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Convergence rates for dispersive approximation schemes to nonlinear Schrödinger equations

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Abstract

This article is devoted to the analysis of the convergence rates of several numerical approximation schemes for linear and nonlinear Schrödinger equations on the real line. Recently, the authors have introduced viscous and two-grid numerical approximation schemes that mimic at the discrete level the so-called Strichartz dispersive estimates of the continuous Schrödinger equation. This allows to guarantee the convergence of numerical approximations for initial data in $L^2(\mathbb{R})$, a fact that cannot be proved in the nonlinear setting for standard conservative schemes unless more regularity of the initial data is assumed. In the present article we obtain explicit convergence rates and prove that dispersive schemes fulfilling the Strichartz estimates are better behaved for $H^s(\mathbb{R})$ data if $0 < s < 1/2$. Indeed, while dispersive schemes ensure a polynomial convergence rate, non-dispersive ones only yield logarithmic ones.

Résumé

Cet article concerne l'analyse de la vitesse de convergence de plusieurs schémas d'approximation numérique pour l'équation de Schrödinger linéaire et non-linéaire en 1-d. Récemment, les auteurs ont introduit des schémas d'approximation numérique visqueux et bi-maille qui satisfont, au niveau de la discrétisation, des estimations dispersives analogues aux estimations de Strichartz pour l'équation de Schrödinger continue. Ceci permet de garantir la convergence des approximations numériques pour des données initiales dans $L^2(\mathbb{R})$, ce qui ne peut pas être montré dans le cadre non-linéaire pour des schémas conservatifs standard, sauf si les données initiales sont plus régulières. On établit aussi les vitesses explicites de convergence et on montre que les schémas dispersifs satisfaisant les estimations de Strichartz ont un meilleur comportement pour des données dans $H^s(\mathbb{R})$, si $0 < s < 1/2$. En effet, alors que les schémas dispersifs garantissent une vitesse polynomiale de convergence, les nondispersifs ne convergent que de manière logarithmique.

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1. Introduction

Let us consider the linear (LSE) and the nonlinear (NSE) Schrödinger equations:

$$\begin{cases} iu_t + \partial_x^2 u = 0, & x \in \mathbb{R},\ t \neq 0, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

and

$$\begin{cases} iu_t + \partial_x^2 u = f(u), & x \in \mathbb{R},\ t \neq 0, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

respectively.

The linear equation (1.1) is solved by $u(x,t) = S(t)\varphi$, where $S(t) = e^{it\Delta}$ is the free Schrödinger operator and has two important properties. First, the conservation of the L^2-norm

$$\|u(t)\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})} \tag{1.3}$$

which shows that it is in fact a group of isometries in $L^2(\mathbb{R})$, and a dispersive estimate of the form:

$$|S(t)\varphi(x)| = |u(t,x)| \leqslant \frac{1}{(4\pi|t|)^{1/2}} \|\varphi\|_{L^1(\mathbb{R})}, \quad x \in \mathbb{R},\ t \neq 0. \tag{1.4}$$

The space–time estimate

$$\|S(\cdot)\varphi\|_{L^6(\mathbb{R}, L^6(\mathbb{R}))} \leqslant C\|\varphi\|_{L^2(\mathbb{R})}, \tag{1.5}$$

due to Strichartz [27], guarantees that the solutions decay as t becomes large and that they gain some spatial integrability.

Inequality (1.5) was generalized by Ginibre and Velo [10]. They proved:

$$\|S(\cdot)\varphi\|_{L^q(\mathbb{R}, L^r(\mathbb{R}))} \leqslant C(q)\|\varphi\|_{L^2(\mathbb{R})} \tag{1.6}$$

for the so-called 1/2-admissible pairs (q,r). We recall that the exponent pair (q,r) is α-admissible (cf. [22]) if $2 \leqslant q, r \leqslant \infty$, $(q,r,\alpha) \neq (2,\infty,1)$, and

$$\frac{1}{q} = \alpha\left(\frac{1}{2} - \frac{1}{r}\right). \tag{1.7}$$

We see that (1.5) is a particular instance of (1.6) in which $\alpha = 1/2$ and $q = r = 6$.

The extension of these estimates to the inhomogeneous linear Schrödinger equation is due to Yajima [30] and Cazenave and Weissler [6]. These estimates can also be extended to a larger class of equations for which the Laplacian is replaced by any self-adjoint operator such that the L^∞-norm of the fundamental solution behaves like $t^{-1/2}$ [22].

