* D *the connection induced by it on* L *. Let* s *be a positive section of* L *and let* α *be a one-form on* M*.*

Then any two of the following assertions imply the third:

(i) $Dc = 0$.

(ii) $Ds = \alpha \otimes s$ *(equivalently,* α *is the local connection form, with respect to* s *, of the connection induced by* D *on* L*).*

(iii) $Dg_s = -2\,\alpha \otimes g_s$.

Proof. This follows quickly by using the fact that $c = s^2 \otimes g_s$. □

A connection D on a conformal manifold (M^m, c) is *conformal* if $Dc = 0$; equivalently, for a representative g of c there exists a one-form α on M^m such that $Dg = -2\,\alpha \otimes g$.

Any conformal connection on (M^m, c) corresponds to a principal connection on the bundle of conformal frames of (M^m, c) .

Definition 2.2.2. A *Weyl connection* on a conformal manifold is a torsion free conformal connection.

A *Weyl space* is a conformal manifold endowed with a Weyl connection.

It is a fundamental fact in Weyl Geometry that if M^m is endowed with a conformal structure c then the Weyl connections on (M^m, c) are determined by the connections they induce on L .

Proposition 2.2.3. *Let D be a Weyl connection on a conformal manifold (M^m, c) ; we shall denote by the same letter D the connection induced by D on L and its powers.*

For any vector fields X, Y, Z on M, we have

$$
\begin{aligned}
2\,c(D_X Y, Z) &= D_X(c(Y,Z)) + D_Y(c(Z,X)) \\
&\quad - D_Z(c(X,Y)) - c(X,[Y,Z]) \\
&\quad + c(Y,[Z,X]) + c(Z,[X,Y]) \;.
\end{aligned}
\tag{2.2.1}
$$

Proof. Let X, Y, Z on M be vector fields on M.

As $Dc = 0$, we have

$$D_X(c(Y,Z)) = c(D_X Y, Z) + c(Y, D_X Z) \;. \tag{2.2.2}$$

By permuting X, Y, Z, in (2.2.2), we also have

(2.2.3)
$$
\begin{aligned}
D_Y(c(Z,X)) &= c(D_Y Z, X) + c(Z, D_Y X)\ ,\\
-D_Z(c(X,Y)) &= -c(D_Z X, Y) - c(X, D_Z Y)\ .
\end{aligned}
$$

If we take the sum of (2.2.2) and (2.2.3), and we use the fact that D is torsion free we obtain

(2.2.4)
$$
\begin{aligned}
D_X(c(Y,Z)) &+ D_Y(c(Z,X)) - D_Z(c(X,Y))\\
&= c(D_X Y, Z) + c(Z, D_Y X)\\
&\quad + c(Y,[X,Z]) + c(X,[Y,Z])\ .
\end{aligned}
$$

But $D_Y X = D_X Y - [X,Y]$ and, hence, (2.2.4) gives

(2.2.5)
$$
\begin{aligned}
D_X(c(Y,Z)) &+ D_Y(c(Z,X)) - D_Z(c(X,Y))\\
&= 2\, c(D_X Y, Z) - c(Z,[X,Y])\\
&\quad + c(Y,[X,Z]) + c(X,[Y,Z])\ ,
\end{aligned}
$$

which, obviously, is equivalent to (2.2.1). □

Let $(M^m,[g])$ be a conformal manifold. Recall (see [20]) that the Levi-Civita connection of g is the unique torsion free connecton ∇ on M^m such that $\nabla g = 0$. Obviously, ∇ is a Weyl connection on $(M^m,[g])$. In fact, Lemma 2.2.1 implies that ∇

is the Weyl connection on $(M^m,[g])$ with respect to which $\nabla s=0$, where s is the positive section of L such that $g=g_s$ (in particular, the connection on L corresponding to ∇ is flat). Hence, as $c=s^2\otimes g$, we have $\nabla_X(c(Y,Z))=\nabla_X(g(Y,Z)s^2)=X(g(Y,Z))s^2$, for any vector fields X, Y, Z on M. Therefore (2.2.1) applied to ∇ gives the classical *Koszul formula*:

(2.2.6)
$$\begin{aligned}2\,g(\nabla_XY,Z)&=X(g(Y,Z))+Y(g(Z,X))\\&\quad-Z(g(X,Y))-g(X,[Y,Z])\\&\quad+g(Y,[Z,X])+g(Z,[X,Y])\,,\end{aligned}$$

for any vector fields X, Y, Z on M.

Two connections on a line bundle differ by a one form on the base manifold of the line bundle. It is useful to know what is the relation between two Weyl connections on a conformal manifold (M^m,c) in terms of the difference of the corresponding connections on L.

Corollary 2.2.4. *Let D' and D'' be two Weyl connections on (M^m,c) for which the corresponding connections on L satisfy $D''=D'+\alpha$, for some one-form α on M.*

Then, for any vector fields X and Y on M, we *have*

$$D''_X Y = D'_X Y + \alpha(X)Y + \alpha(Y)X - c(X,Y)\alpha^\sharp \,.$$

Proof. As $D''(s^2) = D'(s^2) + 2\alpha \otimes s^2$, for any section s of L, the proof is an immediate consequence of (2.2.1). □

See [2] for the result of Quaternionic Geometry corresponding to Corollary 2.2.4.

Let (M^m, c, D) be a Weyl space and let g be a representative of c. The one-form α such that $Dg = -2\alpha \otimes g$ is called *the Lee form of D, with respect to g*.

Remark 2.2.5. Let (M^m, c, D) be a Weyl space.

1) Denote by α^g the Lee form of D with respect to a representative g of c. Then, from the definition of the Lee form it follows quickly that $\alpha^{\lambda^{-2}g} = \alpha^g + \lambda^{-1}\,\mathrm{d}\lambda$, for any positive function λ on M.

Alternatively, this can be proved as follows. Let s be the positive section of L such that $g = g_s$. Then $\lambda^{-2}g = g_{\lambda s}$ whilst, by Lemma 2.2.1, we have that α^g and $\alpha^{\lambda^{-2}g}$ are the local connection forms, with respect to s and λs, respectively, of the connection on L corresponding to D.

2) Let g be a representative of c. Then D is

determined by its Lee form α, with respect to g. Indeed, let s be the positive section of L such that $g = g_s$. As α is the local connection form, with respect to s, of the connection on L corresponding to D, we have $D = \nabla + \alpha$ as connections on L, where ∇ is the Levi-Civita connection of g.

Furthermore, by Corollary 2.2.4, we have

$$D_X Y = \nabla_X Y + \alpha(X)Y + \alpha(Y)X - g(X,Y)\alpha^{\sharp_g} ,$$

for any vector fields X and Y on M^m.

Let b be a covariant tensor field of degree two on a conformal manifold (M^m, c). Then $\mathrm{trace}_{\lambda^{-2}g}\, b = \lambda^2\, \mathrm{trace}_g\, b$ for any representative g of c and any positive function λ on M^m. Therefore there exists a section $\mathrm{trace}_c\, b$ of $(L^*)^2$ given by $\mathrm{trace}_c\, b = (\mathrm{trace}_{g_s} b)s^{-2}$, for any positive section s of L.

Similarly, if b is a section of $E \otimes \big(\otimes^2 T^*M\big)$, for some vector bundle E over M, then $\mathrm{trace}_c\, b$ is a section of $(L^*)^2 \otimes E$.

Another consequence of Proposition 2.2.3 is that on a conformal manifold there is no distinguished Weyl connection. Nevertheless, the next result shows that we can point out such a Weyl connection if the conformal manifold is endowed with certain $(1,1)$-tensor fields.

Proposition 2.2.6. *Let (M^m, c) be a conformal manifold endowed with a $(1,1)$-tensor field P. Denote by $\mathcal{P} = (m-1)P + P^* - \operatorname{trace} P\, \mathrm{Id}_{TM}$, where P^* is the adjoint of P, with respect to c.*

If $\mathcal{P}$ is invertible at each point then there exists a unique Weyl connection D on (M^m, c) such that $\operatorname{trace}_c(DP) = 0$.

PROOF. Let D be any Weyl connection on (M^m, c) and let α be a one-form on M^m.

Then for any local frame $(X_j)_{j=1,\dots,m}$, orthonormal with respect to some representative g of c, we have

$$\begin{aligned}\operatorname{trace}_g\big((D+\alpha)P\big) &= \sum_{j=1}^{m}\big((D+\alpha)_{X_j}P\big)(X_j)\\ &= \sum_{j=1}^{m}\Big((D+\alpha)_{X_j}(PX_j) - P\big((D+\alpha)_{X_j}X_j\big)\Big)\ .\end{aligned}$$

Now, Corollary 2.2.4 gives that $\operatorname{trace}_g\big((D+\alpha)P\big)$ is equal to:

$$\begin{aligned}&\sum_{j=1}^{m}\big(D_{X_j}(PX_j) + \alpha(X_j)PX_j + \alpha(PX_j)X_j\\ &\qquad\qquad\qquad - g(X_j, PX_j)\alpha^{\sharp_g}\big)\\ &- P\Big(\sum_{j=1}^{m}\big(D_{X_j}X_j + \alpha(X_j)X_j + \alpha(X_j)X_j\\ &\qquad\qquad\qquad - g(X_j, X_j)\alpha^{\sharp_g}\big)\Big)\ .\end{aligned}$$

As $\alpha^{\sharp_g}=\sum_{j=1}^m\alpha(X_j)X_j$ and

$$P^*(\alpha^{\sharp_g})=\sum_{j=1}^m\alpha(PX_j)X_j\;,$$

we obtain

$$\operatorname{trace}_g\bigl((D+\alpha)P\bigr)=\operatorname{trace}_g(DP)+\mathcal{P}(\alpha^{\sharp_g})\;, \tag{2.2.7}$$

from which the proof quickly follows. □

Let (M^m,c) be a conformal manifold endowed with a $(1,1)$-tensor field P such that

$$\mathcal{P}=(m-1)P+P^*-\operatorname{trace}P\,\mathrm{Id}_{TM}$$

is invertible at each point. The Weyl connection D on (M^m,c) such that $\operatorname{trace}_c(DP)=0$ is called *the Weyl connection of* (M^m,c,P).

Example 2.2.7. Let (M^m,c) be a conformal manifold, $m\geq 4$, endowed with an *almost Hermitian structure* J; that is, J is an almost complex structure on M^m such that $c(JX,JY)=c(X,Y)$, for any $X,Y\in TM$. Then (M^m,c,J) is an *almost Hermitian (conformal) manifold.* If J is integrable then it is called a *Hermitian (conformal) structure*, whilst (M^m,c,J) is a *Hermitian (conformal) manifold*.

As $J^* = -J$, on taking $P = J$ in Proposition 2.2.6 , we obtain that there exists a unique Weyl connection D on (M^m, c) such that $\mathrm{trace}_c(DJ) = 0$.

To determine the Lee form α of D, with respect to some representative g of c , we use (2.2.7) . Thus, we obtain that $\alpha = -\frac{1}{m-2}\,\mathrm{trace}_g(\nabla J)$, where ∇ is the Levi-Civita connection of g.

This Weyl connection is due to [29] .

On a two-dimensional (oriented) conformal manifold (M^2, c) there are just two (almost) Hermitian structures $\pm J$ (see Example 3.1.5 , below). Then, for any Weyl connection D on (M^2, c) , we have $DJ = 0$.

Let (M^m, c) be a conformal manifold endowed with a distribution $\mathscr{V}$. We denote $\mathscr{H} = \mathscr{V}^\perp$ and call $\mathscr{V}$ and $\mathscr{H}$ the *vertical* and *horizontal distributions*, respectively. Accordingly, a *vertical vector* is a vector tangent to $\mathscr{V}$ whilst a *horizontal vector* is a vector tangent to $\mathscr{H}$.

We shall, also, denote by $\mathscr{V}$ and $\mathscr{H}$ the corresponding orthogonal projections onto $\mathscr{V}$ and $\mathscr{H}$, respectively.

Suppose that (M^m, c) is endowed with a Weyl connection D. The *second fundamental form* of $\mathscr{V}$, with respect to D, is the $\mathscr{H}$-valued covariant tensor of degree two on $\mathscr{V}$ given by

$$B^{\mathscr{V},D}(U,V) = \tfrac{1}{2}\mathscr{H}(D_U V + D_V U) \;,$$

for any vertical vector fields U and V.

The second fundamental form of $\mathscr{V}$ is symmetric. Furthermore, the second fundamental of $\mathscr{V}$ is zero if and only if any geodesic of D which is vertical at some point is vertical everywhere; we then say that $\mathscr{V}$ is *geodesic*, with respect to D.

Then $\mathrm{trace}_c(B^{\mathscr{V},D})^\flat$ is a (horizontal) one-form on M^m (indeed, we have

$$\mathrm{trace}_c(B^{\mathscr{V},D})^\flat = \mathrm{trace}_g(B^{\mathscr{V},D})^{\flat_g},$$

for any representative g of c). This is ($(m-n)$ times) the *mean curvature form* of $\mathscr{V}$, with respect to D (where $n = \dim \mathscr{H}$). The distribution $\mathscr{V}$ is *minimal*, with respect to D, if its mean curvature form is zero.

If ∇ is the Levi-Civita connection of some representative g of c we denote by $B^{\mathscr{V},g}$ the second fundamental form of $\mathscr{V}$, with respect to ∇.

Example 2.2.8. Let (M^m, c) be a conformal manifold endowed with a distribution $\mathscr{V}$. Denote $n = \dim \mathscr{H}$, where $\mathscr{H} = \mathscr{V}^\perp$, and suppose that $0 < n < m$.

If in Proposition 2.2.6 we take $P = \mathscr{V}$ then, by using that $\mathscr{V}^* = \mathscr{V}$, $\operatorname{trace}\mathscr{V} = m - n$ and $\mathscr{V} + \mathscr{H} = \mathrm{Id}_{TM}$, we obtain $\mathcal{P} = n\mathscr{V} - (m-n)\mathscr{H}$ which, obviously, is invertible at each point. Therefore there exists a unique Weyl connection D on (M, c) such that $\operatorname{trace}_c(D\mathscr{V}) = 0$.

Also, as $\mathscr{V} + \mathscr{H} = \mathrm{Id}_{TM}$, the Weyl connections of $(M^m, c, \mathscr{V})$ and $(M^m, c, \mathscr{H})$ are equal.

Let g be a representative of c and let

$$(U_1, \ldots, U_{m-n}, X_1, \ldots, X_n)$$

be an orthonormal local frame on (M^m, g), adapted to the decomposition $TM = \mathscr{V} \oplus \mathscr{H}$, $(\dim \mathscr{H} = n)$. Then

$$\begin{aligned}
\operatorname{trace}_g(D\mathscr{V}) &= \sum_{r=1}^{m-n} (D_{U_r}\mathscr{V})(U_r) + \sum_{a=1}^{n} (D_{X_a}\mathscr{V})(X_a) \\
&= \sum_{r=1}^{m-n} \big(D_{U_r}U_r - \mathscr{V}(D_{U_r}U_r)\big) - \sum_{a=1}^{n} \mathscr{V}(D_{X_a}X_a) \\
&= \sum_{r=1}^{m-n} \mathscr{H}(D_{U_r}U_r) - \sum_{a=1}^{n} \mathscr{V}(D_{X_a}X_a) \\
&= \operatorname{trace}_g(B^{\mathscr{V},D}) - \operatorname{trace}_g(B^{\mathscr{H},D}) \, .
\end{aligned}$$

Thus, D is, also, characterised by the condition that, with respect to it, both $\mathscr{V}$ and $\mathscr{H}$ are minimal.

Let α be the Lee form of D with respect to g.

Then

(2.2.8)

$$\begin{aligned}\operatorname{trace}_g(B^{\mathscr{V},D}) &= \sum_{r=1}^{m-n} \mathscr{H}(D_{U_r}U_r)\\ &= \sum_{r=1}^{m-n} \mathscr{H}(\nabla_{U_r}U_r) - (m-n)\mathscr{H}(\alpha^{\sharp_g}),\end{aligned}$$

where ∇ is the Levi-Civita connection of g (note that, (2.2.8) still holds if we replace D with any Weyl connection on (M^m, c)).

Hence, $\alpha|_{\mathscr{H}} = \frac{1}{m-n}\operatorname{trace}_g(B^{\mathscr{V},g})^{\flat_g}$. Similarly, $\alpha|_{\mathscr{V}} = \frac{1}{n}\operatorname{trace}_g(B^{\mathscr{H},g})^{\flat_g}$ and therefore

$$\alpha = \tfrac{1}{m-n}\operatorname{trace}_g(B^{\mathscr{V},g})^{\flat_g} + \tfrac{1}{n}\operatorname{trace}_g(B^{\mathscr{H},g})^{\flat_g}.$$

This Weyl connection was introduced in [8] .

2.3. Harmonic maps

Recall (see [5]) that the *Laplace-Beltrami equation* for a function f on a Riemannian manifold (M, g) is given by $\operatorname{trace}_g(\nabla \mathrm{d}f) = 0$, where ∇ is the Levi-Civita connection of (M, g) .

Also, a map $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is harmonic if and only if

$$\operatorname{trace}_g(\nabla \mathrm{d}\varphi) = 0 ,$$

where ∇ is the connection on $\operatorname{Hom}(TM, \varphi^*(TN))$ induced by the Levi-Civita connections of (M, g)

and (N,h), and we have, also, denoted by $\mathrm{d}\varphi$ the section of $\mathrm{Hom}(TM,\varphi^*(TN))$ corresponding to the differential of φ.

The following definition is a natural generalization of these classical facts.

Definition 2.3.1. 1) A (real or complex) function f on a Weyl space (M,c,D) is *harmonic* if

$$\mathrm{trace}_c(D\mathrm{d}f)=0\ .$$

2) A map $\varphi:(M,c_M,D^M)\to(N,c_N,D^N)$ between Weyl spaces is *harmonic* if

$$\mathrm{trace}_{c_M}(D\mathrm{d}\varphi)=0\ ,$$

where D is the connection induced by D^M and D^N on $\mathrm{Hom}(TM,\varphi^*(TN))$.

Let $\varphi:(M,c_M,D^M)\to(N,c_N,D^N)$ be a map between Weyl spaces. Let (x^i) and (y^α) be local system of coordinates on M and N, respectively, and let ${}^M\Gamma^i_{jk}$ and ${}^N\Gamma^\alpha_{\beta\gamma}$ be the corresponding Christoffel symbols of D^M and D^N, respectively; that is, $D_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k}={}^M\Gamma^i_{kj}\frac{\partial}{\partial x^i}$ and, similarly, for ${}^N\Gamma^\alpha_{\beta\gamma}$ (note that, $\left({}^M\Gamma^i_{jk}\mathrm{d}x^k\right)_{i,j}$ is the local connection form of D^M with respect to $\left(\frac{\partial}{\partial x^j}\right)_j$).

Then a straightfoward calculation shows that,

for any j , k , and α , we have
(2.3.1)

$$\big(D\mathrm{d}\varphi\big)^{\alpha}_{jk}=\frac{\partial^2\varphi^{\alpha}}{\partial x^j\partial x^k}-{}^{M}\Gamma^{i}_{jk}\frac{\partial\varphi^{\alpha}}{\partial x^i}+{}^{N}\Gamma^{\alpha}_{\beta\gamma}\frac{\partial\varphi^{\beta}}{\partial x^j}\frac{\partial\varphi^{\gamma}}{\partial x^k}\ ;$$

in particular, $D\mathrm{d}\varphi$ is symmetric (cf. [5]).

Let (M,g) and (N,h) be Riemannian manifolds and let ∇^g and ∇^h be the Levi-Civita connections of g and h , respectively. Then a map $\varphi:(M,[g],\nabla^g)\to(N,[h],\nabla^h)$ is harmonic if and only if $\varphi:(M,g)\to(N,h)$ is harmonic.

However, there are natural constructions of harmonic maps on Weyl spaces which do not come from Riemannian manifolds.

Proposition 2.3.2. *Let (M,c,J) be a Hermitian manifold and let D be a Weyl connection on (M,c) .*

Then the following assertions are equivalent:

(i) *D is the Weyl connection of (M,c,J) .*

(ii) *The holomorphic functions of (M,J) are harmonic functions of (M,c,D) .*

Proof. Any holomorphic function (locally defined) on (M,J) whose differential is nowhere zero is, locally, a coordinate function of some local system of complex coordinates on (M,J) . Therefore it is sufficient to consider in (ii) only coordinate functions of local system of complex coordinates on (M,J) .

Let $(z^1,\dots,z^n)$ be a local system of complex coordinates on (M,J), where $m=2n$, and let g be a representative of c. The fact that J is (almost) Hermitian is equivalent to the fact that $T^{0,1}M$ is isotropic with respect to the (complexification of) g (that is, $g|_{T^{0,1}M}=0$).

Thus, $g_{jk}=g_{\bar{j}\,\bar{k}}=0$, where $g_{jk}=g\big(\frac{\partial}{\partial z^j},\frac{\partial}{\partial z^k}\big)$ and $g_{\bar{j}\,\bar{k}}=g\big(\frac{\partial}{\partial \bar{z}^j},\frac{\partial}{\partial \bar{z}^k}\big)$, for any $j,k=1,\dots,n$.

Let $g_{j\bar{k}}=g\big(\frac{\partial}{\partial z^j},\frac{\partial}{\partial \bar{z}^k}\big)$, for $j,k=1,\dots,n$, and denote by $\big(g^{AB}\big)$ the inverse matrix of (g_{AB}), where $A,B\in\{1,\dots,n,\overline{1},\dots,\overline{n}\}$.

By applying (2.3.1), we obtain

$$
\begin{aligned}
\mathrm{trace}_g(D\mathrm{d}z^j) &= g^{k\bar{l}}\big(D\mathrm{d}z^j\big)_{k\bar{l}}+g^{\bar{l}k}\big(D\mathrm{d}z^j\big)_{\bar{l}k}\\
&= 2\,g^{k\bar{l}}\big(D\mathrm{d}z^j\big)_{k\bar{l}}=-2\,g^{k\bar{l}}\,\Gamma^j_{k\bar{l}}\;,
\end{aligned}
\tag{2.3.2}
$$

for any $j=1,\dots,n$.

On the other hand, $\mathrm{trace}_g(DJ)$ is equal to:

$$
\begin{aligned}
&g^{k\bar{l}}\big(D_{\frac{\partial}{\partial z^k}}J\big)\left(\frac{\partial}{\partial \bar{z}^l}\right)+g^{\bar{l}k}\big(D_{\frac{\partial}{\partial \bar{z}^l}}J\big)\left(\frac{\partial}{\partial z^k}\right)\\
&=g^{k\bar{l}}\left(-\mathrm{i}D_{\frac{\partial}{\partial z^k}}\frac{\partial}{\partial \bar{z}^l}-JD_{\frac{\partial}{\partial z^k}}\frac{\partial}{\partial \bar{z}^l}\right)\\
&\quad+g^{\bar{l}k}\left(\mathrm{i}D_{\frac{\partial}{\partial \bar{z}^l}}\frac{\partial}{\partial z^k}-JD_{\frac{\partial}{\partial \bar{z}^l}}\frac{\partial}{\partial z^k}\right)\;.
\end{aligned}
$$

Therefore

(2.3.3)

$$
\begin{aligned}
&\mathrm{trace}_g(DJ)\\
&= g^{k\bar{l}}\left(-\mathrm{i}\,\Gamma^{j}_{\bar{l}k}\frac{\partial}{\partial z^j} - \mathrm{i}\,\Gamma^{\bar{j}}_{\bar{l}k}\frac{\partial}{\partial\overline{z^j}} - \mathrm{i}\,\Gamma^{j}_{\bar{l}k}\frac{\partial}{\partial z^j} + \mathrm{i}\,\Gamma^{\bar{j}}_{\bar{l}k}\frac{\partial}{\partial\overline{z^j}}\right)\\
&+ g^{\bar{l}k}\left(\mathrm{i}\,\Gamma^{j}_{k\bar{l}}\frac{\partial}{\partial z^j} + \mathrm{i}\,\Gamma^{\bar{j}}_{k\bar{l}}\frac{\partial}{\partial\overline{z}^j} - \mathrm{i}\,\Gamma^{j}_{k\bar{l}}\frac{\partial}{\partial z^j} + \mathrm{i}\,\Gamma^{\bar{j}}_{k\bar{l}}\frac{\partial}{\partial\overline{z}^j}\right)\\
&= -2\,\mathrm{i}\,g^{k\bar{l}}\,\Gamma^{j}_{\bar{l}k}\frac{\partial}{\partial z^j} + 2\,\mathrm{i}\,g^{\bar{l}k}\,\Gamma^{\bar{j}}_{k\bar{l}}\frac{\partial}{\partial\overline{z}^j}\;.
\end{aligned}
$$

As $\Gamma^{j}_{\bar{l}k} = \Gamma^{j}_{k\bar{l}}$ and $\overline{\Gamma^{j}_{\bar{k}l}} = \Gamma^{\bar{j}}_{k\bar{l}}$, $(j,k,l=1,\dots,n)$, the proof is an immediate consequence of (2.3.2) and (2.3.3) . $\square$

See Remark 4.2.5(3) , below, for explicit examples of (hyper-)Hermitian manifolds whose Weyl connections are not, even locally, Levi-Civita connections of representatives of conformal structures.

Remark 2.3.3. Let (M^2,c,D) be a two-dimensional Weyl space and let J be the positive Hermitian structure on (M^2,c) .

As $DJ=0$, the proof of Proposition 2.3.2 shows that any holomorphic function of (M^2,J) is a harmonic function of (M^2,c,D) .

2.4. Harmonic morphisms

We start this section by recalling ([21] , cf. [5]) the following definition.

Definition 2.4.1. Let (M, c_M, D^M) and (N, c_N, D^N) be Weyl manifolds.

A map $\varphi : (M, c_M, D^M) \to (N, c_N, D^N)$ is called a *harmonic morphism* if for any harmonic function f defined on some open set U of N, such that $\varphi^{-1}(U) \neq \emptyset$, the function $f \circ \varphi|_{\varphi^{-1}(U)}$ is harmonic.

For harmonic morphisms between Riemannian manifolds the interested reader should consult [5], [26] and the references therein.

Let D be the Weyl connection of a Hermitian manifold (M, c, J). Then the holomorphic functions of (M, J) are harmonic morphisms (locally defined) on (M, c, D). Indeed, this is an immediate consequence of Proposition 2.3.2 and the fact that a function, defined on some open set of $\mathbb{C}$, is harmonic if and only if, locally, it is the real part of a holomorphic function.

A linear isomorphism $A : (V, c_V) \to (W, c_W)$ between conformal vector spaces is *conformal* if there exists g and h representatives of c_V and c_W, respectively, such that $A^*h = g$.

Definition 2.4.2. A map $\varphi : (M, c_M) \to (N, c_N)$ between conformal manifolds is *horizontally weakly conformal* if, for any $x \in M$, either $\mathrm{d}\varphi_x = 0$ or

$\mathrm{d}\varphi_x|_{(\ker \mathrm{d}\varphi_x)^\perp}$ is a conformal linear isomorphism from $\big((\ker \mathrm{d}\varphi_x)^\perp, c_M|_{(\ker \mathrm{d}\varphi_x)^\perp}\big)$ onto $\big(T_{\varphi(x)}N, (c_N)_{\varphi(x)}\big)$.

The following result is basic for harmonic morphisms.

Theorem 2.4.3. *A map between Weyl spaces is a harmonic morphism if and only if it is a harmonic map which is horizontally weakly conformal.*

The proof of Theorem 2.4.3 is based on the following two lemmas.

Lemma 2.4.4. *Let (M, c, D) be a Weyl space and let $x \in M$.*

Then for any $\alpha \in T^*_xM$ and any trace free symmetric bilinear form b on (T_xM, c_x) there exists a harmonic function f defined on some open neighbourhood of x such that $\mathrm{d}f_x = \alpha$ and $(D\mathrm{d}f)_x = b$.

Proof. This is essentially the same as for harmonic functions on Riemannian manifolds (see [5] and the references therein).

We shall give a straightforward proof assuming (M, c, D) real analytic. Let U be the domain of a normal coordinate system $x^1, \ldots, x^m$ for D, centred at x , where $m = \dim M$. Obviously, the hypersurface $S = \{x^m = 0\}$ is noncharacteristic for the second order linear differential operator $f \mapsto$

$\mathrm{trace}_g(D\mathrm{d}f)$.

Let $p = b_{ij}x^ix^j + \alpha_i x^i$. Then, by further restricting U, if necessary, and by applying the Cauchy–Kovalevskaya theorem, we can find a harmonic function f defined on U such that f and p are equal up to the first derivatives along S; in particular, $\mathrm{d}f_x = \alpha$. Hence, possibly excepting $\frac{\partial^2 f}{(\partial x^m)^2}(x)$, all the second order partial derivatives of f, at x, are equal to the corresponding derivatives of p, at x. As f is harmonic, b is trace free, with respect to g, and x is the centre of the normal system of coordinates $x^1, \dots, x^m$, for D, the derivatives $\frac{\partial^2 f}{(\partial x^m)^2}(x)$ and $\frac{\partial^2 p}{(\partial x^m)^2}(x)$ are determined by the other second order partial derivatives, at x, of f and p, respectively, and hence must be equal. Thus, $(D\mathrm{d}f)_x = b$.

$\square$

Lemma 2.4.5. *Let $\varphi : (M, [g], D^M) \to (N, [h], D^N)$ be a map between Weyl spaces. Then, for any function f on N, we have*

$$\mathrm{trace}_g\big(D\mathrm{d}(f \circ \varphi)\big) = \mathrm{d}f\big(\mathrm{trace}_g(D\mathrm{d}\varphi)\big) + h\big(D^N\mathrm{d}f, \big((\mathrm{d}\varphi)^T\big)^*(g)\big) ,$$

where, at each point, $\big((\mathrm{d}\varphi)^T\big)^*(g)$ is the pull back of g through the adjoint $(\mathrm{d}\varphi)^T$ of $\mathrm{d}\varphi$, with respect to g and h.

PROOF. Let $x_0 \in M$ and let (x^j) and (y^α) be normal coordinate systems of D^M and D^N, about x_0 and $\varphi(x_0)$, respectively.

Then

(2.4.1)
$$\frac{\partial^2(f\circ\varphi)}{\partial x^j\partial x^k}=\frac{\partial^2 f}{\partial y^\alpha\partial y^\beta}\,\frac{\partial\varphi^\alpha}{\partial x^j}\,\frac{\partial\varphi^\beta}{\partial x^k}+\frac{\partial f}{\partial y^\alpha}\,\frac{\partial^2\varphi^\alpha}{\partial x^j\partial x^k}\ ,$$

for any j and k.

Also, as $h\Big(\mathrm{d}\varphi\big(\frac{\partial}{\partial x^j}\big)\,,\frac{\partial}{\partial y^\alpha}\Big)=g\Big(\frac{\partial}{\partial x^j},(\mathrm{d}\varphi)^T\Big(\frac{\partial}{\partial y^\alpha}\Big)\Big)$, for any j and α, we have $\big((\mathrm{d}\varphi)^T\big)^j_{\ \alpha}=g^{jk}\,\frac{\partial\varphi^\alpha}{\partial x^k}\,h_{\alpha\beta}$, for any j and α. Hence, for any α and β, we have

(2.4.2)
$$\begin{aligned}\big((\mathrm{d}\varphi)^T\big)^*(g)\Big(\frac{\partial}{\partial y^\alpha},\frac{\partial}{\partial y^\beta}\Big)&\\&=g\Big((\mathrm{d}\varphi)^T\Big(\frac{\partial}{\partial y^\alpha}\Big)\,,(\mathrm{d}\varphi)^T\Big(\frac{\partial}{\partial y^\beta}\Big)\Big)\\&=\big((\mathrm{d}\varphi)^T\big)^j_{\ \alpha}\,\big((\mathrm{d}\varphi)^T\big)^k_{\ \beta}\,g_{jk}\\&=g^{jk}\,\frac{\partial\varphi^\gamma}{\partial x^j}\,\frac{\partial\varphi^\eta}{\partial x^k}\,h_{\gamma\alpha}\,h_{\eta\beta}\ .\end{aligned}$$

By using (2.4.2), we obtain

(2.4.3)
$$\begin{aligned}&h\big(D^N\mathrm{d}f,\big((\mathrm{d}\varphi)^T\big)^*(g)\big)\\&=\big(D^N\mathrm{d}f\big)\Big(\frac{\partial}{\partial y^\alpha},\frac{\partial}{\partial y^\beta}\Big)\big((\mathrm{d}\varphi)^T\big)^*(g)\Big(\frac{\partial}{\partial y^\gamma},\frac{\partial}{\partial y^\eta}\Big)h_{\alpha\gamma}h_{\beta\eta}\\&=g^{jk}\frac{\partial f}{\partial y^\alpha}\,\frac{\partial^2\varphi^\alpha}{\partial x^j\partial x^k}\ .\end{aligned}$$

By taking into account that ${}^M\Gamma^i_{jk}(x_0) = 0$ and ${}^N\Gamma^\alpha_{\beta\gamma}(\varphi(x_0)) = 0$, the proof follows quickly from (2.3.1), (2.4.1) and (2.4.3). $\square$

Proof of Theorem 2.4.3. Using Lemma 2.4.4 with $b = 0$ and for all $\alpha \in T^*_xN$, $(x \in N)$, from Lemma 2.4.5 we obtain that φ is a harmonic map.

Then, by applying again Lemmas 2.4.4 and 2.4.5, we obtain that for any trace free $b \in \odot^2 T^*N$ we have $h\big(b, ((\mathrm{d}\varphi)^T)^*(g)\big) = 0$.

Therefore, at each $x \in M$, there exists a non-negative number $\lambda(x)$ such that $\big((\mathrm{d}\varphi)^T\big)^*(g_x) = \lambda(x)^2\, h_{\varphi(x)}$; equivalently, φ is horizontally weakly conformal. $\square$

2.5. The fundamental equation

We are mainly interested in submersive harmonic morphisms. According to Theorem 2.4.3 these are just harmonic maps which are horizontally conformal (submersions). Therefore the next step to do is to characterise the horizontally conformal submersions which are harmonic. This will be done in this section.

Firstly, we take a closer look to horizontally conformal submersions.

If $\varphi : (M, c) \to N$ is a submersion, from a conformal (or Riemannian) manifold, we denote $\mathscr{V} =$

$\ker \mathrm{d}\varphi$ and $\mathscr{H} = \mathscr{V}^{\perp}$. Note that, there exists a vector bundle isomorphism $\mathscr{H} = \varphi^*(TN)$ under which the pull back, through φ, of a vector field X on N is just the horizontal lift of X. Also, under this isomorphism, the section of $\mathrm{Hom}(TM, \varphi^*(TN))$ corresponding to $\mathrm{d}\varphi$ is the orthogonal projection onto $\mathscr{H}$.

A *Riemannian submersion* $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds is a map such that $\mathrm{d}\varphi_x|_{\mathscr{H}_x} : \big(\mathscr{H}_x, g|_{\mathscr{H}_x}\big) \to \big(T_{\varphi(x)}N, h_{\varphi(x)}\big)$ is a linear isometry, for any $x \in M$.

Proposition 2.5.1. *Let (M, c_M) and (N, c_N) be conformal manifolds and let $\varphi : M \to N$ be a submersion.*

The following assertions are equivalent:

(i) *$\varphi : (M, c_M) \to (N, c_N)$ is horizontally conformal.*

(ii) *There exists representatives g and h of c_M and c_N, respectively, such that $\varphi : (M, g) \to (N, h)$ is a Riemannian submersion.*

(iii) *For any representative h of c_N there exists a representative g of c_M such that $\varphi : (M, g) \to (N, h)$ is a Riemannian submersion.*

Proof. It is obvious that (iii)$\Longrightarrow$(ii)$\Longrightarrow$(i).

Suppose that $\varphi : (M, c_M) \to (N, c_N)$ is horizontally conformal and let g and h be representatives of c_M and c_N , respectively. Then there exists a positive (smooth) function λ on M such that $\varphi^*(h) = \lambda^2 g|_{\mathscr{H}}$; equivalently, $\varphi : (M, \lambda^2 g) \to (N, h)$ is a Riemannian submersion. $\square$

Let L_M and L_N be the line bundles of the conformal manifolds (M, c_M) and (N, c_N) , respectively. Assertion (iii) of Proposition 2.5.1 is equivalent to the following:

(iii′) *There exists a (unique) isomorphism of oriented line bundles $L_M = \varphi^*(L_N)$ under which, for any positive section s of L_N , we have that the map φ from (M, g_s) to (N, h_s) is a Riemannian submersion, where g_s and h_s are the representatives of c_M and c_N , respectively, corresponding to s .*

The next result will be used later on.

Lemma 2.5.2. *Let $\varphi : (M, g) \to (N, h)$ be a Riemannian submersion and let ∇^g and ∇^h be the Levi-Civita connections of g and h , respectively.*

Then, under the isomorphism of vector bundles $\varphi^*(TN) = \mathscr{H}$, we have $\varphi^*(\nabla^h)_X Y = \mathscr{H}(\nabla^g_X Y)$ for any horizontal vector fields X and Y on M.

Proof. Let X, Y, Z be horizontal vector fields on M which are projectable, with respect to φ .

As φ is a Riemannian submersion and $\mathrm{d}\varphi([Y, Z]) =$

$[\mathrm{d}\varphi(Y), \mathrm{d}\varphi(Z)]$ we have

$$g\bigl(X, [Y, Z]\bigr) = h\bigl(\mathrm{d}\varphi(X), [\mathrm{d}\varphi(Y), \mathrm{d}\varphi(Z)]\bigr) \circ \varphi \, .$$

Therefore (2.2.6) gives

$$g(\nabla^g_X Y, Z) = h\bigl(\nabla^h_{\mathrm{d}\varphi(X)} \mathrm{d}\varphi(Y), \mathrm{d}\varphi(Z)\bigr) \circ \varphi \, .$$

But $\varphi^*(\nabla^h)_X Y$ is the horizontal lift of $\nabla^h_{\mathrm{d}\varphi(X)} \mathrm{d}\varphi(Y)$. Thus, we obtain $\varphi^*(\nabla^h)_X Y = \mathscr{H}(\nabla^g_X Y)$ and the proof is complete. □

From (2.2.6) it also follows that if $\varphi : (M, g) \to (N, h)$ is a Riemannian submersion then $\mathscr{H}$ is geodesic (with respect to g).

Furthermore, from (2.2.1) it follows that if $\varphi : (M, c_M) \to (N, c_N)$ is a horizontally conformal submersion then the trace free part of $B^{\mathscr{H},D}$ is zero, for any Weyl connection D on (M, c_M) (equivalently, $\mathscr{H}$ is *umbilical*).

Next, we prove *the fundamental equation* for horizontally conformal submersions between Weyl spaces.

Proposition 2.5.3. *Let φ be a horizontally conformal submersion from (M, c_M, D^M) to (N, c_N, D^N) . Denote by L_M and L_N the line bundles of M and N, respectively, and let α be the one-form on M*

such that $\varphi^*(D^N) = D^M + \alpha$ *as connections on* $\varphi^*(L_N) = L_M$ *. Then*

(2.5.1)

$$\mathrm{trace}_{c_M}\big(D\mathrm{d}\varphi\big)^\flat + \mathrm{trace}_{c_M}\Big(B^{\mathscr{V},D^M}\Big)^\flat + (n-2)\alpha|_{\mathscr{H}} = 0\ ,$$

where $n = \dim N$*.*

Proof. Let g and h be representatives of c_M and c_N , respectively, such that $\varphi : (M,g) \to (N,h)$ is a Riemannian submersion. Denote by α_M and α_N the Lee forms of D^M and D^N with respect to g and h , respectively. From Lemma 2.5.2 it follows that $\alpha = \varphi^*(\alpha_N) - \alpha_M$.