The Strichartz estimates play an important role in the proof of the well-posedness of the nonlinear Schrödinger equation. Typically they are used for nonlinearities for which the energy methods fail to provide well-posedness results. In this way, Tsutsumi [29] proved the existence and uniqueness for $L^2(\mathbb{R})$-initial data for power-like nonlinearities $F(u) = |u|^p u$, in the range of exponents $0 \leqslant p \leqslant 4$. More precisely it was proved that the NSE is globally well posed in $L^\infty(\mathbb{R}, L^2(\mathbb{R})) \cap L^q_{\mathrm{loc}}(\mathbb{R}, L^r(\mathbb{R}))$, where (q,r) is a 1/2-admissible pair depending on the exponent p. This result was complemented by Cazenave and Weissler [7] who proved the local existence in the critical case $p = 4$. The case of H^1-solutions was analyzed by Baillon, Cazenave and Figueira [1], Lin and Strauss [23], Ginibre and Velo [8,9], Cazenave [4], and, in a more general context, by Kato [20,21].

This analysis has been extended to semi-discrete numerical schemes for Schrödinger equations by Ignat and Zuazua in [16,17,19]. In these articles it was first pointed out that conservative numerical schemes often fail to be dispersive, in the sense that numerical solutions do not fulfill the integrability properties above. This is due to the pathological behavior of high frequency spurious numerical solutions. Then several numerical schemes were developed fulfilling the dispersive properties, uniformly in the mesh-parameter. In the sequel these schemes will be referred to as being *dispersive*. As proved in those articles these schemes may be used in the nonlinear context to prove convergence towards the solutions of the NSE, for the range of exponents p and the functional setting above. The analysis of fully discrete schemes was later developed in [13] where necessary and sufficient conditions were given guaranteeing that the dispersive properties of the continuous model are maintained uniformly with respect to the mesh-size parameters at the discrete level. The present paper is devoted to further analyze the convergence of these numerical schemes, the main goal being the obtention of convergence rates.

Despite of the fact that non-dispersive schemes (in the sense that they do not satisfy the discrete analogue of (1.5)) cannot be applied directly in the L^2-setting for nonlinear equations one could still use them by first approximating the L^2-initial data by smooth ones. This paper is devoted to prove that, even if this is done, dispersive schemes are better behaved than the non-dispersive ones in what concerns the order of convergence for rough initial data.

The main results of the paper are as follows. In Theorem 3.1 we prove that the error committed when the LSE is approximated by a dispersive numerical scheme in the $L^q(0,T;l^r(h\mathbb{Z}))$-norms is of the same order as the one classical consistency + stability analysis yields. Using the ideas of [3, Chapter 6], we can also estimate the error in the $L^q(0,T;l^r(h\mathbb{Z}))$-norms, $r>2$, for non-dispersive schemes; for example for the classical three-point second order approximation of the Laplace operator. In this case, in contrast with the good properties of dispersive schemes, for $H^s(\mathbb{R})$-initial data with small s, $1/2-1/r \leqslant s \leqslant 4+1/2-1/r$, the error losses a factor of order $h^{3/2(1/2-1/r)}$ with respect to the case $L^\infty(0,T;l^2(h\mathbb{Z}))$ which can be handled by classical energy methods (see Example 1 in Section 3.2). Summarizing, we see that the dispersive properties of numerical schemes are needed to guarantee that the convergence rate of numerical solutions is kept in the spaces $L^q(0,T;l^r(h\mathbb{Z}))$.

In the context of the NSE we prove that the dispersive methods introduced in this paper converge to the solutions of NSE with the same order as in the linear problem. To be more precise, in Theorem 5.4 we prove a polynomial order of convergence, $h^{s/2}$, in the case of a dispersive approximation scheme of order two for the Laplace operator for initial data $H^s(\mathbb{R}^d)$ when $0<s<4$. In the case of the classical non-dispersive schemes this convergence rate can only be guaranteed for smooth enough initial data, $H^s(\mathbb{R})$, $1/2<s<4$ (see Theorem 6.1).