Let $(U_1,\dots,U_{m-n},X_1,\dots,X_n)$ be an orthonormal local frame on (M,g) , adapted to the decomposition $TM = \mathscr{V} \oplus \mathscr{H}$, where $m = \dim M$. By using the fact that the section of $\mathrm{Hom}(TM, \varphi^*(TN))$ corresponding to $\mathrm{d}\varphi$ is $\mathscr{H}$, we obtain

(2.5.2)

$$\begin{aligned}
\mathrm{trace}_g(D\mathrm{d}\varphi) &= \sum_{r=1}^{m-n}(D_{U_r}\mathrm{d}\varphi)(U_r) + \sum_{j=1}^{n}\big(D_{X_j}\mathrm{d}\varphi)(X_j) \\
&= \sum_{r=1}^{m-n}\big(\varphi^*(D^N)_{U_r}(\mathscr{H}U_r) - \mathscr{H}(D^M_{U_r}U_r)\big) \\
&+ \sum_{j=1}^{n}\big(\varphi^*(D^N)_{X_j}(\mathscr{H}X_j) - \mathscr{H}(D^M_{X_j}X_j)\big)\ .
\end{aligned}$$

Now, we have

(2.5.3)

$$\sum_{j=1}^{n}\mathscr{H}(D^M_{X_j}X_j)=\sum_{j=1}^{n}\bigl(\nabla_{X_j}X_j-(n-2)\mathscr{H}(\alpha_M^{\sharp_g})\bigr)\;,$$

where ∇ is the Levi-Civita connection of g.

On the other hand, by applying Lemma 2.5.2, we obtain

(2.5.4)

$$\sum_{j=1}^{n}\varphi^*(D^N)_{X_j}(\mathscr{H}X_j)=\sum_{j=1}^{n}\bigl(\nabla_{X_j}X_j-(n-2)\varphi^*(\alpha_N)^{\sharp_g}\bigr)\;.$$

Then (2.5.1) is an immediate consequence of (2.5.2), (2.5.3) and (2.5.4). □

From Theorem 2.4.3, Proposition 2.5.3 and (2.2.1) we immediately obtain the following result.

Corollary 2.5.4. *Let* $\varphi:(M,c_M,D^M)\to(N,c_N,D^N)$ *be a horizontally conformal submersion between Weyl spaces; denote by* L_M *and* L_N *the line bundles of* M *and* N*, respectively.*

(a) *If* $\dim N=2$ *then* φ *is a harmonic morphism if and only if its fibres are minimal.*

(b) *If* $\dim N\neq 2$ *then any two of the following assertions imply the third:*

(i) φ *is a harmonic morphism.*

(ii) φ *has minimal fibres.*

(iii) $\varphi^*(D^N)_X Y = \mathscr{H}(D^M_X Y)$ *for any horizontal vector fields* X *and* Y *on* M.

Let D be the Weyl connection of the Hermitian manifold (M,c,J). Then, by Corollary 2.5.4, the regular fibres of any holomorphic function $\varphi : (M,J) \to \mathbb{C}$ are minimal submanifolds of (M,c,D).

Corollary 2.5.5. *Let* $\varphi : (M,c_M) \to (N,c_N)$ *be a horizontally conformal submersion,* $\dim N > 2$.

Then for any Weyl connection D^N *on* (N,c_N) *there exists a Weyl connection* D^M *on* (M,c_M) *such that* $\varphi : (M,c_M,D^M) \to (N,c_N,D^N)$ *is a harmonic morphism.*

PROOF. Let g and h be representatives of c_M and c_N, respectively, such that $\varphi : (M,g) \to (N,h)$ is a Riemannian submersion.

Let D^M be any Weyl connection on (M,c_M). By using (2.2.8), we obtain that the fundamental equation (2.5.1) is equivalent to the following relation:

(2.5.5)

$$\begin{aligned}&\mathrm{trace}_{c_M}(D\mathrm{d}\varphi)^\flat\\ &= (m-2)\alpha_M|_{\mathscr{H}} - (n-2)\varphi^*(\alpha_N) - \mathrm{trace}_g\big(B^{\mathscr{V},g}\big)^{\flat_g},\end{aligned}$$

where α_M and α_N are the Lee forms of D^M and D^N with respect to g and h, respectively, and $m =$

$\dim M$, $n = \dim N$.

The proof follows. $\square$

Note that, in Corollary 2.5.5, unless $\dim M = \dim N$, the resulting Weyl connection on (M, c_M) is not unique.

Example 2.5.6. 1) There exists unique Riemannian metrics on $\mathbb{C}P^n$ and $\mathbb{H}P^n$ with respect to which the projections of the Hopf bundles $(S^{2n+1}, \mathbb{C}P^n, S^1)$ and $(S^{4n+3}, \mathbb{H}P^n, S^3)$, respectively, are Riemannian submersions (here, we have identified S^3 with the Lie group of unit quaternions). Indeed, this follows from the fact that S^1 and S^3 act by isometries on S^{2n+1} and S^{4n+3}, respectively.

2) The Hopf polynomial map $\mathbb{R}^4 \to \mathbb{R}^3$, $(z_1, z_2) \mapsto (|z_1|^2 - |z_2|^2, 2z_1\overline{z_2})$ is horizontally weakly conformal (note that, it has a critical point at 0).

3) The radial projection $\mathbb{R}^{n+1} \setminus \{0\} \to S^n$, $x \mapsto \frac{1}{|x|}\,x$ is horizontally conformal (to see this, write the Euclidean metric in polar coordinates).

4) The projections $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$ and $\mathbb{H}^{n+1} \setminus \{0\} \to \mathbb{H}P^n$ are horizontally conformal (these are the compositions of radial projections followed by projections of Hopf bundles).

5) The projections of the Hopf bundles induce

a map $\mathbb{C}P^{2n+1} \to \mathbb{H}P^n$ which is a Riemannian submersion.

All of the maps of Example 2.5.6 are harmonic morphisms between Riemannian manifolds (for (1), (3) and (5) this follows from Corollary 2.5.4, for (2) this can be proved directly, whilst (4) are compositions of harmonic morphisms).

Also, by endowing the codomains of these horizontally conformal submersions with Weyl connections and by using Corollary 2.5.5, we obtain harmonic morphisms between Weyl spaces (not necessarily coming from Riemannian manifolds).

CHAPTER 3

Twistorial structures and maps

3.1. Twistorial structures

As before, unless otherwise stated, all the manifolds are assumed connected, smooth and oriented, and all the maps are smooth.

We start this chapter by presenting the definition of almost twistorial structures, at a level of generality suitable for the purposes of this book (see [22] for a more general definition).

Definition 3.1.1. An *almost twistorial structure* on a manifold M is a quadruple $\tau = (Q, M, \pi, \mathcal{C})$, where $\pi : Q \to M$ is a locally trivial fibre space and $\mathcal{C}$ is an almost co-CR structure on Q which induces almost complex structures on the fibres of π.

The almost twistorial structure $\tau = (Q, M, \pi, \mathcal{C})$ is *integrable* if $\mathcal{C}$ is integrable. A *twistorial structure* is an integrable almost twistorial structure.

A twistorial structure $\tau = (Q, M, \pi, \mathcal{C})$ is *simple* if $\mathcal{C}$ is simple; if τ is simple, with $\pi_Z : (Q, \mathcal{C}) \to Z$ the corresponding holomorphic submersion, the complex manifold Z is the *twistor space* of τ .

$$\begin{tikzcd} & Q \arrow[dl, "\pi_Z"'] \arrow[dr, "\pi"] & \\ Z & & M \end{tikzcd}$$

Proposition 3.1.2. *Let $\tau = (Q, M, \pi, \mathcal{C})$ be a simple twistorial structure and let $\pi_Z : (Q, \mathcal{C}) \to Z$ be the corresponding holomorphic submersion onto the twistor space of τ.*

Then, for any $z \in Z$, the map

$$\pi|_{\pi_Z^{-1}(z)} : \pi_Z^{-1}(z) \to M$$

is an immersion whose normal bundle is endowed with a linear complex structure, induced by $\mathcal{C}$.

Proof. If $q \in \pi_Z^{-1}(z)$ and $X \in T_q\big(\pi_Z^{-1}(z)\big)$ then $\mathrm{d}\pi(X) = 0$ if and only if X is tangent to the fibre of π through q . As $\mathcal{C}$ induces complex structures on the fibres of π and $T\big(\pi_Z^{-1}(z)\big) = (\mathcal{C} \cap \overline{\mathcal{C}})|_{\pi_Z^{-1}(z)}$, we obtain that $X = 0$.

By using the isomorphism of vector bundles

$$\pi^*(TM) = TQ/(\ker \mathrm{d}\pi)$$

we obtain that the normal bundle of $\pi_Z^{-1}(z)$ in M is isomorphic to the restriction to $\pi_Z^{-1}(z)$ of

$$TQ/\big((\mathcal{C} \cap \overline{\mathcal{C}}) \oplus \ker \mathrm{d}\pi\big) \ .$$

Now, the last assertion follows from the fact that $\mathcal{C}+\overline{\mathcal{C}}=TQ$. $\square$

A particular kind of almost twistorial structure $\tau=(Q,M,\pi,\mathcal{C})$ is given by the condition that $\mathcal{C}$ be an almost complex structure. Then, if integrable, τ is simple and the immersed submanifolds of Proposition 3.1.2 are just points of M; in particular, $Z=Q$ and $\pi_Z=\mathrm{Id}_Z$.

Note that, essentially the same proof as for Proposition 3.1.2 shows that if $(Q,M,\pi,\mathcal{C})$ is a twistorial structure then the restriction of π to any leaf of $\mathcal{C}\cap\overline{\mathcal{C}}$ is an immersion whose normal bundle in endowed with a linear complex structure.

Proposition 3.1.2 shows that to any *twistor* $z\in Z$ it corresponds a pair (P_z,J_z) where P_z is an immersed submanifold of M and J_z is a linear complex structure on the normal bundle of P_z in M.

In particular, the linear twistors are just linear co-CR structures. Therefore to find examples of twistorial structures, we have, firstly, to look for spaces of linear co-CR structures. Furthermore, we, also, need an adequate compatibility condition between conformal and co-CR structures. This is provided by the following simple fact.

Proposition 3.1.3. *Let (V,g) be an Euclidean space.*

Then any coisotropic vector space $C \subseteq V^{\mathbb{C}}$ (that is, $C^{\perp}$ is isotropic; equivalently, $g|_{C^{\perp}} = 0$) is a linear co-CR structure on V; that is,

$$C + \overline{C} = V\,.$$

PROOF. As $C^{\perp}$ is isotropic, we have $\bigl(C+\overline{C}\bigr)^{\perp} = C^{\perp} \cap \overline{C^{\perp}} = \{0\}$. □

Dually to Proposition 3.1.3, if (V,g) is an Euclidean space and $C \subseteq V^{\mathbb{C}}$ is isotropic then C is a linear CR structure (that is, $C \cap \overline{C} = \{0\}$).

The following result is an immediate consequence of Proposition 3.1.3.

Corollary 3.1.4. *Let (V,g) be an Euclidean space and let $C \subseteq V^{\mathbb{C}}$.*

Then the following assertions are equivalent:

(i) *C is the eigenspace, corresponding to $-\mathrm{i}$, of an orthogonal complex structure on (V,g).*

(ii) *C is both isotropic and coisotropic.*

Here, we shall be interested in coisotropic spaces in dimensions two, three and four.

Example 3.1.5. On a two-dimensional oriented vector space (V,g) there are two nontrivial (co)isotropic spaces. These are spanned by $u \pm \mathrm{i}v$, where (u,v)

is any positive orthonormal basis of (V,g).

Accordingly, there are two orthogonal complex structures $\pm J$ on (V,g), given by the rotations of angles $\pm\pi/2$, respectively.

Example 3.1.6. Let (V,g) be a three-dimensional oriented Euclidean space. Then the space S of two-dimensional coisotropic spaces on (V,g) can be identified with $S^2\,(=\mathbb{C}P^1)$, as follows: to any $u\in S^2$ we associate the span q_u of $\{u, v+\mathrm{i}w\}$, where (u,v,w) is a positive orthonormal basis of (V,g).

Conversely, if $q\subseteq V^{\mathbb{C}}$ is a two-dimensional coisotropic space then $q^{\perp}$ is isotropic (and one-dimensional). Hence, $q^{\perp}\oplus\overline{q^{\perp}}$ is (the complexification) of a two-dimensional vector subspace p of (V,g) which we orient so that $q^{\perp}$ be the eigenspace corresponding to $-\mathrm{i}$ of the rotation through $\pi/2$. Next, we orient $p^{\perp}\,(=\ q\cap\overline{q})$ such that the decomposition $V=p^{\perp}\oplus p$ be orientation preserving. Then $q=q_u$, where u is the positive orthonormal basis of $p^{\perp}$.

This correspondence may be described, more conceptually, by observing that $\mathrm{SO}(3,\mathbb{R})$ acts transitively on S with isotropy group S^1, where the embedding $S^1\subseteq\mathrm{SO}(3,\mathbb{R})$ is given by

$$\mathrm{e}^{\mathrm{i}t}\longmapsto\begin{pmatrix}1 & 0 & 0\\ 0 & \cos t & -\sin t\\ 0 & \sin t & \cos t\end{pmatrix}.$$

Therefore

$$S = \frac{\mathrm{SO}(3,\mathbb{R})}{S^1} = \frac{\mathrm{Sp}(1)/\{\pm 1\}}{S^1/\{\pm 1\}} = \frac{\mathrm{Sp}(1)}{S^1} = S^2\,,$$

where:

• $\mathrm{Sp}(1)$ is the group of unit quaternions and the isomorphism $\mathrm{Sp}(1)/\{\pm 1\} = \mathrm{SO}(3,\mathbb{R})$ is induced by the action of $\mathrm{Sp}(1)$ on $V = \mathrm{Im}\mathbb{H}$ given by $q \cdot v = qvq^{-1}$, for any unit quaternion q and any $v \in V$;

• the isomorphism $S^1/\{\pm 1\} = S^1$ is induced by the morphism of Lie groups $S^1 \to S^1$, $z \mapsto z^2$.

Example 3.1.7. Let (V, g) be a four-dimensional oriented Euclidean space. A complex vector subspace of $V^{\mathbb{C}}$ is called *(anti-)self-dual* if it is the eigenspace, corresponding to $-\mathrm{i}$, of a positive (negative) orthogonal complex structure on (V, g) (a linear complex structure J on V is positive if some (and, hence, any) basis of the form (u, Ju, v, Jv) is positive).

Any (anti-)self-dual space on (V, g) has complex dimension two. Conversely, any two-dimensional (co)isotropic space on (V, g) is either self-dual or anti-self-dual.

Let S be the space of self-dual spaces on (V, g). We claim that $S = S^2$. Indeed, think of S as the space of positive orthogonal complex structures on

(V, g) and let $U \subseteq V$ be a three-dimensional vector subspace. Choose some orientation on U and orient $U^\perp$ so that the decomposition $V = U^\perp \oplus U$ be orientation preserving. Denote by v the positive orthonormal basis of $U^\perp$ and identify S^2 with the unit sphere in U.

Then the map $S \to S^2$, $J \mapsto Jv$, $(J \in S)$, gives the claimed correspondence. The inverse map associates to any $u \in S^2$ the positive orthogonal complex structure J on (V, g) such that $Jv = u$ and J is the rotation through $\pi/2$ on $u^\perp \cap U$.

Note that, the orthogonal projection $V \to U$ maps (anti-)self-dual spaces on (V, g) onto two-dimensional coisotropic spaces on $(U, g|_U)$.

Also, on identifying $V = \mathbb{H}$ such that $U = \mathrm{Im}\mathbb{H}$, any $q \in S^2$ corresponds to the orthogonal complex structure J_q on (V, g) given by $J_q(u) = qu$, for any $u \in V$. Therefore, under the isomorphism of Lie groups, $\big(\mathrm{Sp}(1) \times \mathrm{Sp}(1)\big)/\{\pm 1\} = \mathrm{SO}(4, \mathbb{R})$ induced by the action of $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ on V, given by $(p_+, p_-) \cdot v = p_+ v p_-^{-1}$, we have

$$S = \frac{\mathrm{SO}(4, \mathbb{R})}{\mathrm{U}(2)} = \frac{\mathrm{Sp}(1) \times \mathrm{Sp}(1)}{S^1 \times \mathrm{Sp}(1)} = S^2 \,,$$

where $U(2) \big(= (S^1 \times \mathrm{Sp}(1))/\{\pm 1\}\big)$ is the group of orthogonal linear isomorphisms of (V, g) which are linear complex, with respect to J_{i}.

Note that, this way, any $\pm(p_+, p_-) \in \mathrm{SO}(4, \mathbb{R})$ maps the self-dual space corresponding to $q \in S^2$ (where $S^2 \subseteq \mathrm{Im}\mathbb{H}$) onto the self-dual space corresponding to $p_+ q p_+^{-1}$.

We end this section with the simplest example of a twistorial structure.

Example 3.1.8. Let (M^2, c) be a two-dimensional (oriented) conformal manifold and let J be the positive Hermitian structure on (M^2, c) .

Then $(M^2, M^2, \mathrm{Id}_M, J)$ is *the twistorial structure of* (M^2, c) ; its twistor space is (M^2, J) .

3.2. Einstein–Weyl spaces of dimension three and anti-self-dual manifolds

In this section we present the almost twistorial structures associated to the Weyl spaces of dimensions three and four and we discuss their integrability.

Example 3.2.1. Let (M, c, D) be a three-dimensional Weyl space. Let $\pi : Q \to M$ be the space of two-dimensional coisotropic spaces on (M^3, c) . Obviously, Q is also the bundle of two-dimensional coisotropic spaces on $(L^* \otimes TM, c)$, where L is the line bundle of M^3.

From Example 3.1.6 it follows that we may identify Q with the sphere bundle of $(L^* \otimes TM, c)$.

Hence, D induces a connection $\mathscr{H}(\subseteq TQ)$ on Q.

Define the complex vector subbundle $\mathcal{B}$ of $\mathscr{H}^{\mathbb{C}}$ such that $\mathrm{d}\pi(\mathcal{B}_q) = q$, for any $q \in Q$.

Then $\mathcal{C} = \mathcal{B} \oplus (\ker \mathrm{d}p)^{0,1}$ is an almost co-CR structure on Q and $(Q, M, \pi, \mathcal{C})$ is *the almost twistorial structure of* (M^3, c, D).

The following result shows that the almost twistorial structure of a three-dimensional Weyl space determines the underlying Weyl connection.

Proposition 3.2.2. *Let (M^3, c) be a three-dimensional conformal manifold endowed with two Weyl connections D' and D''. Let τ' and τ'' be the almost twistorial structures of (M^3, c, D') and (M^3, c, D''), respectively.*

Then $D' = D''$ if and only if $\tau' = \tau''$.

Proof. If $\tau' = (Q, M, \pi, \mathcal{C}')$, $\tau'' = (Q, M, \pi, \mathcal{C}'')$ then $\tau' = \tau''$ if and only if $\mathcal{C}' = \mathcal{C}''$.

Let $x_0 \in M$ and let q be a section of Q defined in some open neighbourhood U of x_0; denote $q_0 = q_{x_0}$.

If $X \in q_0$ then, by Proposition 1.2.1, we have $\mathrm{d}q(X) \in \mathcal{C}'_{q_0}$ if and only if $D'_X Y \in q_0$, for any section Y of q (here, we think of q as a complex vector subbundle of $T^{\mathbb{C}}M|_U$). Similarly, $\mathrm{d}q(X) \in \mathcal{C}''_{q_0}$ if and only if $D''_X Y \in q_0$, for any section Y of q.

Thus, if α is the one-form on M^3 such that $D'' =$

$D' + \alpha$, as connections on the line bundle of M^3, then $\mathcal{C}' = \mathcal{C}''$ if and only if, for any $q \in Q$ and any $X, Y \in q$, we have $\alpha(X)Y + \alpha(Y)X - c(X,Y)\alpha^{\sharp}_{\pi(q)} \in q$; equivalently, $\alpha^{\sharp}_{\pi(q)} \in q$, for any $q \in Q$, which, obviously, holds if and only if $\alpha = 0$. $\square$

Next, we present the four dimensional version of Example 3.2.1.

Example 3.2.3. Let (M^4, c) be a four-dimensional (oriented) conformal manifold endowed with a Weyl connection D. Let $\pi : Q \to M$ be the bundle of self-dual spaces on (M^4, c). Obviously, Q is also the bundle of self-dual spaces on $(L^* \otimes TM, c)$, where L is the line bundle of M^4.

Let $\rho : \mathrm{SO}(4, \mathbb{R}) \to \mathrm{SO}(3, \mathbb{R})$ be the morphism of Lie groups which to any $\pm(p_+, p_-) \in \mathrm{SO}(4, \mathbb{R})$ $\big(= (\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\{\pm 1\}\big)$ it associates $\pm p_+ \in \mathrm{SO}(3)$ $\big(= Sp(1)/\{\pm 1\}\big)$.

Denote by E the oriented Riemannian vector bundle of rank three associated, through ρ, to the bundle of positive orthonormal frames on $(L^* \otimes TM, c)$.

From Example 3.1.7 it follows that Q can be identified with the sphere bundle of E. Hence, D induces a connection $\mathscr{H} (\subseteq TQ)$ on Q.

Define the complex vector subbundle $\mathcal{B}$ of $\mathscr{H}^{\mathbb{C}}$

such that $\mathrm{d}\pi(\mathcal{B}_q) = q$, for any $q \in Q$.

Then $\mathcal{C} = \mathcal{B} \oplus (\ker \mathrm{d}p)^{0,1}$ is an almost complex structure on Q and $(Q, M, \pi, \mathcal{C})$ is an almost twistorial structure on M^4.

In contrast to Proposition 3.2.2 we have the following result.

Proposition 3.2.4. *Let (M^4, c) be a four-dimensional conformal manifold endowed with two Weyl connections D' and D''. Let τ' and τ'' be the almost twistorial structures of Example 3.2.3, associated to (M^4, c, D') and (M^4, c, D''), respectively.*

Then $\tau' = \tau''$.

Proof. Let α be the one-form on M^3 such that $D'' = D' + \alpha$, as connections on the line bundle of M^4.

Then, similarly to the proof of Proposition 3.2.2, we obtain that $\tau' = \tau''$ if and only if, for any $q \in Q$ and any $X, Y \in q$, we have $\alpha(X)Y + \alpha(Y)X - c(X,Y)\alpha^{\sharp}_{\pi(q)} \in q$. But, this is always satisfied, as any $q \in Q$ is isotropic. $\square$

Let (M^4, c) be a four-dimensional conformal manifold and let τ be the almost twistorial structure of Example 3.2.3, associated to (M^4, c, D), where D is any Weyl connection on (M^4, c). We call τ *the* *almost twistorial structure of* (M^4, c) .

Next, we characterise the integrability of the almost twistorial structures of Examples 3.2.1 and 3.2.3 .

The following result is due to [16] .

Theorem 3.2.5. *Let* (M^3, c, D) *be a three-dimensional Weyl space and let* τ *be its almost twistorial structure.*

Then the following assertions are equivalent:

(i) τ *is integrable.*

(ii) (M^3, c, D) *is Einstein–Weyl.*

Proof. This is an immediate consequence of Theorem 1.3.1 and Proposition A.1.9 . □

Next, we prove the following result of [4] .

Theorem 3.2.6. *Let* (M^4, c) *be a four-dimensional (oriented) conformal manifold and* τ *be its almost twistorial structure.*

Then the following assertions are equivalent:

(i) τ *is integrable.*

(ii) (M^4, c) *is anti-self-dual.*

Proof. This is an immediate consequence of Theorem 1.3.1 and Proposition A.1.11 . □

To give examples of twistor spaces we, firstly, observe that:

(i) Let (M^3, c, D) be a three-dimensional Einstein–Weyl space. Then the twistors on τ are pairs (γ, J) , where γ is a (maximal unparametrized) geodesic of D and J is an orthogonal complex structure on the normal bundle of γ (this follows from Proposition 1.2.1). Obviously, on the normal bundle of any geodesic there are just two orthogonal complex structures. Thus, as M^3 is assumed oriented, we may identify any twistor on (M^3, c, D) with an oriented geodesic.

(ii) The twistors of an anti-self-dual manifold (M^4, c) are pairs (x, J) , where $x \in M^4$ and J is a positive orthogonal complex structure on (T_xM, c_x) . Thus the twistors of an anti-self-dual manifold are just positive orthogonal complex structures.

Secondly, note that, by Corollary A.1.7 , for a constant curvature Riemannian manifold (M^m, g) we have that $(M^m, [g])$ is flat (that is, $W = 0$) and $(M^m, [g], \nabla)$ is Einstein–Weyl, where ∇ is the Levi-Civita connection of g ; in particular, if $m = 4$, then $(M^4, [g])$ is anti-self-dual.

We can now describe, at least the smooth structures, of some twistors spaces (see [5]).

Example 3.2.7. Let $(Q, M, \pi, \mathcal{C})$ be the twistorial structure of the Einstein–Weyl space determined by the three-dimensional Euclidean space $\mathbb{R}^3$, endowed with its canonical Riemannian metric.

Then $Q = S^2 \times \mathbb{R}^3$ and the foliation $\mathcal{C} \cap \overline{\mathcal{C}}$ on it is characterised by the fact that its restriction to $\{u\} \times \mathbb{R}^3$ is the foliation formed by the lines parallel to u, $(u \in S^2)$.

Therefore, as a smooth manifold, the twistor space Z of $\mathbb{R}^3$ is the quotient of the trivial vector bundle $S^2 \times \mathbb{R}^3$ through the tautological line bundle $\mathcal{L} \subseteq S^2 \times \mathbb{R}^3$ whose fibre, over any $u \in S^2$, is $\{u\} \times \mathbb{R}u$. Hence, $Z = TS^2$, embedded in $S^2 \times \mathbb{R}^3$ as $\mathcal{L}^{\perp}$.

This way, to any $(u, v) \in Z$ (that is, $(u, v) \in S^2 \times \mathbb{R}^3$ with u orthogonal on v) we associate the oriented line which passes through v and is parallel to u.

It can be proved that, as a complex manifold, $Z = \mathcal{O}(2)\,(= \otimes^2\mathcal{O}(1))$, where $\mathcal{O}(1)$ is the dual of the tautological holomorphic line bundle over $\mathbb{C}P^1$ (that is, $\mathcal{O}(-1) = \{(l, u) \,|\, l \in \mathbb{C}P^1,\, u \in l\} \subseteq \mathbb{C}P^1 \times \mathbb{C}^2$).

Example 3.2.8. Let $(Q, M, \pi, \mathcal{C})$ be the twistorial structure of the Einstein–Weyl space determined by the three-dimensional sphere S^3, endowed with its canonical Riemannian metric.

It is convenient to identify $\mathbb{R}^4 = \mathbb{H}$ and, in particular, S^3 with the group of unit quaternions $\mathrm{Sp}(1)$. Then $TS^3 = S^3 \times \mathrm{Im}\mathbb{H}$ such that the constant sections of $S^3 \times \mathrm{Im}\mathbb{H}$ correspond to left invariant vector fields on S^3.

Also, the geodesics of S^3 are just left translations of the subgroups with one parameter of S^3; equivalently, the geodesics of S^3 are the integral curves of the left invariant vector fields.

Therefore, $Q = S^3 \times S^2$ and the foliation $\mathcal{C} \cap \overline{\mathcal{C}}$ on it is characterised by the fact that its restriction to $S^3 \times \{u\}$ is the foliation formed by the integral curves of the left invariant vector field determined by u, $(u \in S^2)$.

Now, the flow of the left invariant vector field determined by any $u \in S^2 \,(\subseteq \mathrm{Im}\mathbb{H})$ is given by $S^3 \times \mathbb{R} \to S^3$, $(q,t) \mapsto q(\cos t + u \sin t)$. Hence, the leaf space of the foliation formed by its integral curves is S^2 and the projection $S^3 \to S^2$ is the projection of a Hopf bundle.

Therefore, as a smooth manifold, the twistor space Z of S^3 is $S^2 \times S^2$. It can be proved that, also as a complex manifold, $Z = S^2 \times S^2$ (where, for example, $S^2 = \mathbb{C}P^1$ via the stereographic projection from the South pole).

Example 3.2.9. Let $(Q, M, \pi, \mathcal{J})$ be the twistorial structure of $\mathbb{R}^4$. From Example 3.1.7 it follows that

$Q = S^2 \times \mathbb{R}^4$. Hence, as a smooth manifold, the twistor space Z, of $\mathbb{R}^4$, is $S^2 \times \mathbb{R}^4$.

Note that, at each $(u,v) \in Z$, the linear complex structure $\mathcal{J}_{(u,v)}$ on $T_{(u,v)}Z = T_uS^2 \times T_v\mathbb{R}^4$ is given as follows: on T_uS^2 is rotation though $\pi/2$ whilst on $T_v\mathbb{R}^4 = \mathbb{H}$ it acts by left multiplication with u.

It can be proved that, as a complex manifold, $Z = \mathcal{O}(1) \oplus \mathcal{O}(1)$.

Furthermore, by using that $S^4 \setminus \{\text{point}\} = \mathbb{R}^4$, as conformal manifolds, it can be shown that the twistor space of S^4 is $\mathbb{C}P^3$.

We strongly recommend, to the interested reader, that before searching for proofs of the statements, from Examples 3.2.7, 3.2.8 and 3.2.9, regarding the complex structures of the corresponding twistor spaces, to try to understand the Twistor Theory specific to the complex analytic category (see [28]).

Then the twistor space of a three-dimensional Einstein–Weyl space M is an open set of the twistor space of its complexification $M^{\mathbb{C}}$ which, in turn, is the space of coisotropic geodesic (complex) surfaces in $M^{\mathbb{C}}$. In fact, any oriented geodesic on M is the intersection of M with a coisotropic geodesic surface in $M^{\mathbb{C}}$.

For example, the complexification of S^3 is the group $\mathrm{SL}(2,\mathbb{C})$ of 2×2 complex matrices of determinant 1, endowed with the complex-Riemannian biinvariant metric which, on its Lie algebra, is given by the determinant. Then the space of (connected) coisotropic geodesic complex surfaces of $\mathrm{SL}(2,\mathbb{C})$ is $\mathbb{C}P^1\times\mathbb{C}P^1$.

Indeed, as the complex-Riemannian metric of $\mathrm{SL}(2,\mathbb{C})$ is biinvariant, any coisotropic geodesic surface on it is the left translation of a coisotropic (connected) Lie subgroup of $SL(2,\mathbb{C})$. In particular, the orthogonal complement of any isotropic element of $\mathfrak{sl}(2,\mathbb{C})$ (the Lie algebra of $SL(2,\mathbb{C})$) is a Lie subalgebra of $\mathfrak{sl}(2,\mathbb{C})$.

Now, $y\in\mathfrak{sl}(2,\mathbb{C})\setminus\{0\}$ is isotropic if and only if $\ker y$ is one-dimensional (this gives another proof of the fact that the space of isotropic directions on a three-dimensional Euclidean space is a complex projective line). Furthermore, $y^{\perp}$ is made of those elements $x\in\mathfrak{sl}(2,\mathbb{C})$ such that $x(\ker y)\subseteq\ker y$. Consequently, $a\in\exp(y^{\perp})\,(\subseteq\mathrm{SL}(2,\mathbb{C}))$ if and only if $a(\ker y)=\ker y$.

Therefore, similarly to Example 3.2.8, any coisotropic geodesic surface in $\mathrm{SL}(2,\mathbb{C})$ is the projection of a unique fibre of the surjective holomorphic submersion $\varphi:\mathrm{SL}(2,\mathbb{C})\times\mathbb{C}P^1\to\mathbb{C}P^1\times\mathbb{C}P^1$, characterised by $\varphi(a,\ker y)=\big(a(\ker y),\ker y\big)$, for any

$a \in \mathrm{SL}(2, \mathbb{C})$ and any isotropic $y \in \mathfrak{sl}(2, \mathbb{C}) \setminus \{0\}$.

Finally, note that for any isotropic $y \in \mathfrak{sl}(2, \mathbb{C}) \setminus \{0\}$, the restriction to S^3 of the map $\mathrm{SL}(2, \mathbb{C}) \to \mathbb{C}P^1$, $a \mapsto a(\ker y)$, is the projection of the Hopf bundle. Thus, the twistor space of S^3 is $\mathbb{C}P^1 \times \mathbb{C}P^1$.

3.3. Twistorial maps (basic facts and first examples)

We start this section with the definition of twistorial maps, suitably formulated for the purposes of this book (cf. [22]).

Definition 3.3.1. Let $\tau_M = (Q_M, M, \pi_M, \mathcal{C}^M)$ and $\tau_N = (Q_N, N, \pi_N, \mathcal{C}^N)$ be almost twistorial structures and let $\varphi : M \to N$ be a map. Suppose that there exists a locally trivial fibre subspace $\pi_{M,\varphi} : Q_{M,\varphi} \to M$ of $\pi_M : Q_M \to M$ and a map $\varPhi : Q_{M,\varphi} \to Q_N$ with the properties:

1) $\mathcal{C}^M$ induces an almost co-CR structure $\mathcal{C}^{M,\varphi}$ on $Q_{M,\varphi}$ and almost complex structures on each fibre of $\pi_{M,\varphi}$.

2) $\mathrm{d}\pi_M(\mathcal{C}^M_q) = \mathrm{d}\pi_{M,\varphi}(\mathcal{C}^{M,\varphi}_q)$, for any $q \in Q_{M,\varphi}$.

3) $\varphi \circ \pi_{M,\varphi} = \pi_N \circ \varPhi$.

$$\begin{tikzcd}
Q_M \arrow[dr, "\pi_M"'] & Q_{M,\varphi} \arrow[l, hook'] \arrow[r, "\varPhi"] \arrow[d, "\pi_{M,\varphi}"] & Q_N \arrow[d, "\pi_N"] \\
& M \arrow[r, "\varphi"'] & N
\end{tikzcd}$$

Then $\varphi : (M, \tau_M) \to (N, \tau_N)$ is a *twistorial map (with respect to Φ)* if the map $\Phi : (Q_{M,\varphi}, \mathcal{C}^{M,\varphi}) \to (Q_N, \mathcal{C}^N)$ is holomorphic. If, further, $\mathcal{C}^{M,\varphi}$ and $\mathcal{C}^N$ are simple co-CR structures, with $(Q_{M,\varphi}, \mathcal{C}^{M,\varphi}) \to Z_{M,\varphi}$ and $(Q_N, \mathcal{C}^N) \to Z_N$, respectively, the corresponding holomorphic submersions onto complex manifolds, then Φ induces a holomorphic map $Z_\varphi : Z_{M,\varphi} \to Z_N$ which is called the *twistorial representation* of φ.

Remark 3.3.2. With the same notations as in Definition 3.3.1, let $\tau_{M,\varphi} = (Q_{M,\varphi}, M, \pi_{M,\varphi}, \mathcal{C}^{M,\varphi})$.

Then $\tau_{M,\varphi}$ is an almost twistorial structure on M which is integrable if τ_M is integrable.

Furthermore, if τ_M is simple then, also, $\tau_{M,\varphi}$ is simple and the twistor space $Z_{M,\varphi}$ of $\tau_{M,\varphi}$ is a complex submanifold of the twistor space Z_M of τ_M. Moreover, for any $z \in Z_{M,\varphi}$, the corresponding immersed submanifolds of M endowed with linear complex structures on their normal bundles of Proposition 3.1.2, determined by τ_M and $\tau_{M,\varphi}$, are equal (for this is required condition (2) of Definition 3.3.1).

Next, we start giving examples of twistorial maps between manifolds endowed with the almost twistorial structures of Examples 3.1.8, 3.2.1 or 3.2.3.

Example 3.3.3. Let (M^2, c_M) and (N^2, c_N) be two-dimensional conformal manifolds.

Denote by $\tau_M = (M, M, \mathrm{Id}_M, J^M)$ and $\tau_N = (N, N, \mathrm{Id}_N, J^N)$ the twistorial structures of Example 3.1.8 associated to (M^2, c_M) and (N^2, c_N), respectively.

Let $\varphi : M^2 \to N^2$ be a map. Obviously, $\varphi : (M^2, \tau_M) \to (N^2, \tau_N)$ is twistorial (with respect to φ) if and only if $\varphi : (M^2, J^M) \to (N^2, J^N)$ is holomorphic.

Example 3.3.4. Let (M^3, c, D) be a three-dimensional Weyl space and let N^2 be an (oriented) surface in M^3. Let $\tau_M = (Q, M, \pi, \mathcal{C})$ be the almost twistorial structure of Example 3.2.1, associated to (M^3, c, D), and let $\tau_N = (N, N, \mathrm{Id}_N, J)$ be the twistorial structure of Example 3.1.8, associated to $(N^2, c|_N)$.

Let q be the section of Q^5 over N^2 such that, $q_x = \left(T_x^{0,1}N\right)^{\perp}$, for any $x \in N$. Then the following assertions are equivalent:

(i) The inclusion $(N^2, \tau_N) \to (M^3, \tau_M)$ is twistorial (with respect to q).

(ii) $q : (N^2, J) \to (P^5, \mathcal{C})$ is holomorphic.

(iii) N^2 is an umbilical submanifold of (M^3, c).

Indeed, (i)$\Longleftrightarrow$(ii) by definition, whilst (ii) holds if and only if, for any $X \in T^{0,1}N$, we have $\mathrm{d}q(X) \in$

$\mathcal{C}$. Hence, from Proposition 1.2.1 it follows that (ii) is equivalent to $c(D_X Y, X) = 0$, for any nowhere zero local section X of $T^{0,1}N$ and any section Y of the orthogonal complement of X in $T^{\mathbb{C}}M|_N$; equivalently, $c(U, D_X X) = -c(D_X U, X) = 0$ for any nowhere zero local section X of $T^{0,1}N$ and any U orthogonal to N^2.

Thus, we have shown that (ii) holds if and only if the second fundamental form of N^2 in (M^3, c, D) is zero along isotropic directions; that is, (ii)$\Longleftrightarrow$(iii).

Example 3.3.5. Let (M^4, c) be a four-dimensional conformal manifold and let N^2 be an oriented surface in M^4. Let $\tau_M = (Q, M, \pi, \mathcal{J})$ be the almost twistorial structure of Example 3.2.3, associated to (M^4, c), and let $\tau_N = (N, N, \mathrm{Id}_N, J)$ be the twistorial structure of Example 3.1.8, associated to $(N^2, c|_N)$.

Let q be the section of Q^6 over N^2 such that $q_x|_{T_xN} = J_x$, for any $x \in N$. Then the following assertions are equivalent:

(i) The inclusion $(N^2, \tau_N) \to (M^4, \tau_M)$ is twistorial (with respect to q).

(ii) $q : (N^2, J) \to (Q^6, \mathcal{J})$ is holomorphic.

If we endow M^4 with the opposite orientation then Example 3.2.3 gives another almost twistorial structure $\widetilde{\tau}_M = (\widetilde{Q}, M, \widetilde{\pi}, \widetilde{\mathcal{J}})$.