In Section 6 we show that non-dispersive numerical schemes with rough data behave badly. Indeed, when using non-dispersive numerical schemes, combined with a $H^1(\mathbb{R})$-approximation of the initial data $\varphi \in H^s(\mathbb{R})\setminus H^1(\mathbb{R})$, one gets an order of convergence $|\log h|^{-s/(1-s)}$ which is much weaker than the $h^{s/2}$-one that dispersive schemes ensure.

The paper is organized as follows. In Section 2 we first obtain a quite general result which allows us to estimate the difference of two families of operators that admit Strichartz estimates. We then particularize it to operators acting on discrete spaces $l^p(h\mathbb{Z})$, obtaining results which will be used in the following sections to get the order of convergence for approximations of the NSE. In Sections 3 and 4 we revisit the dispersive schemes for LSE introduced in [15–17,19] which are based, respectively, on the use of artificial numerical viscosity and a two-grid preconditioning technique of the initial data.

Section 5 is devoted to analyze approximations of the NSE based on the dispersive schemes analyzed in previous sections. Section 6 contains classical material on conservative schemes that we include here in order to emphasize the advantages of the dispersive methods. Finally, Section 7 contains some technical results used along the paper.

The analysis in this paper can be extended to fully discrete dispersive schemes introduced and analyzed in [13] and to the multidimensional case. However, several technical aspects need to be dealt with carefully. In particular, one has to take care of the well-posedness of the NSE (see [5,24]). Furthermore, suitable versions of the technical harmonic analysis results employed in the paper (see, for instance, Section 7) would also be needed (see [12]). This will be the object of future work.

Our methods use Fourier analysis techniques in an essential manner. Adapting this theory to numerical approximation schemes in non-regular meshes is by now a completely open subject.

2. Estimates on linear semigroups

In this section we will obtain $L^q_t L^r_x$ estimates for the difference of two semigroups $S_A(t)$ and $S_B(t)$ which admit Strichartz estimates. Once this result is obtained in an abstract setting we particularize it to the discrete spaces $l^p(h\mathbb{Z})$.

2.1. An abstract result

First we state a well-known result by Keel and Tao [22].

Proposition 2.1. *(See [22, Theorem 1.2].) Let H be a Hilbert space, (X, dx) be a measure space and $U(t): H \to L^2(X)$ be a one parameter family of mappings with $t \in \mathbb{R}$, which obey the energy estimate*

$$\left\| U(t) f \right\|_{L^2(X)} \leqslant C \| f \|_H \tag{2.1}$$

and the decay estimate

$$\left\| U(t) U(s)^* g \right\|_{L^\infty(X)} \leqslant C |t-s|^{-\alpha} \| g \|_{L^1(X)} \tag{2.2}$$

for some $\alpha > 0$. Then

$$\left\| U(t) f \right\|_{L^q(\mathbb{R}, L^r(X))} \leqslant C \| f \|_H, \tag{2.3}$$

$$\left\| \int_{\mathbb{R}} \big(U(s)^* F(s, \cdot) \big)\, ds \right\|_H \leqslant C \| F \|_{L^{q'}(\mathbb{R}, L^{r'}(X))}, \tag{2.4}$$

$$\left\| \int_0^t U(t-s) F(s)\, ds \right\|_{L^q(\mathbb{R}, L^r(X))} \leqslant C \| F \|_{L^{\tilde{q}'}(\mathbb{R}, L^{\tilde{r}'}(X))} \tag{2.5}$$

for all (q, r) and $(\tilde{q}, \tilde{r})$, α-admissible pairs.

The following theorem provides the key estimate in obtaining the order of convergence when the LSE is approximated by a dispersive scheme.

Theorem 2.1. *Let (X, dx) be a measure space, $A: D(A) \to L^2(X)$, $B: D(B) \to L^2(X)$ two linear m-dissipative operators with $D(A) \hookrightarrow D(B)$ continuously and satisfying $AB = BA$. Assume that $(S_A(t))_{t \geqslant 0}$ and $(S_B(t))_{t \geqslant 0}$ the semigroups generated by A and B satisfy assumptions (2.1) and (2.2) with $H = L^2(X)$. Then for any two α-admissible pairs (q, r), $(\tilde{q}, \tilde{r})$ the following hold:*

(i) *There exists a positive constant $C(q)$ such that*

$$\left\| S_A(t)\varphi - S_B(t)\varphi \right\|_{L^q(I, L^r(X))} \leqslant C(q) \min\left\{ \|\varphi\|_{L^2(X)}, |I| \left\| (A-B)\varphi \right\|_{L^2(X)} \right\} \tag{2.6}$$

for all bounded intervals I and $\varphi \in D(A)$.