Then an argument as in Example 3.3.4 shows that the following assertions are equivalent:
(a) $(N^2, \tau_N) \to (M^4, \tau_M)$ and $(N^2, \tau_N) \to (M^4, \widetilde{\tau}_M)$ are twistorial.
(b) N^2 is an umbilical submanifold of (M^4, c).

3.4. Twistorial maps to two-dimensional conformal manifolds

In this section we discuss the twistorial maps from three-dimensional Weyl spaces, or four-dimensional conformal manifolds to two-dimensional conformal manifolds.

Also, we prove that the Weyl connection D of a four-dimensional Hermitian manifold (M^4, c, J) is characterised by $DJ = 0$.

Example 3.4.1. Let (M^3, c_M, D) be a three-dimensional Weyl space and let (N^2, c_N) be a two-dimensional conformal manifold. Denote by $\tau_M = (Q, M, \pi, \mathcal{C})$ the almost twistorial structure of Example 3.2.1, associated to (M^3, c_M, D), and by $\tau_N = (N, N, \mathrm{Id}_N, J)$ the twistorial structure of Example 3.1.8, associated to (N^2, c_N).

Let $\varphi : M^3 \to N^2$ be a submersion. Orient $\mathscr{V} = \ker \mathrm{d}\varphi$ and $\mathscr{H} = \mathscr{V}^{\perp}$ such that the isomorphisms $\mathscr{H} = \varphi^*(TN)$ and $TM = \mathscr{V} \oplus \mathscr{H}$ be orientation preserving. Let $\mathscr{H}_+ \subseteq \mathscr{H}^{\mathbb{C}}$ be the eigenbundle, corresponding to $-\mathrm{i}$, of the positive orthogonal complex structure on $(\mathscr{H}, (c_M)|_{\mathscr{H}})$.

Then $\mathcal{C}^\varphi = \mathscr{V} \oplus \mathscr{H}_+$ is a coisotropic almost co-CR structure on M^3. Denote by q_φ the corresponding section of Q and define $\varPhi = \varphi \circ {q_\varphi}^{-1} : q_\varphi(M) \to N$.

$$\begin{tikzcd} Q \arrow[dr, "\pi"'] & q_\varphi(M) \arrow[l, hook'] \arrow[r, "\varPhi"] & N \arrow[d, "\mathrm{Id}_N"] \\ & M \arrow[u, "q_\varphi"] \arrow[r, "\varphi"'] & N \end{tikzcd}$$

The following assertions are equivalent:

(i) $\varphi : (M^3, \tau_M) \to (N^2, \tau_N)$ is twistorial (with respect to $\varPhi$).

(ii) $\varphi : (M^3, c_M) \to (N^2, c_N)$ is horizontally conformal and $q_\varphi : (M^3, \mathcal{C}^\varphi) \to (Q^5, \mathcal{C})$ is holomorphic.

(iii) $\varphi : (M^3, c_M) \to (N^2, c_N)$ is horizontally conformal and the fibres of $\varPhi$ are tangent to the connection induced by D on Q^5.

(iv) $\varphi : (M^3, c_M, D) \to (N^2, c_N)$ is a horizontally conformal submersion with geodesic fibres.

Indeed, firstly, note that (i) is equivalent to the following:

(i$'$) $\mathcal{C}$ induces an almost co-CR structure on $q_\varphi(M)$ which is mapped by $\pi|_{q_\varphi(M)}$ onto $\mathcal{C}^\varphi$, and with respect to which $\varPhi$ is a holomorphic map onto (N^2, J).

Obviously, the first condition of (i$'$) holds if and only if the map $q_{\varphi} : (M^3, \mathcal{C}^{\varphi}) \to (Q^5, \mathcal{C})$ is holomorphic. Then, assuming that this holds, the second condition of (i′) is equivalent to $\varphi : (M^3, \mathcal{C}_{\varphi}) \to (N^2, J)$ is holomorphic; that is, $T^{0,1}N = \mathrm{d}\varphi(\mathcal{C}^{\varphi}) = \mathrm{d}\varphi(\mathscr{H}_+)$.

To complete the proof of (i)$\Longleftrightarrow$(ii), just note that a nonconstant submersion, between conformal manifolds, is horizontally conformal if and only if it maps horizontal isotropic directions onto isotropic directions.

For the proof of (ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) we may assume φ horizontally conformal.

Now, as we have seen before proving Proposition 2.5.3, if φ is horizontally conformal then $\mathscr{H}$ is umbilical; equivalently, $D_X X$ is horizontal for any section X of $\mathscr{H}_+$.

Let U be a nowhere zero section of $\mathscr{V}$. Then, by applying Proposition 1.2.1, we obtain that $q_{\varphi} : (M^3, \mathcal{C}^{\varphi}) \to (Q^5, \mathcal{C})$ is holomorphic if and only if $c_M(D_X U, X) = c_M(D_U U, X) = 0$, for any section X of $\mathscr{H}_+$. As φ is horizontally conformal, this holds if and only if $c_M(D_U U, X) = 0$, for any section X of $\mathscr{H}_+$; that is, the fibres of φ are geodesics. Hence, (ii)$\Longleftrightarrow$(iv).

Finally, the fibres of $\varPhi$ are tangent to the connection induced by D on Q^5 if and only if $q_{\varphi}(\mathscr{V})$

is horizontal on Q^5 ; that is, $c_M(D_U U, X) = 0$, for any section X of $\mathscr{H}_+$. Hence, (ii)$\Longleftrightarrow$(iii) .

From Example 3.4.1 we immediately obtain the following:

Corollary 3.4.2. *Let (M^3, c, D) be a three-dimensional Weyl space and let $(Q, M, \pi, \mathcal{C})$ be its almost twistorial structure.*

Let q be a section of Q^5 and let $\mathcal{C}^q$ be the corresponding coisotropic almost co-CR structure on (M^3, c) .

Then the following assertions are equivalent:

(i) *$q : (M^3, \mathcal{C}^q) \to (Q^5, \mathcal{C})$ is holomorphic.*

(ii) *$\mathcal{C}^q$ is integrable and $\mathcal{C}^q \cap \overline{\mathcal{C}^q}$ is a foliation by geodesics on (M^3, D) .*

Also, note that, a coisotropic almost co-CR structure $\mathcal{C}$ on a three-dimensional conformal manifold (M^3, c) is integrable if and only if the foliation $\mathcal{C} \cap \overline{\mathcal{C}}$ is locally defined by horizontally conformal submersions (that is, $\mathcal{C} \cap \overline{\mathcal{C}}$ is a *conformal foliation*).

Example 3.4.3. Let (M^4, c_M) and (N^2, c_N) be conformal manifolds of dimensions four and two, respectively. Let $\tau_M = (Q, M, \pi, \mathcal{J})$ be the almost twistorial structure of Example 3.2.3 , associated to (M^4, c_M) , and let $\tau_N = (N, N, \mathrm{Id}_N, J)$ be the twistorial structure of Example 3.1.8 , associated to

(N^2, c_N) .

Let $\varphi : M^4 \to N^2$ be a submersion. Orient $\mathscr{V} = \ker \mathrm{d}\varphi$ and $\mathscr{H} = \mathscr{V}^\perp$ such that the isomorphisms $\mathscr{H} = \varphi^*(TN)$ and $TM = \mathscr{V} \oplus \mathscr{H}$ be orientation preserving.

Let J^φ be the positive almost Hermitian structure on (M^4, c_M) such that $J^\varphi|_{\mathscr{V}}$ and $J^\varphi|_{\mathscr{H}}$ are the rotations of angle $\pi/2$.

Denote by q_φ the section of Q^6 corresponding to J^φ and define $\varPhi = \varphi \circ {p_\varphi}^{-1} : p_\varphi(M) \to N$. Note that, q_φ restricted to any fibre $\varphi^{-1}(y)$ is equal to $q_{\varphi^{-1}(y)}$ of Example 3.3.5 , $(y \in \varphi(M)\,)$.

The following assertions are equivalent:

(i) $\varphi : (M^4, \tau_M) \to (N^2, \tau_N)$ is twistorial (with respect to $\varPhi$).

(ii) $\varphi : (M^4, c_M) \to (N^2, c_N)$ is horizontally conformal and $q_\varphi : (M^4, J^\varphi) \to (Q^6, \mathcal{J})$ is holomorphic.

(iii) $\varphi : (M^4, c_M) \to (N^2, c_N)$ is horizontally conformal and its fibres are twistorial, in the sense of Example 3.3.5 .

(iv) $\varphi : (M^4, c_M) \to (N^2, c_N)$ is horizontally conformal and J^φ is integrable.

Indeed, firstly, note that (i) is equivalent to the following:

(i′) $\mathcal{J}$ induces an almost co-CR structure on

$q_\varphi(M)$ with respect to which $\varPhi$ is a holomorphic map onto (N^2, J) .

From the definition of $\mathcal{J}$ it follows quickly that q_φ is holomorphic if and only if $\mathcal{J}$ induces an almost co-CR structure on $q_\varphi(M)$.

Also, from the definition of J^φ it follows quickly that $\varPhi$ is holomorphic if and only if $\mathrm{d}\varphi(\mathscr{H}_+)$ is isotropic on (N^2, c_N) , where $\mathscr{H}_+$ is the eigenbundle, corresponding to $-\mathrm{i}$, of the positive orthogonal complex structure on $(\mathscr{H}, (c_M)|_{\mathscr{H}})$.

We have, thus, proved that (i)$\Longleftrightarrow$(ii) .

For the proof of (ii)$\Longleftrightarrow$(iii)$\Longleftrightarrow$(iv) we may assume φ horizontally conformal. Hence, $\mathscr{H}$ is umbilical; equivalently, $D_X X$ is horizontal for any section X of $\mathscr{H}_+$, where D is any Weyl connection on (M^4, c_M) .

As $\mathscr{H}_+$ is isotropic and D is conformal, for any section X of $\mathscr{H}_+$, we have $c_M(D_X X, X) = 0$. Thus, for any section X of $\mathscr{H}_+$, we have that, also, $D_X X$ is a section of $\mathscr{H}_+$. Together with Proposition 1.2.1 , this shows that the differential of q_φ , restricted to $\mathscr{H}$, intertwines $J^\varphi|_{\mathscr{H}}$ and $\mathcal{J}$.

On the other hand, the fibres of φ are twistorial if and only if the differential of q_φ , restricted to $\mathscr{V}$, intertwines $J^\varphi|_{\mathscr{V}}$ and $\mathcal{J}$. Therefore (ii)$\Longleftrightarrow$(iii) .

For (ii)$\Longleftrightarrow$(iv) , it is sufficient to prove that q_φ

is holomorphic if and only if J^{φ} is integrable.

For this, firstly, note that, by Proposition 1.2.1 , we have that q_{φ} is holomorphic if and only if $D_X Y$ is a section of $T^{0,1}M$ for any sections X and Y of $T^{0,1}M$, where $T^{0,1}M$ is the eigenbundle of J^{φ}, corresponding to $-\mathrm{i}$.

Thus, if q_{φ} is holomorphic then $[X,Y] = D_X Y - D_Y X$ is a section of $T^{0,1}M$, for any sections X and Y of $T^{0,1}M$. Hence, J^{φ} is integrable.

Finally, as $T^{0,1}M$ is isotropic, by applying (2.2.1) , we obtain

$$\begin{aligned}
&2\,c_M(D_X Y, Z) \\
&\quad = -c_M(X,[Y,Z]) + c_M(Y,[Z,X]) + c_M(Z,[X,Y])\,,
\end{aligned}$$

for any sections X, Y and Z of $T^{0,1}M$.

Therefore if J^{φ} is integrable then q_{φ} is holomorphic.

Next, we prove the following result.

Corollary 3.4.4. *Let (M^4, c) be a four-dimensional conformal manifold and let $(Q, M, \pi, \mathcal{J})$ be its almost twistorial structure.*

Let J be a positive almost Hermitian structure on (M^4, c) and let q be the corresponding section of Q.

Then the following assertions are equivalent:

(i) *$q : (M^4, J) \to (Q^6, \mathcal{J})$ is holomorphic.*

(ii) $D_{JX}J = JD_XJ$, *for any* $X \in TM$, *where* D *is any Weyl connection on* (M^4, c) *.*

(iii) J *is integrable.*

PROOF. In Example 3.4.3 we have seen that assertions (i) and (iii) are equivalent. Also, assertion (i) holds if and only if D_XY is a section of $T^{0,1}M$, for any sections X and Y of $T^{0,1}M$.

As $T^{0,1}M$ is generated by all $X + \mathrm{i}JX$, with X a vector field on M^4, assertion (i) is equivalent to the fact that, for any vector fields X and Y on M^4, we have $JD_{X+\mathrm{i}JX}(Y + \mathrm{i}JY) = -\mathrm{i}\, D_{X+\mathrm{i}JX}(Y + \mathrm{i}JY)$; equivalently,

$$J\big(D_XY - D_{JX}(JY)\big) = D_X(JY) + D_{JX}Y \ .$$

The proof of (i)$\Longleftrightarrow$(ii) follows quickly. $\square$

Remark 3.4.5. Corollary 3.4.4 implies that on an anti-self-dual manifold (M^4, c) , locally, there exist many positive Hermitian structures. Indeed, if J is a positive Hermitian structure on (M^4, c) then $q(M)$ is a complex surface in $(Q, \mathcal{J})$, where q is the section of Q corresponding to J. Moreover, any complex surface in $(Q, \mathcal{J})$ which intersects each fibre of π at most once and transversely is the image of the section of Q corresponding to a local positive Hermitian structure on (M^4, c) .

Similarly, it can be shown, by using Corollary 3.4.2, that on a three-dimensional Einstein–Weyl space, locally, there exist many horizontally conformal submersions with geodesic fibres.

Furthermore, these properties characterise the anti-self-dual manifolds and the three-dimensional Einstein–Weyl spaces among the four-dimensional conformal manifolds and the three-dimensional Weyl spaces, respectively.

Corollary 3.4.4, with assertion (i) suitably adapted, holds for any even-dimensional conformal manifold. However, the next result is specific to dimension four.

Proposition 3.4.6. *Let (M^4, c, J) be a four-dimensional almost Hermitian manifold and let D be a Weyl connection on (M^4, c).*

Then the following assertions are equivalent:

(i) *D is the Weyl connection of (M^4, c, J).*

(ii) *$D_{JX}J = -JD_XJ$, for any $X \in TM$.*

Proof. By Example 3.2.3, the bundle of positive orthogonal complex structures on (M^4, c) is the sphere bundle of an oriented Riemannian vector bundle E, of rank three, on which D induces a Riemannian connection.

Furthermore, any positive orthonormal frame on E satisfies the quaternionic identities. Hence, locally, there exist almost Hermitian structures I and K on (M^4, c) such that $IJK = -\mathrm{Id}_{TM}$.

Also, there exist one-forms a and c such that

$$DJ = c \otimes I - a \otimes K \; ; \tag{3.4.1}$$

in particular, we have

$$\begin{aligned} D_{JX}J &= c(JX)I - a(JX)K \; , \\ -J(D_X J) &= c(X)K + a(X)I \; , \end{aligned} \tag{3.4.2}$$

for any $X \in TM$.

Relations (3.4.2) show that (ii) holds if and only if $a = c \circ J$.

On the other hand, if g is a representative of c and X is a (local) vector field such that $g(X, X) = 1$ then (X, IX, JX, KX) is a positive orthonormal frame on (M^4, g). Therefore

$$\begin{aligned} \mathrm{trace}_g(DJ) =& (D_X J)(X) + (D_{IX}J)(IX) \\ & + (D_{JX}J)(JX) + (D_{KX}J)(KX) \; . \end{aligned}$$

Then, by using (3.4.1), after a straightforward calculation, we obtain

$$\begin{aligned} \mathrm{trace}_g(DJ) =& \big(a(KX) - c(IX)\big)X \\ & + \big(a(JX) + c(X)\big)IX \\ & - \big(a(IX) + c(KX)\big)JX \\ & - \big(a(X) - c(JX)\big)KX \; . \end{aligned} \tag{3.4.3}$$

Relation (3.4.3) shows that (i) holds if and only if $a = c \circ J$.

The proof is complete. □

We end this section with the following immediate consequence of Corollary 3.4.4 and Proposition 3.4.6 .

Corollary 3.4.7. *Let (M^4, c, J) be a four-dimensional Hermitian manifold and let D be a Weyl connection on (M^4, c).*

Then the following assertions are equivalent:

(i) *D is the Weyl connection of (M^4, c, J) .*

(ii) *$DJ = 0$.*

3.5. Twistorial maps from 4-dim conformal manifolds to 3-dim Weyl spaces

Let (M^4, c_M) be a four-dimensional oriented conformal manifold and let (N^3, c_N, D^N) be a three-dimensional Weyl space.

Denote by $\tau_M = (Q_M, M, \pi_M, \mathcal{J})$ the almost twistorial structure of Example 3.2.3 , associated to (M^4, c_M) , and by $\tau_N = (Q_N, N, \pi_N, \mathcal{F})$ the almost twistorial structure of Example 3.2.1 , associated to (N^3, c_N, D^N) .

Let $\varphi : M^4 \to N^3$ be a submersion. Denote, as usual, $\mathscr{V} = \ker \mathrm{d}\varphi$ and $\mathscr{H} = \mathscr{V}^{\perp}$. Orient $\mathscr{V} = \ker \mathrm{d}\varphi$ and $\mathscr{H} = \mathscr{V}^{\perp}$ such that the isomorphisms

$\mathscr{H} = \varphi^*(TN)$ and $TM = \mathscr{V} \oplus \mathscr{H}$ be orientation preserving. Let U be a positive (nowhere zero) section of $\mathscr{V}$.

We define a map $\Phi : Q_M \to Q_N$ as follows: if $J \in Q_M$ then $\Phi(J)$ is the two-dimensional coisotropic space on $\big(T_{\varphi(\pi(J))}N, h\big)$ determined by $\mathrm{d}\varphi(JU)$, via Example 3.1.6, where h is the representative of $(c_N)_{\varphi(\pi(J))}$ with respect to which $\mathrm{d}\varphi(JU)$ has length 1.

It is easy to check that Φ doesn't depend of U.

Also, if $\varphi : (M^4, c_M) \to (N^3, c_N)$ is horizontally conformal then, for any $J \in Q_M$, we have $\Phi(J) = \mathrm{d}\varphi(T^J M)$, where $T^J M$ is the eigenspace of J corresponding to $-\mathrm{i}$.

Next, we prove the main result of this section.

Theorem 3.5.1. *The following assertions are equivalent:*

(i) $\varphi : (M^4, \tau_M) \to (N^3, \tau_N)$ *is twistorial (with respect to* Φ*).*

(ii) φ *is horizontally conformal and*

$$(D\mathrm{d}\varphi)(T^J M, T^J M) \subseteq \Phi(J) \ ,$$

for any $J \in Q_M$*, where* D *is the connection on* $\mathrm{Hom}(TM, \varphi^*(TN))$ *induced by* D^N *and some Weyl connection on* (M^4, c_M)*.*

To prove Theorem 3.5.1 we need some preliminary results.

Lemma 3.5.2. *For* $A \in \mathrm{GL}(3, \mathbb{R})$ *let* $\psi_A : S^2 \to S^2$ *be defined by* $\psi_A(x) = \frac{1}{||Ax||} Ax$ *, for any* $x \in S^2$.

Then the following assertions are equivalent:

(i) ψ_A *is holomorphic.*

(ii) $A \in \mathrm{CO}(3, \mathbb{R})$ *and* $\det A > 0$.

Proof. Assertion (i) is equivalent to the following:

(i′) ψ_A is conformal and preserves the orientations.

By the polar decomposition, there exists a linear isometry U of $\mathbb{R}^3$ and positive numbers λ_1 , λ_2 , λ_3 such that $A = UP$, where $P_{ij} = \lambda_i \delta_{ij}$, for any $i, j \in \{1, 2, 3\}$.

We claim that ψ_P is conformal if and only if $\lambda_1 = \lambda_2 = \lambda_3$. Indeed, firstly note that, ψ_P preserves the vectors of the canonical basis of $\mathbb{R}^3$. Furthermore, a straightforward calculation shows that, under the identification $T_{(1,0,0)} S^2 = \mathbb{R}^2$, where $\mathbb{R}^2 \subseteq \mathbb{R}^3$ is given by $x_1 = 0$, we have

$$(\mathrm{d}\psi_P)_{(1,0,0)} = \begin{pmatrix} \lambda_1^{-1}\lambda_2 & 0 \\ 0 & \lambda_1^{-1}\lambda_3 \end{pmatrix} .$$

Thus, $(\mathrm{d}\psi_P)_{(1,0,0)}$ is conformal if and only if $\lambda_2 = \lambda_3$. Similarly, $(\mathrm{d}\psi_P)_{(0,1,0)}$ is conformal if and only if

$\lambda_1 = \lambda_3$. Therefore $\psi_A = U \circ \psi_P$ is conformal if and only if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda\,(>0)$. Then $\psi_A = U|_{S^2}$ and, as $\det A = \lambda^3 \det U$, the proof is complete. $\square$

Proof of Theorem 3.5.1. Denote by $\tau_M = (Q_M, M, \pi_M, \mathcal{J})$ and $\tau_N = (Q_N, N, \pi_N, \mathcal{F})$ the almost twistorial structures associated to (M^4, c_M) and (N^3, c_N, D^N), respectively, where (M^4, c_M) is endowed with a Weyl connection D^M.

If we apply Lemma 3.5.2, with A equal to the differential of φ, at each point of M^4, we obtain that Φ restricted to each fibre of π_M is holomorphic if and only if φ is horizontally conformal.

Thus, we may assume φ horizontally conformal. Then $\Phi : Q_M \to Q_N$ is given by $\Phi(J) = \mathrm{d}\varphi(T^J M)$, for any $J \in Q_M$.

Therefore (i) holds if and only if $\mathrm{d}\Phi(\mathscr{B}) \subseteq \mathcal{F}$, where $\mathscr{B}$ is the complex vector subbundle of $T^{\mathbb{C}}Q_M$ such that $\mathscr{B}_J$ is the horizontal lift, with respect to D^M, of $T^J M\ (\subseteq T^{\mathbb{C}}_{\pi_M(J)}M)$, for any $J \in Q_M$.

Let $J_0 \in Q_M$, $Y_0 \in T^{J_0}M$ and denote $x_0 = \pi_M(J_0)$. Let P^2 be a surface in M^4 which is tangent to Y_0; equivalently, $x_0 \in P$ and $T^{\mathbb{C}}_{x_0}P$ is spanned by Y_0 and $\overline{Y_0}$. As $\mathrm{d}\varphi(T_{x_0}P)$ is a two-dimensional vector space tangent to N^3, by passing, if necessary, to an open neighbourhood of x_0 in P^2, we may suppose that $\varphi(P)$ is a surface in N^3. Furthermore, we may suppose that there exists a section q of Q_M over P^2 such that $q_{x_0} = J_0$ and $q(P)$ is horizontal at J_0, with respect to D^M; equivalently, $\mathrm{d}q(Y_0) \in \mathscr{B}_{J_0}$.

Then there exists a section $\check{q}$ of Q_N over $\varphi(P)$ such that $\Phi \circ q = \check{q} \circ \varphi$. Furthermore, if Y is a section of q (over P^2), such that $Y_{x_0} = Y_0$, then
(3.5.1)

$$(D\mathrm{d}\varphi)(Y,Y) = D^N_{\mathrm{d}\varphi(Y)}\big(\mathrm{d}\varphi(Y)\big) - \mathrm{d}\varphi(D^M_Y Y)\,.$$

As $q(P)$ is horizontal at x_0, we have $D^M_{Y_0}Y \in T^{J_0}M$, which implies that $\mathrm{d}\varphi(D^M_{Y_0}Y) \in \Phi(J_0)$. Thus, by (3.5.1), we have $(D\mathrm{d}\varphi)(Y_0,Y_0) \in \Phi(J_0)$ if and only if $D^N_{\mathrm{d}\varphi(Y_0)}\big(\mathrm{d}\varphi(Y)\big) \in \Phi(J_0)$. Now, if Y_0 is not horizontal then this is equivalent to $\mathrm{d}\check{q}\big(\mathrm{d}\varphi(Y_0)\big) \in \mathcal{C}^N_{\check{q}(\varphi(x_0))}$. But $\Phi \circ q = \check{q} \circ \varphi$ and, hence, if Y_0 is not horizontal then $(D\mathrm{d}\varphi)(Y_0,Y_0) \in \Phi(J_0)$ if and only if $\mathrm{d}\Phi\big(\mathrm{d}q(Y_0)\big) \in \mathcal{C}^N_{\Phi(q_0)}$. (Note that, if Y_0 is horizontal then $(D\mathrm{d}\varphi)(Y_0,Y_0) \in \Phi(J_0)$, due to the horizontal conformality of φ.)

The proof is complete. $\square$

The next result shows that the twistorial maps from anti-self-dual manifolds to three-dimensional Einstein–Weyl spaces behave like morphisms.

Proposition 3.5.3. *Let (M^4, c_M) be an anti-self-dual manifold and let (N^3, c_N, D^N) be a three-dimensional Einstein–Weyl space. Also, let φ be a* *horizontally conformal submersion from* (M^4, c_M) *to* (N^3, c_N) *.*

Then the following assertions are equivalent:

(i) $\varphi : (M^4, \tau_M) \to (N^3, \tau_N)$ *is twistorial, where* τ_M *and* τ_N *are the almost twistorial structures of* (M^4, c_M) *and* (N^3, c_N, D^N) *, respectively.*

(ii) *For any horizontally conformal submersion* ψ *with geodesic fibres, defined on* (N^3, c_N, D^N) *, locally, we have that* $\psi \circ \varphi$ *is twistorial (in the sense of Example 3.4.3).*

Proof. Let $\tau_M = (Q_M, M, \pi_M, \mathcal{J})$ and $\tau_N = (Q_N, N, \pi_N, \mathcal{F})$, and let $\psi : (N^3, c_N, D^N) \to (P^2, c_P)$ be a horizontally conformal submersion with geodesic fibres (recall that, by Example 3.4.1 , we have that ψ is twistorial).

Denote by q_ψ and $q_{\psi\circ\varphi}$ the sections of Q_N and Q_M determined by ψ and $\psi\circ\varphi$, as in Examples 3.4.1 and 3.4.3 , respectively. Note that, $q_\psi \circ \varphi = \Phi \circ q_{\psi\circ\varphi}$ and $q_{\psi\circ\varphi}(M) = \Phi^{-1}(q_\psi(N))$.

If (i) holds then $q_{\psi\circ\varphi}(M) = \Phi^{-1}(q_\psi(N))$ is a complex surface in $(Q_M, \mathcal{J})$. Hence, by Example 3.4.3 , we have that $\psi \circ \varphi$ is twistorial.

Conversely, if $\psi\circ\varphi$ is twistorial then $q_{\psi\circ\varphi}(M)$ is a complex surface in $(Q_M, \mathcal{J})$ and, from the fact that

$q_\psi \circ \varphi = \Phi \circ q_{\psi\circ\varphi}$, it follows that $\Phi|_{q_{\psi\circ\varphi}(M)}$ is holomorphic. As φ is horizontally conformal, together with Lemma 3.5.2, this shows that (ii)$\Longrightarrow$(i). $\square$

Let $\mathscr{V}$ be a one-dimensional oriented foliation on a conformal manifold (M^m, c). Then, for any positive section U of $\mathscr{V}$, there exists a unique representative g of c such that $g(U,U) = 1$. Consequently, c induces an isomorphism of oriented line bundles between $\mathscr{V}$ and the line bundle of M^m. Furthermore, as any representative of c corresponds to a representative of $c|_{\mathscr{H}}$, where $\mathscr{H} = \mathscr{V}^\perp$, we also have an isomorphism of oriented line bundles between $\mathscr{V}$ and the line bundle of $\mathscr{H}$.

Therefore the Hodge $*$-operator $*_{\mathscr{H}}$ of $(\mathscr{H}, c|_{\mathscr{H}})$ defines, for example, an isomorphism of vector bundles between $\Lambda^2\mathscr{H}^*$ and $\mathscr{V}^{m-4} \otimes \Lambda^{m-3}\mathscr{H}^*$. We shall apply $*_{\mathscr{H}}$ to the integrability tensor $I^{\mathscr{H}}$ of $\mathscr{H}$ which, by definition, is the $\mathscr{V}$-valued two-form on M^m such that $I^{\mathscr{H}}(X,Y) = -\mathscr{V}[X,Y]$, for any horizontal vector fields X and Y. In particular, if $m = 4$ then $*_{\mathscr{H}} I^{\mathscr{H}}$ is a (horizontal) one-form on M^4.

The following result is due to [8].

Corollary 3.5.4. *Let* $\varphi : (M^4, c_M) \to (N^3, c_N, D^N)$ *be a horizontally conformal submersion. Then the following assertions are equivalent:*

(i) $\varphi : (M^4, \tau_M) \to (N^3, \tau_N)$ *is twistorial, where* τ_M *and* τ_N *are the almost twistorial structures of* (M^4, c_M) *and* (N^3, c_N, D^N) *, respectively.*

(ii) *For any* $X \in \mathscr{H}$*, we have* $\varphi^*(D^N)_X = D_X + (*_{\mathscr{H}} I^{\mathscr{H}})(X)$ *, as connections on* $\mathscr{V}$*, where* D *is the Weyl connection of* $(M^4, c_M, \mathscr{V})$ *.*

Proof. Let g and h be representatives of c_M and c_N , respectively, with respect to which $\varphi : (M^4, g) \to (N^3, h)$ is a Riemannian submersion. Let ∇ be the Levi-Civita connection of (M^4, g) and let (U, X_1, X_2, X_3) be a positive local orthonormal frame on (M^4, g) such that U is vertical. We may suppose that X_j , $(j = 1, 2, 3)$, are projectable with respect to φ ; in particular, $[U, X_1]$ is vertical.

Let J be the positive almost Hermitian structure, locally defined on (M^4, g) , whose eigenbundle corresponding to $-\mathrm{i}$ is spanned by $U + \mathrm{i}X_1$ and $X_2 + \mathrm{i}X_3$.

Then $(D\mathrm{d}\varphi)(T^J M, T^J M) \subseteq \varPhi(J)$ if and only if the following relation holds $g\big((D\mathrm{d}\varphi)(U + \mathrm{i}X_1, U + \mathrm{i}X_1), X_2 + \mathrm{i}X_3\big) = 0$; equivalently,

(3.5.2)

$$g\big(\varphi^*(D^N)_{X_1} X_1 + \nabla_{U+\mathrm{i}X_1}(U + \mathrm{i}X_1), X_2 + \mathrm{i}X_3\big) = 0 \,.$$

If α is the Lee form of D^N with respect to h then

(3.5.3)
$$g\big(\varphi^*(D^N)_{X_1}X_1-\nabla_{X_1}X_1, X_2+\mathrm{i}X_3\big)=-\varphi^*(\alpha)(X_2+\mathrm{i}X_3).$$

Also, we have

(3.5.4)
$$g(\nabla_U U, X_2+\mathrm{i}X_3)=g(\mathrm{trace}_g(B^{\mathscr{V},g}), X_2+\mathrm{i}X_3)\ .$$

By using the fact that φ is a Riemannian submersion we, also, obtain

(3.5.5)
$$\begin{aligned} g(\nabla_U X_1+\nabla_{X_1}U, X_2+\mathrm{i}X_3) & \\ &= g(2\nabla_{X_1}U+[X_1,U], X_2+\mathrm{i}X_3) \\ &= 2g(\nabla_{X_1}U, X_2+\mathrm{i}X_3) \\ &= -2g(U, \nabla_{X_1}(X_2+\mathrm{i}X_3)) \\ &= -g(U, [X_1, X_2+\mathrm{i}X_3])\ . \end{aligned}$$

Now, relations (3.5.3), (3.5.4) and (3.5.5) show that (3.5.2) holds if and only if

$$\varphi^*(\alpha)(X_j)=g(\mathrm{trace}_g(B^{\mathscr{V},g}), X_j)+(*_{\mathscr{H}}I^{\mathscr{H}})(X_j)\ ,$$

for $j=2,3$.

The proof follows from Theorem 3.5.1. □

Next, we prove another result of [8].

Corollary 3.5.5. *Let $\varphi : (M^4, c_M) \to (N^3, c_N, D^N)$ be a surjective twistorial map. Then the following assertions are equivalent:*

(i) (M^4, c_M) *is anti-self-dual.*

(ii) (N^3, c_N, D^N) *is Einstein–Weyl.*

Proof. Let $\tau_M = (Q_M, M, \pi_M, \mathcal{J})$ and $\tau_N = (Q_N, N, \pi_N, \mathcal{F})$ be the almost twistorial structures of (M^4, c_M) and (N^3, c_N, D^N), respectively; denote by $\mathcal{C}$ the eigenbundle of $\mathcal{J}$ corresponding to $-\mathrm{i}$.

From the fact that the fibres of Φ are mapped by π_M diffeomorphically onto the fibres of φ it follows that the sum $\mathcal{C} + (\ker \mathrm{d}\pi_M)^{1,0} + \ker \mathrm{d}\Phi$ is direct.

Now, as $\mathrm{d}\Phi(\mathcal{C}) = \mathcal{F}$ we, obviously, have (i)$\Longrightarrow$(ii).

Conversely, if (ii) holds then $(\mathrm{d}\Phi)^{-1}(\mathcal{F}) = \mathcal{C} \oplus \ker \mathrm{d}\Phi$ is integrable. Together with Remark 1.3.2, this gives that the bracket of any two sections of $\mathcal{C}$ is a section of $\big(\mathcal{C} \oplus \ker \mathrm{d}\Phi\big) \cap \big(\mathcal{C} \oplus (\ker \mathrm{d}\pi_M)^{1,0}\big) = \mathcal{C}$.

The proof is complete. □

Next, we give a construction, due to [18] (cf. [14]; see Example 4.2.3, below), of a twistorial map from a four-dimensional conformal manifold to a three-dimensional Weyl space.

Example 3.5.6. Let V be a nowhere zero *conformal vector field* on a four-dimensional conformal manifold (M^4, c_M); that is, the local flow of V is formed of conformal diffeomorphisms (equivalently, for any representative g of c_M there exists a function f on M such that $\mathcal{L}_V g = fg$, where $\mathcal{L}$ denotes the Lie derivation).

Suppose that the orbits of V are the fibres of a surjective submersion $\varphi : M \to N$ (locally, this always holds). Then, as V is conformal, there exists a unique conformal structure c_N on N with respect to which φ is horizontally conformal.

Let g and h be representatives of c_M and c_N, respectively, with respect to which φ is a Riemannian submersion.

We claim that V is a *Killing vector field* on (M^4, g); that is, the local flow of V is formed of (local) isometries of g (equivalently, $\mathcal{L}_V g = 0$). Moreover, this holds in any dimension.

Indeed, a straightforward calculation (see [5]) shows that

$$(\mathcal{L}_V g)(X,Y) = -2g\big(B^{\mathscr{H},g}(X,Y), V\big)\,, \tag{3.5.6}$$

for any $X, Y \in \mathscr{H}$. Together with the fact that V is conformal and φ is a Riemannian submersion, this shows that $\mathcal{L}_V g = 0$.

Now, let $\rho = g(V,V)$. Then, as $V(g(V,V)) = (\mathcal{L}_V g)(V,V) = 0$, we have that ρ is constant along the fibres of φ. Hence, $\rho = \sigma \circ \varphi$ for some positive function σ on N. Thus, by replacing, if necessary, g and h with $\rho^{-1}g$ and $\sigma^{-1}h$, respectively, we may assume $g(V,V)=1$.

If X is horizontal then

$$g(V,[V,X]) = -(\mathcal{L}_V g)(V,X) = 0 \ .$$

Hence, if, further, X is projectable, with respect to φ, then $[V,X]=0$; thus,

$$\begin{aligned} g(\mathrm{trace}_g(B^{\mathscr{V},g}),X) &= g(\nabla_V V,X) = -g(V,\nabla_V X) \\ &= -g(V,\nabla_X V) = -\tfrac{1}{2}X(g(V,V)) \\ &= 0 \ . \end{aligned}$$

Let $\theta = V^{\flat_g}$. Then $(\mathrm{d}\theta)(X,Y) = -g(V,[X,Y])$, for any horizontal X and Y. Also, by using the fact that V commutes with the projectable horizontal vector fields, we obtain $(\mathrm{d}\theta)(V,X)=0$, for any horizontal X. Hence, $I^{\mathscr{H}} = V \otimes \mathrm{d}\theta$.

Also, it follows quickly that there exists a two-form F on N such that $\mathrm{d}\theta = \varphi^*(F)$. Moreover, $*_{\mathscr{H}} I^{\mathscr{H}} = \varphi^*(*_h F)$, where $*_h$ is the Hodge $*$-operator of (N,h).

Let D^N be the Weyl connection on (N^3,c_N) whose Lee form with respect to h is $*_h F$. Then, by Corollary 3.5.4, $\varphi : (M^4,c_M) \to (N^3,c_N,D^N)$ is twistorial.

With the same notations as in Example 3.5.6, note that, as $\mathrm{d}F = 0$, the Lee form α of D^N with respect to h satisfies $\mathrm{d}{*_h}\alpha = 0$; equivalently, α is a co-closed one-form on (N^3, h). This fact can be used to show that (M^4, c_M) can be, locally, retrieved from (N^3, c_N, D^N), as we shall now explain.

Remark 3.5.7. Let $(N^3, [h], D)$ be a three-dimensional Weyl space such that the Lee form α of D, with respect to h, is a co-closed one-form on (N^3, h). Then, locally, there exists a one-form A on N^3 such that $*_h\alpha = \mathrm{d}A$ (equivalently, $\alpha = *_h \mathrm{d}A$).

Let $M^4 = \mathbb{R} \times N^3$ and let $g = h + (\mathrm{d}t + A)^2$, where $t : M^4 \to \mathbb{R}$ is the projection. Then $\partial/\partial t$ is a conformal vector field on $(M^4, [g])$ and the projection $(M^4, [g]) \to (N^3, [h], D)$ is the corresponding twistorial map.

CHAPTER 4

Harmonic morphisms and twistorial maps

In this chapter we discuss the conditions under which the examples of twistorial maps presented in Chapter 3 are harmonic morphisms.

Also, we prove that any harmonic morphism from a four-dimensional Einstein–Weyl space to a two-dimensional conformal manifold is twistorial, with respect to a suitable orientation on its domain. It follows that, also, any harmonic morphism between Einstein–Weyl spaces of dimensions four and three is twistorial, with respect to a suitable orientation on its domain.

4.1. Twistorial harmonic morphisms to two-dimensional conformal manifolds

In this section we study the conditions under which the twistorial maps of Examples 3.3.3, 3.4.1 and 3.4.3 are harmonic morphisms.

Corollary 4.1.1. *Let $\varphi : (M^2, c_M) \to (N^2, c_N)$ be a map between two-dimensional Weyl spaces.*

Then the following assertions are equivalent:

(i) *φ is a harmonic morphism, with respect to some (and, hence, any) Weyl connections on its domain and codomain.*

(ii) *φ is twistorial, with respect to suitable orientations of its domain and codomain.*

Proof. Let J^M and J^N be the positive Hermitian structures on (M^2, c_M) and (N^2, c_N), respectively. Assertion (ii) is equivalent to the fact that $\varphi : (M^2, J^M) \to (N^2, J^N)$ is either holomorphic or anti-holomorphic.

The proof is an immediate consequence of Remark 2.3.3. □

Next, we prove the following result.

Corollary 4.1.2. *Let $\varphi : (M^3, c_M, D) \to (N^2, c_N)$ be a submersion from a three-dimensional Weyl space to a two-dimensional conformal manifold.*

Then the following assertions are equivalent:

(i) *φ is a harmonic morphism.*

(ii) *φ is twistorial.*

Proof. By Example 3.4.1, assertion (ii) is equivalent to the fact that φ is a horizontally conformal submersion with geodesic fibres.

The proof is an immediate consequence of Corollary 2.5.4(a). □

To state the next result we use the same notations as in Example 3.4.3 .

Corollary 4.1.3. *Let* $\varphi : (M^4, c_M, D) \to (N^2, c_N)$ *be a submersion from a four-dimensional Weyl space to a two-dimensional conformal manifold.*

Then any two of the following assertions imply the third:

(i) φ *is a harmonic morphism.*

(ii) φ *is twistorial.*

(iii) *The fibres of* Φ *are tangent to the connection induced by* D *on* Q*.*

To prove Corollary 4.1.3 we involve the following almost twistorial structure, due to [13] .

Example 4.1.4. Let (M^4, c, D) be a four-dimensional Weyl space. With the same notations as in Example 3.2.3 , let $\mathcal{J}'$ be the almost complex structure on Q whose eigenbundle corresponding to $-\mathrm{i}$ is $\mathcal{B} \oplus (\ker \mathrm{d}p)^{1,0}$. From [20, Proposition III.2.3] it follows quickly that $\mathcal{J}'$ is nonintegrable (that is, always not integrable).

Then $(Q, M, \pi, \mathcal{J}')$ is the *nonintegrable almost twistorial structure of* (M^4, c, D) .

The almost twistorial structure of Example 4.1.4 provides the setting for the twistorial interpretation of minimal surfaces in four-dimensional Weyl spaces (cf. [13]):

Example 4.1.5. Let (M^4,c,D) be a four-dimensional Weyl space and let N^2 be an oriented surface in M^4. Let $\tau'_M=(Q,M,\pi,\mathcal{J}')$ be the nonintegrable almost twistorial structure of (M^4,c,D) , and let $\tau_N=(N,N,\mathrm{Id}_N,J)$ be the twistorial structure of Example 3.1.8 , associated to $(N^2,c|_N)$.

As in Example 3.3.5 , let q be the section of Q^6 over N^2 such that $q_x|_{T_xN}=J_x$, for any $x\in N$. Then the following assertions are equivalent:

(i) The inclusion $(N^2,\tau_N)\to(M^4,\tau'_M)$ is twistorial (with respect to q).

(ii) $q:(N^2,J)\to(P^6,\mathcal{J}')$ is holomorphic.

(iii) N^2 is minimal in (M^4,c,D) .

Indeed, Proposition 1.2.1 implies that (ii) holds if and only if, for any sections X and Y of q with X tangent to N^2 and Y normal to N^2, we have that $D_{\overline{X}}X$ and $D_{\overline{X}}Y$ are sections of q ; equivalently, $c(D_{\overline{X}}X,Y)=0$. Furthermore, as $[X,\overline{X}]$ is tangent to N^2, we have that $c(D_{\overline{X}}X,Y)=0$ if and only if $c(D_{\overline{X}}X,\overline{Y})=0$.

Thus, we have proved that (ii) holds if and only if, for any section X of q tangent to N^2, we have that $D_{\overline{X}}X$ is tangent to N^2.

The equivalence (ii)$\Longleftrightarrow$(iii) follows quickly.

Let J be an almost Hermitian structure on a four-dimensional Weyl space (M^4, c, D). With the same notations as above, let q be the section of Q corresponding to J. From Proposition 3.4.6 it follows that the following assertions are equivalent:

(i) D is the Weyl connection of (M^4, c, J).

(ii) $q : (M^4, J) \to (Q^6, \mathcal{J}')$ is holomorphic.

We end this section with the following:

PROOF OF COROLLARY 4.1.3. By using Corollary 2.5.4(a) and Example 4.1.5, we obtain that assertion (i) is equivalent to the following:

(i′) φ is horizontally conformal and its fibres are twistorial, in the sense of Example 4.1.5.

By Example 3.3.5, assertion (ii) is equivalent to the following:

(ii′) φ is horizontally conformal and its fibres are twistorial, in the sense of Example 3.3.5.

Let P^2 be a fibre of φ and let J be the positive Hermitian structure on $(P^2, c|_P)$. Also, let q_P be the section of Q over P^2 such that $(q_P)_x|_{T_xP} = J_x$, for any $x \in P$.

Then, the inclusion $P^2 \to M^4$ is twistorial, in the sense of Example 3.3.5, if and only if $q_P : (P^2, J) \to (Q^6, \mathcal{J})$ is holomorphic.

Similarly, the inclusion $P^2 \to M^4$ is twistorial, in the sense of Example 4.1.5, if and only if $q_P : (P^2, J) \to (Q^6, \mathcal{J}')$ is holomorphic.

Therefore the inclusion $P^2 \to M^4$ is twistorial, both in the sense of Example 3.3.5 and Example 4.1.5, if and only if $q_P(P)$ is tangent to the connection induced by D on Q^6.

To complete the proof just note that any fibre of Φ is of the form $q_P(P)$ for some fibre P of φ. $\square$

4.2. Twistorial harmonic morphisms between Weyl spaces of dimensions four and three

Throughout this section, for simplicity, the map $\varphi : (M^4, c_M) \to (N^3, c_N)$ will be assumed to be a horizontally conformal submersion between conformal manifolds of dimensions four and three.

We shall use the same notations as in Section 3.5. In particular, $\mathscr{V} = \ker \mathrm{d}\varphi$, $\mathscr{H} = \mathscr{V}^{\perp}$, and $I^{\mathscr{H}}$ denotes the integrability tensor of $\mathscr{H}$. Also, $\pi_M : Q_M \to M$ is the bundle of positive orthogonal complex structures on (M^4, c_M), whilst $\pi_N : Q_N \to N$ is the bundle of coisotropic spaces on (N^3, c_N).

As φ is horizontally conformal, the natural lift $\Phi : Q_M \to Q_N$ is given by $\Phi(J) = \mathrm{d}\varphi(T^J M)$, where $T^J M$ is the eigenspace of J corresponding to $-\mathrm{i}$.

Proposition 4.2.1. *Let φ be a horizontally conformal submersion from a four-dimensional Weyl space (M^4, c_M, D^M) to a three-dimensional conformal manifold (N^3, c_N).*

Then the following assertions are equivalent:

(i) *The fibres of Φ are tangent to the connection induced by D^M on Q_M.*

(ii) *For any $X \in \mathscr{H}$, we have*

$$D^M_X = D_X + \tfrac{1}{2}(*_{\mathscr{H}} I^{\mathscr{H}})(X) \,,$$

as connections on $\mathscr{V}$, where D is the connection of $(M^4, c_M, \mathscr{V})$.

Proof. Let g and h be representatives of c_M and c_N, respectively, with respect to which $\varphi : (M^4, g) \to (N^3, h)$ is a Riemannian submersion. Let (U, X_1, X_2, X_3) be a positive local orthonormal frame on (M^4, g) such that U is vertical. We may suppose that X_j, $(j = 1, 2, 3)$, are projectable with respect to φ; in particular, $[U, X_1]$ is vertical.

Let J be the positive almost Hermitian structure, locally defined on (M^4, g), whose eigenbundle corresponding to $-\mathrm{i}$ is spanned by $U + \mathrm{i}X_1$ and $X_2 + \mathrm{i}X_3$; denote by q_J the local section of Q_M corresponding to J.

Obviously, $\mathrm{d}\varphi(T^J M)$ is constant along the fibres of φ ; equivalently, q_J maps the fibres of φ onto fibres of Φ .

From Proposition 1.2.1 it follows that assertion (i) holds, on the domain of (U, X_1, X_2, X_3) , if and only if $g\big(D^M_U(U + \mathrm{i}X_1), X_2 + \mathrm{i}X_3\big) = 0$.

Let ∇ be the Levi-Civita connection of (M^4, g) and let α_M be the Lee form of D^M, with respect to g . We have

$$\begin{aligned}
g\big(D^M_U&(U + \mathrm{i}X_1), X_2 + \mathrm{i}X_3\big) \\
&= g(D^M_U U, X_2 + \mathrm{i}X_3) + \mathrm{i}g(D^M_U X_1, X_2 + \mathrm{i}X_3) \\
&= g(\nabla_U U, X_2 + \mathrm{i}X_3) - \alpha_M(X_2 + \mathrm{i}X_3) \\
&\qquad\qquad\qquad - \mathrm{i}g\big(U, D^M_{X_1}(X_2 + \mathrm{i}X_3)\big) \\
&= \big(\mathrm{trace}_g(B^{\mathscr{V},g})^{\flat_g} - \alpha_M\big)(X_2 + \mathrm{i}X_3) \\
&\qquad\qquad\qquad + \tfrac{\mathrm{i}}{2} g\big(U, I^{\mathscr{H}}(X_1, X_2 + \mathrm{i}X_3)\big) .
\end{aligned}$$

As $g\big(U, I^{\mathscr{H}}(X_1, X_2 + \mathrm{i}X_3)\big) = (*_{\mathscr{H}} I^{\mathscr{H}})(X_3 - \mathrm{i}X_2)$, it follows quickly that (i) holds if and only if

$$\alpha_M|_{\mathscr{H}} = \mathrm{trace}_g(B^{\mathscr{V},g})^{\flat_g} + \tfrac{1}{2} {*_{\mathscr{H}}} I^{\mathscr{H}} . \tag{4.2.1}$$

The proof is complete. □

Now, we can prove the main result of this section.

Theorem 4.2.2. *Let* (M^4, c_M, D^M) *and* (N^3, c_N, D^N) *be Weyl spaces of dimensions four and three. Let*

$\varphi : M^4 \to N^3$ be a submersion.

Then any two of the following assertions imply the third:

(i) *$\varphi : (M^4, c_M, D^M) \to (N^3, c_N, D^N)$ is a harmonic morphism.*

(ii) *$\varphi : (M^4, c_M) \to (N^3, c_N, D^N)$ is twistorial.*

(iii) *The fibres of Φ are tangent to the connection induced by D^M on Q_M.*

Proof. We may assume φ horizontally conformal, as this is a necessary condition for both (i) and (ii).

Let g and h be representatives of c_M and c_N, respectively, with respect to which $\varphi : (M^4, g) \to (N^3, h)$ is a Riemannian submersion. Denote by α_M and α_N the Lee forms of D^M and D^N with respect to g and h, respectively.

By (2.5.5) and Theorem 2.4.3, assertion (i) is equivalent to the following relation

$$2\,\alpha_M|_{\mathscr{H}} = \varphi^*(\alpha_N) + \mathrm{trace}_g(B^{\mathscr{V},g})^{\flat_g} \,. \tag{4.2.2}$$

By Corollary 3.5.4, assertion (ii) is equivalent to the following relation

$$\varphi^*(\alpha_N) = \mathrm{trace}_g(B^{\mathscr{V},g})^{\flat_g} + *_{\mathscr{H}} I^{\mathscr{H}} \,. \tag{4.2.3}$$

Now, the proof quickly follows from relations (4.2.2), (4.2.3) and the fact that, by Proposition 4.2.1, assertion (iii) is equivalent to (4.2.1). □

In the remaining of this section we present two basic examples of maps which satisfy the conditions (i), (ii), (iii) of Theorem 4.2.2.

Let (N^3,h) be a (connected) open set of $\mathbb{R}^3$ or the three-dimensional sphere of radius $2/c$, for some $c>0$. By using suitable left invariant vector fields, we fix three Riemannian submersions with geodesic fibres $\psi_j:(N^3,h)\to(P^2,k)$ such that, at each point, the fibres of ψ_j are orthogonal onto each other, $j=1,2,3$ (if $N^3\subseteq\mathbb{R}^3$ then ψ_j are restrictions of orthogonal projections whilst if $N^3\subseteq S^3$ then ψ_j are restrictions of projections of Hopf bundles, $j=1,2,3$).

By Corollary 4.1.2, we have that

$$\psi_j:(N^3,[h],\nabla^h)\to(P^2,[k])$$

are both harmonic morphisms and twistorial maps, $(j=1,2,3)$, where ∇^h is the Levi-Civita connection of (N^3,h).

The next example is based on a construction of [14].

Example 4.2.3. Let u be a positive function on N^3 and let A be a one-form on N^3 which satisfy the *monopole equation* (see [22]) $\mathrm{d}u = -\ast_h \mathrm{d}A$.

Let $M^4 = \mathbb{R} \times N^3$ and define the Riemannian metric g on M^4 by

$$g = uh + u^{-1}(\mathrm{d}t + A)^2 \ ,$$

where $t : M^4 \to \mathbb{R}$ is the projection.

Then the projection $\varphi : (M^4, [g]) \to (N^3, [h], \nabla^h)$ is twistorial. Indeed, $V = \partial/\partial t$ is a Killing vector field on (M^4, g) . Moreover, V is a Killing vector field with geodesic orbits on (M^4, ug) .

Let α be the Lee form of ∇^h with respect to u^2h . By using Remark 2.2.5 we obtain that $\alpha = -u^{-1}\,\mathrm{d}u = u^{-1} \ast_h \mathrm{d}A = \ast_{u^2h} \mathrm{d}A$.

Thus, by Remark 3.5.7 , we have that

$$\varphi : (M^4, [g]) \to (N^3, [h], \nabla^h)$$

is twistorial. Consequently, by Proposition 3.5.3 , also, $\psi_j \circ \varphi$ are twistorial, $(j = 1, 2, 3)$.

Now, denote by J_j the Hermitian structures on $(M^4, [g])$ determined by $\psi_j \circ \varphi$, $(j = 1, 2, 3)$.

Then, possibly up to a renumbering, J_j , $(j = 1, 2, 3)$, satisfy the quaternionic identities; equivalently, (M^4, J_1, J_2, J_3) is a (four-dimensional) *hypercomplex manifold*.

From Corollary 3.4.7 and the fact that $J_jJ_k = -J_kJ_j$, if $j \neq k$, we obtain that $(M^4, [g], J_j)$, $(j =$

$1,2,3)$, have the same Weyl connection D.

Then D is the *Obata connection* of (M^4, J_1, J_2, J_3). Note that, we have proved that D is the unique torsion free connection such that $DJ_j = 0$, $(j = 1,2,3)$. Moreover, as J_j, $(J = 1,2,3)$, determine $[g]$ (use, for example, the fact that if $X \neq 0$ then (X, J_1X, J_2X, J_3X) is a conformal frame on $(M^4, [g])$), the above argument applies to prove the existence and uniqueness of the Obata connection for any four-dimensional hyper-complex manifold (see [1] for the Obata connection of a higher dimensional hyper-complex manifold).

Finally, as $DJ_j = 0$, $(j = 1,2,3)$, we have that φ satisfies (iii) of Theorem 4.2.2. Therefore $\varphi : (M^4, [g], D) \to (N^3, [h], \nabla^h)$ is a harmonic morphism.

The next example is based on a construction of [27].

Example 4.2.4. Let A be a one-form on N^3 which satisfies the *Beltrami fields equation* (see [19])

$$\mathrm{d}A = 2 *_h A \, .$$

Let $M^4 = (0,\infty) \times N^3$ and define the Riemannian metric g on M^4 by

$$g = \rho^2 h + \rho^{-2}(\rho\,\mathrm{d}\rho + A)^2 \, ,$$

where $\rho : M^4 \to (0, \infty)$ is the projection.

Then the projection $\varphi : (M^4, [g]) \to (N^3, [h], \nabla^h)$ is twistorial. Indeed, let $V = \rho^{-1}\,\partial/\partial\rho$ and let $\theta = \rho\,\mathrm{d}\rho + A$. Then, similarly to Example 3.5.6 , we have $I^{\mathscr{H}} = V \otimes \mathrm{d}\theta$. Thus, by using that $g(V, V) = \rho^{-2}$, we obtain

$$*_{\mathscr{H}} I^{\mathscr{H}} = \rho^{-1} *_{\rho^2 h} \mathrm{d}A = \rho^{-2} *_h \mathrm{d}A = 2\rho^{-2} A \; .$$

On the other hand, $\mathscr{V}$ is a foliation by geodesics on $(M^4, \rho^2 g)$, and the 'Lee form' α of $\varphi^*(\nabla^h)$ with respect to the metric induced by $\rho^2 g$ on $\mathscr{H}$ is given by $\alpha = -\rho^{-2}\,\mathrm{d}^{\mathscr{H}}(\rho^2) = -2\rho^{-1}\,\mathrm{d}^{\mathscr{H}}\rho$, where $\mathrm{d}^{\mathscr{H}}\rho$ is equal to $\mathrm{d}\rho$ on $\mathscr{H}$, and is zero on $\mathscr{V}$.

Define, similarly, $\mathrm{d}^{\mathscr{V}}\rho$ and, note that, $\mathrm{d}\rho = \mathrm{d}^{\mathscr{V}}\rho + \mathrm{d}^{\mathscr{H}}\rho$. Furthermore, as $(\mathrm{d}^{\mathscr{V}}\rho)(V) = V(\rho) = \rho^{-1}$, we have $\mathrm{d}^{\mathscr{V}}\rho = \rho^{-1}\theta$ and, consequently, $\mathrm{d}^{\mathscr{H}}\rho = \mathrm{d}\rho - \rho^{-1}\theta$.

Therefore $\alpha = -2\rho^{-1}(\mathrm{d}\rho - \rho^{-1}\theta) = 2\rho^{-2}A = *_{\mathscr{H}} I^{\mathscr{H}}$, and Corollary 3.5.4 implies that

$$\varphi : (M^4, [g]) \to (N^3, [h], \nabla^h)$$

is twistorial.

Then, similarly to Example 4.2.3 ,

$$\varphi : (M^4, [g], D) \to (N^3, [h], \nabla^h)$$

is a harmonic morphism, where D is the Obata connections of a hyper-complex structure on M^4, compatible with $[g]$.

We end this section with the following:

Remark 4.2.5. 1) For the maps $\varphi : (M^4, [g]) \to (N^3, [h], \nabla^h)$ of Examples 4.2.3 and 4.2.4 to be twistorial it is not necessary to assume (N^3, h) of constant curvature.

2) Let (M^4, J_1, J_2, J_3) be a four-dimensional hyper-complex manifold and let c be the induced conformal structure on M^4.

Then (M^4, c) is anti-self-dual with respect to the orientation determined by J_j , $(j = 1, 2, 3)$.

Indeed, if $a \in S^2$ then $J = a^1 J_1 + a^2 J_2 + a^3 J_3$ is a Hermitian structure on (M^4, c) . Moreover, $DJ = 0$, where D is the Obata connection of (M^4, J_1, J_2, J_3) ; in particular, $R\big(\Lambda^2(T^J M)\big)(T^J M) \subseteq T^J M$, where R is the curvature form of D.

As this holds for any $a \in S^2$, from Proposition A.1.11 it quickly follows that (M^4, c) is anti-self-dual.

Furthermore, a straightforward calculation shows that the symmetrized Ricci tensor of D is zero; in particular, (M^4, c, D) is Einstein–Weyl.

3) Let $(M^4, [g], D)$ be as in Example 4.2.3 , with

u nonconstant. Then D is, locally, the Levi-Civita connection of some representative of $[g]$ if and only if (N^3, h) is flat.

Similarly, if $(M^4, [g], D)$ is as in Example 4.2.4 then D is, locally, the Levi-Civita connection of some representative of $[g]$ if and only if (N^3, h) has constant sectional curvature equal to 1.

These two facts can be proved either directly or by using [26, Corollary 3.5.5, Proposition 3.7.2].

4.3. Harmonic morphisms from four-dimensional Einstein–Weyl spaces

The following result was proved in [30] for Einstein manifolds and then generalized to Einstein–Weyl spaces in [21].

Theorem 4.3.1. *Let (M^4, c_M, D) be a four-dimensional Einstein–Weyl space and let φ be a submersive harmonic morphism from (M^4, c_M, D) to a two-dimensional conformal manifold (N^2, c_N).*

Then φ is twistorial, with respect to a suitable orientation on M^4.

Proof. As Einstein–Weyl spaces are real analytic (see [10]) we may assume φ real-analytic (cf. [5, Proposition 4.7.11]).

As before, orient $\mathscr{V} = \ker \mathrm{d}\varphi$ and $\mathscr{H} = \mathscr{V}^{\perp}$ so that the isomorphisms $\mathscr{H} = \varphi^*(TN)$ and $TM =$

$\mathscr{V} \oplus \mathscr{H}$ be orientation preserving.

Let U and Y be nowhere zero local sections of the eigenbundles, corresponding to $-\mathrm{i}$, of the rotations of angles $\pi/2$ on $\mathscr{V}$ and $\mathscr{H}$, respectively; in particular, U and Y are isotropic.

As $\mathscr{H}$ is umbilical, from (3.5.6), we quickly obtain that

$$g([U,Y],Y) = 0 = g([\overline{U},Y],Y)\,. \tag{4.3.1}$$

Let $J^{\pm}$ be the positive/negative almost Hermitian structure on (M^4, c_M) with respect to which $\varphi : (M^4, J^{\pm}) \to (N^2, J)$ is holomorphic, where J is the positive Hermitian structure on (N^2, c_N).

We have to prove that either J^+ or J^- is integrable. In fact, as $\{U, Y\}$ and $\{\overline{U}, Y\}$ are sets of eigenvectors of J^+ and J^-, respectively, we have to prove that one of these sets span an integrable (complex) distribution; by (4.3.1), this holds if and only if either $g([U,Y],U) = 0$ or $g([\overline{U},Y],\overline{U}) = 0$.

We may assume that $g = 2(U \odot \overline{U} + Y \odot \overline{Y})$ is a local representative of c_M. Note that, for any $E \in TM$, we have

$$E = g(E,\overline{U})U + g(E,U)\overline{U} + g(E,\overline{Y})Y + g(E,Y)\overline{Y}\,.$$

Therefore if R is the curvature form and Ric is the Ricci tensor of D then

(4.3.2)
$$\begin{aligned}\mathrm{Ric}(Y,Y) =& g(R(U,Y)Y,\overline{U}) + g(R(\overline{U},Y)Y,U)\\ &+ g(R(Y,Y)Y,\overline{Y}) + g(R(\overline{Y},Y)Y,Y)\,.\end{aligned}$$

As (M^4, c_M, D) is Einstein–Weyl, $\mathrm{Ric}(Y,Y) = 0$. Further, as $R(Y,Y) = 0$ and D is a conformal connection, the last two terms of the right hand side of (4.3.2) are zero. Also, from (A.1.4) and Lemma A.1.3 it, quickly, follows that the first two terms of the right hand side of (4.3.2) are equal.

Hence, (4.3.2) gives $g(R(U,Y)Y,\overline{U}) = 0$.

By using, again, that $\mathscr{H}$ is umbilical we obtain that $D_Y Y$ is horizontal. Also, by Corollary 2.5.4(a), we have that φ has minimal fibres; equivalently, $g(D_U\overline{U},Y) = g(D_U\overline{U},\overline{Y}) = 0$.

Thus, $g(D_U D_Y Y,\overline{U}) = -g(D_Y Y, D_U\overline{U}) = 0$.

We claim that, also, $g(D_Y D_U Y,\overline{U}) = 0$. Indeed, we have

$$\begin{aligned}D_U Y =& g(D_U Y,\overline{U})U + g(D_U Y,U)\overline{U}\\ &+ g(D_U Y,\overline{Y})Y + g(D_U Y,Y)\overline{Y}\\ =& g(D_U Y,U)\overline{U} + g(D_U Y,\overline{Y})Y\,.\end{aligned}$$

Hence,

$$\begin{aligned} g(D_Y D_U Y, \overline{U}) =& g\big(D_Y\big(g(D_U Y, U)\overline{U} \\ &\quad + g(D_U Y, \overline{Y})Y\big), \overline{U}\big) \\ =& 0 \,, \end{aligned}$$

as $\overline{U}$ is isotropic and $D_Y\big(g(D_U Y, \overline{Y})Y\big)$ is horizontal.

Similar calculations show that

$$g(D_{[U,Y]}Y, \overline{U}) = g([U,Y], U)g([\overline{U}, Y], \overline{U}) \,.$$

We have, thus, proved that

$$g(R(U,Y)Y, \overline{U}) = -g([U,Y], U)g([\overline{U}, Y], \overline{U}) = 0 \,,$$

By analyticity, we obtain that either $g([U,Y], U)$ or $g([\overline{U}, Y], \overline{U})$ is zero, and the proof is complete. □

Next, we prove the following result.

Corollary 4.3.2. *Let φ be a harmonic morphism between the Einstein–Weyl spaces (M^4, c_M, D^M) and (N^3, c_N, D^N) of dimensions four and three, respectively.*

Then φ is twistorial and (M^4, c_M) is anti-self-dual, with respect to a suitable orientation on M^4.

Proof. The fact that φ is twistorial, with respect to a suitable orientation on M^4, follows from

Corollary 4.1.2, Theorem 4.3.1, and Proposition 3.5.3. Then, by Corollary 3.5.5, (M^4, c_M) is anti-self-dual. $\square$

Note that, Examples 4.2.3 and 4.2.4 give harmonic morphisms between Einstein–Weyl spaces of dimensions four and three (consequence of Remark 4.2.5(2)).

APPENDIX A

The curvature of Weyl spaces

Let (M^m, c, D) be a Weyl space, $m = \dim M$, and let R be the curvature form of D.

Also, let F be the curvature form of the connection, corresponding to D, on the line bundle L of M^m. Note that, Lemma 2.2.1 implies that if α is the Lee form of D, with respect to some representative of c, then $F = \mathrm{d}\alpha$.

The curvature form of the connection induced by D on $L^* \otimes TM$ is a two form with values in the endomorphism bundle of $L^* \otimes TM$. Therefore there exists a two-form R^0 with values in the endomorphism bundle of TM such that $\mathrm{Id}_{L^*} \otimes R^0$ is the curvature form of the connection induced by D on $L^* \otimes TM$. Moreover, as $Dc = 0$, we have

$$\text{(A.1.3)} \quad c(R^0(U,V)X, Y) = -c(X, R^0(U,V)Y) \ ,$$

for any $U, V, X, Y \in TM$.

Furthermore,

$$\begin{aligned}(\,\mathrm{Id}_{L^*} \otimes R^0\,)&(X,Y)(s^* \otimes Z) \\ &= -F(X,Y)s^* \otimes Z + s^* \otimes R(X,Y)Z \ ,\end{aligned}$$

for any $X,Y,Z\in TM$ and $\alpha^*\in L^*$ and, hence,

$$R=R^0+F\otimes\mathrm{Id}_{TM}\;. \tag{A.1.4}$$

We shall, also, denote by R^0 the section of $L^2\otimes\bigl(\otimes^4T^*M\bigr)$, given by

$$R^0(T,X,Y,Z)=-c(R^0(T,X)Y,Z)\;,$$

for any $T,X,Y,Z\in TM$. Note that, by (A.1.3), we have that R^0 is a section of $\otimes^2\bigl(\Lambda^2(T^*M)\bigr)$.

In this appendix we shall show that R^0 admits a decomposition [9] similar to the decomposition of the Riemannian curvature tensor (see [7] and the references therein).

For this, recall that is h and k are covariant tensors of second degree on M^m then $h \owedge k$ is the section of $\otimes^2\bigl(\Lambda^2(T^*M)\bigr)$, given by

$$\begin{aligned}(h\owedge k)(T,X,Y,Z)&\\ &=h(T,Y)k(X,Z)+h(X,Z)k(T,Y)\\ &\quad-h(T,Z)k(X,Y)-h(X,Y)k(T,Z)\;,\end{aligned}$$

for any $T,X,Y,Z\in TM$.

Obviously, $h\owedge k=k\owedge h$. Also, if h and k are both (skew-)symmetric then $h\owedge k$ is a section of $\odot^2\bigl(\Lambda^2(T^*M)\bigr)$, where $\odot$ denotes the second symmetric power. On the other hand, if h is symmetric and k is skew-symmetric then $h\owedge k$ is a section of

$\Lambda^2\big(\Lambda^2(T^*M)\big)$.

Let $\odot^2 R^0$ and $\Lambda^2 R^0$ be the symmetric and skew-symmetric parts of R^0, respectively; that is,

$$\begin{aligned}(\odot^2 R^0)(T,X,Y,Z) =& \tfrac{1}{2}\big(R^0(T,X,Y,Z)\\ &+ R^0(Y,Z,T,X)\big)\ ,\\ (\Lambda^2 R^0)(T,X,Y,Z) =& \tfrac{1}{2}\big(R^0(T,X,Y,Z)\\ &- R^0(Y,Z,T,X)\big)\ ,\end{aligned}$$

for any $T,X,Y,Z \in TM$.

Lemma A.1.3. *We have* $\Lambda^2 R^0 = -\frac{1}{2} F \owedge c$.

Proof. From the algebraic Bianchi identity and (A.1.4) we obtain

(A.1.5)
$$\begin{aligned}&R^0(T,X,Y,Z) + R^0(X,Y,T,Z) + R^0(Y,T,X,Z)\\ &= -F(T,X)c(Y,Z) - F(X,Y)c(T,Z)\\ &\qquad\qquad - F(Y,T)c(X,Z)\ ,\end{aligned}$$

for any $T,X,Y,Z \in TM$.

The proof follows by taking the sum of the relations obtained from (A.1.5) by applying circular permutations to (T,X,Y,Z). $\square$

Lemma A.1.4. $\odot^2 R^0$ *satisfies the algebraic Bianchi identity.*

Proof. As $R^0 = \odot^2 R^0 + \Lambda^2 R^0$, from Lemma A.1.3 we obtain

(A.1.6)
$$\begin{aligned}
2(\odot^2 R^0)(T,X,Y,Z) = 2R^0(T,X,Y,Z)& \\
+F(T,Y)c(X,Z) + F(X,Z)c(T,Y)& \\
-F(T,Z)c(X,Y) - F(X,Y)c(T,Z)& \;,
\end{aligned}$$

for any $T,X,Y,Z \in TM$.

Now, the proof follows quickly from (A.1.5) and (A.1.6). □

To carry on, recall (see [20]) that the *Ricci tensor* of D is the covariant tensor of second degree Ric on M^m defined by

$$\mathrm{Ric}(X,Y) = \mathrm{trace}\big(Z \mapsto R(Z,X)Y\big) \;,$$

for any $X,Y \in TM$.

Unlike the Riemannian case, the Ricci tensor of a Weyl space is not symmetric.

Lemma A.1.5. *We have* $\Lambda^2\,\mathrm{Ric} = -\frac{m}{2}\,F$.

Proof. By the algebraic Bianchi identity, we have

$$R(Z,X)Y + R(X,Y)Z + R(Y,Z)X = 0 \;,$$

for any $X,Y,Z \in TM$.

Now, $\mathrm{Ric}(X,Y) = \mathrm{trace}\big(Z \mapsto R(Z,X)Y\big)$ and, as $R(Y,Z) = -R(Z,Y)$, we have

$$\operatorname{trace}\bigl(Z \mapsto R(Y,Z)X\bigr) = -\operatorname{Ric}(Y,X)\ ,$$

for any $X, Y \in TM$.

On the other hand, (A.1.4) implies that

$$\operatorname{trace}\bigl(Z \mapsto R(X,Y)Z\bigr) = mF(X,Y)\ ,$$

for any $X, Y \in TM$.

Hence, $\operatorname{Ric}(X,Y) + mF(X,Y) - \operatorname{Ric}(Y,X) = 0$, for any $X, Y \in TM$, and the proof is complete. □

Lemma A.1.6. *We have*

$$(\odot^2 \operatorname{Ric})(X,Y) = \operatorname{trace}_c\bigl((\odot^2 R^0)(\cdot, X, \cdot, Y)\bigr)\ ,$$

for any $X, Y \in TM$.

Proof. From (A.1.4) we obtain

$$\operatorname{Ric}(X,Y) = F(Y,X) + \operatorname{trace}\bigl(Z \mapsto R^0(Z,X)Y\bigr)\ ,$$

for any $X, Y \in TM$.

Let L be the line bundle of M^m. If A is a tensor of degree $(1,1)$ on M^m then $\operatorname{trace} A = \operatorname{trace}_c b$, where b is the L^2-valued covariant tensor of second degree defined by $b(X,Y) = c(AX,Y)$, for any $X, Y \in TM$.

Therefore

$$
\begin{aligned}
2(\odot^2\mathrm{Ric})(X,Y) &= \mathrm{trace}_c\big(c(R^0(\cdot,X)Y,\cdot) \\
&\quad + c(R^0(\cdot,Y)X,\cdot)\big) \\
&= \mathrm{trace}_c\big(R^0(\cdot,X,\cdot,Y) \\
&\quad + R^0(\cdot,Y,\cdot,X)\big) \\
&= 2\,\mathrm{trace}_c\big((\odot^2 R^0)(\cdot,X,\cdot,Y)\big)\ ,
\end{aligned}
$$

for any $X, Y \in TM$. □

We can now prove the following result of [9].

Corollary A.1.7. *If (M^m, c) is a Weyl space, where $m = \dim M$, then*

(A.1.7)

$$
\begin{aligned}
R^0 = &\frac{1}{2m(m-1)}\, Sc \owedge c + \frac{1}{m-2}\,(\odot^2_0\mathrm{Ric}) \owedge c \\
&- \frac{1}{2}\, F \owedge c + W\ ,
\end{aligned}
$$

where $S = \mathrm{trace}_c \mathrm{Ric}$, and $\odot^2_0$ denotes the trace free symmetric part, whilst W is a section of $L^2 \otimes \big(\odot^2\big(\Lambda^2(T^*M)\big)\big)$ satisfying the algebraic Bianchi identity and

$$
\mathrm{trace}_c\big(W(\cdot,X,\cdot,Y)\big) = 0\ ,
$$

for any $X, Y \in TM$.

Proof. This follows quickly from Lemmas A.1.3, A.1.4 and A.1.6 (cf. [7, 1.116]). □

The section W of $L^2 \otimes \big(\odot^2\big(\Lambda^2(T^*M)\big)\big)$ is called the *Weyl tensor* of (M^m, c, D). It depends only of (M^m, c). Indeed, by (A.1.4), (A.1.7) and [25, Proposition 2.2] (see [26, Proposition A.4.2]), it is sufficient to show that, for some representative g of c, we have

$$g(R(X,Y)X,Y) = g(R^g(X,Y)X,Y) \; ,$$

for any $X, Y \in T^{\mathbb{C}}M$ spanning an isotropic space, where R^g is the curvature form of the Levi-Civita connection of g. This follows by computing R in terms of R^g and the Lee form of D with respect to g and by using Remark 2.2.5(2) (cf. [7, Theorem 1.159(b)]).

The following notion is essential in three-dimensional Twistor Theory.

Definition A.1.8. An *Einstein–Weyl space* is a Weyl space (M^m, c, D) for which the trace free symmetric part of its Ricci tensor is zero.

The three-dimensional Einstein–Weyl spaces can be characterised as follows.

Proposition A.1.9. *Let* (M^3, c, D) *be a three-dimensional Weyl space. Then the following assertions are equivalent:*

(i) (M^3, c, D) *is Einstein–Weyl;*

(ii) $R(\Lambda^2 q)q \subseteq q$, *for any two-dimensional coisotropic space* q *on* (M^3, c) , *where* R *is the curvature form of* D.

Proof. Any two-dimensional coisotropic space on (M^3, c) is spanned by some $X, Y \in T^{\mathbb{C}}M$, where Y is isotropic and $X \in Y^{\perp}$. Together with (A.1.4) this shows that assertion (ii) is equivalent to:

(ii') $c(R^0(X,Y)X, Y) = 0$ for any isotropic $Y \in T^{\mathbb{C}}M$ and $X \in Y^{\perp}$.

In dimension three the Weyl tensor is zero. Thus, (A.1.7) becomes

$$R^0 = \frac{1}{12}\, Sc \owedge c + (\odot_0^2 \mathrm{Ric}) \owedge c - \frac{1}{2}\, F \owedge c \,,$$

which implies that (i)$\Longleftrightarrow$(ii') . $\square$

If (V, g) is an oriented Euclidean space then the *Hodge* $*$*-operator* acting on k-forms is characterised by $\omega \wedge *\eta = g(\omega, \eta) v_g$, for any $\omega, \eta \in \Lambda^k V^*$, where v_g is the volume form of (V, g) .

For example, if $(X_1, \ldots, X_m)$ is an orthonormal basis of (V, g) , with $m = \dim V$, then

$$*(X_1 \wedge \ldots \wedge X_k) = \pm X_{k+1} \wedge \ldots \wedge X_m \ ,$$

according to whether or not $(X_1, \ldots, X_m)$ is positive.

The Hodge $*$-operator can be defined on any oriented Riemannian vector bundle. We shall apply it to $(L^*\otimes TM, c)$, where (M^m, c) is a conformal manifold and L is its line bundle.

As $\Lambda^k(L^*\otimes TM)^* = L^k\otimes\Lambda^k T^*M$ we obtain the operator

$$*: L^k\otimes\Lambda^k T^*M \longrightarrow L^{n-k}\otimes\Lambda^{n-k}T^*M\ .$$

By tensorising this with $\mathrm{Id}_{(L^*)^k}$ we obtain the operator

$$*: \Lambda^k T^*M \longrightarrow L^{n-2k}\otimes\Lambda^{n-k}T^*M\ .$$

Furthermore, the Hodge $*$-operator extends to bundle valued forms.

Now, note that the Weyl tensor of a conformal manifold (M^m, c) corresponds to a two-form on M^m, also, denoted by W, with values in the endomorphism bundle of TM; that is, $c(W(T,X)Y,Z) = -W(T,X,Y,Z)$, for any $T,X,Y,Z\in TM$.

The following notion is essential in four-dimensional Twistor Theory.

Definition A.1.10. A four-dimensional (oriented) conformal manifold (M^4, c) is *anti-self-dual* if its Weyl tensor is anti-self-dual; that is,

$$*W = -W\ .$$

Similarly to Proposition A.1.9 , we have the following result.

Proposition A.1.11. *Let (M^4,c) be a four-dimensional conformal manifold. Then the following assertions are equivalent:*

(i) *(M^4,c) is anti-self-dual;*

(ii) *$R(\Lambda^2 q)q \subseteq q$, for any self-dual space q on (M^4,c) , where R is the curvature form of some (and, hence, any) Weyl connection on (M^4,c) .*

Proof. As any self-dual space is isotropic, (A.1.7) implies that assertion (ii) is equivalent to:

(ii') $W(\Lambda^2 q)q \subseteq q$, for any self-dual space q on (M^4,c) .

Next, we look to the Weyl tensor of (M^4,c) as an L^2-valued quadratic form on the bundle $\Lambda^2 TM$ of bivectors on M^4.

Then straightforward calculations show that, in dimension four, the condition $\mathrm{trace}_c\big(W(\cdot,X,\cdot,Y)\big) = 0$, for any $X,Y\in TM$, implies that $W|_{\Lambda^2_+\otimes\Lambda^2_-} = 0$, where Λ^2_+ and Λ^2_- are the bundles of self-dual bivectors and anti-self-dual bivectors on (M^4,c) , respectively; moreover, $\mathrm{trace}_c\big(W|_{\Lambda^2_\pm}\big) = 0$ (see [7, 1.128]).

Therefore (i) holds if and only if $W|_{\Lambda^2_+} = 0$; equivalently, W restricted to the line spanned by an isotropic self-dual form is zero. But any such line is spanned by some $X \wedge Y$, where $\{X, Y\}$ span a self-dual space on (M^4, c).

The proof is complete. $\square$

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