(ii) *There exists a positive constant $C(q, \tilde{q})$ such that*

$$\begin{aligned} &\left\| \int_0^t S_A(t-s) f(s)\, ds - \int_0^t S_B(t-s) f(s)\, ds \right\|_{L^q(I, L^r(X))} \\ &\quad \leqslant C(q, \tilde{q}) \min\left\{ \| f \|_{L^{\tilde{q}'}(I, L^{\tilde{r}'}(X))}, |I| \left\| (A-B) f \right\|_{L^{\tilde{q}'}(I, L^{\tilde{r}'}(X))} \right\} \end{aligned} \tag{2.7}$$

for all bounded intervals I and $f \in L^{\tilde{q}'}(I, L^{\tilde{r}'}(X))$ such that $(A-B) f \in L^{\tilde{q}'}(I, L^{\tilde{r}'}(X))$.

Proof. Using that the operators S_A and S_B verify hypotheses (2.1) and (2.2) of Proposition 2.1 with $H = L^2(X)$, by (2.3) we obtain

$$\left\| S_A(t)\varphi - S_B(t)\varphi \right\|_{L^q(I, L^r(X))} \leqslant C(q) \|\varphi\|_{L^2(X)} \tag{2.8}$$

and, by (2.5),

$$\left\| \int_0^t S_A(t-s) f(s)\, ds - \int_0^t S_B(t-s) f(s)\, ds \right\|_{L^q(\mathbb{R}, L^r(X))} \leqslant C(q, \tilde{q}) \| f \|_{L^{\tilde{q}'}(\mathbb{R}, L^{\tilde{r}'}(X))}. \tag{2.9}$$

In view of (2.8) and (2.9) it is then sufficient to prove the following estimates:

$$\left\| S_A(t)\varphi - S_B(t)\varphi \right\|_{L^q(I, L^r(X))} \leqslant C(q) |I| \left\| (A-B)\varphi \right\|_{L^2(X)} \tag{2.10}$$

and

$$\left\| \int_0^t S_A(t-s)f(s)\,ds - \int_0^t S_B(t-s)f(s)\,ds \right\|_{L^q(I,L^r(X))} \leqslant C(q,\tilde{q})|I| \left\|(A-B)f\right\|_{L^{\tilde{q}'}(I,L^{\tilde{r}'}(X))}. \tag{2.11}$$

In the case of (2.10) we write the difference $S_A(\cdot)-S_B(\cdot)$ as follows

$$S_A(t)\varphi - S_B(t)\varphi = \int_0^t S_B(t-s)(A-B)S_A(s)\varphi\,ds. \tag{2.12}$$

In order to justify this identity let us recall that for any $\varphi \in D(A) \hookrightarrow D(B)$ we have that $u(t) = S_A(t)\varphi \in C([0,\infty), D(A)) \cap C^1([0,\infty), L^2(X))$ and $v(t) = S_B(t)\varphi \in C([0,\infty), D(B)) \cap C^1([0,\infty), L^2(X))$ verify the systems $u_t = Au$, $u(0) = \varphi$, and $v_t = Bv$, $v(0) = \varphi$ respectively. Thus $w = u - v \in C([0,\infty), D(B)) \cap C^1([0,\infty), L^2(X))$ satisfy the system $w_t = Bw + (A-B)u$, $w(0) = 0$. Since $(A-B)u \in C([0,\infty), L^2(X))$ we obtain that w satisfies (2.12).

Going back to (2.12) and using that A and B commute we get the following identity which is the key of our estimates:

$$S_A(t)\varphi - S_B(t)\varphi = \int_0^t S_B(t-s)S_A(s)(A-B)\varphi\,ds. \tag{2.13}$$

We apply Proposition 2.1 to the semigroup $S_B(\cdot)$ and function $F(s) = S_A(s)(A-B)\varphi$ in this identity and, by (2.5) with $\tilde{r} = 2$ and $\tilde{q} = \infty$, we get

$$\begin{aligned} \left\|S_A(t)\varphi - S_B(t)\varphi\right\|_{L^q(I,L^r(X))} &\leqslant C(q)\left\|S_A(s)(A-B)\varphi\right\|_{L^1(I,L^2(X))} \\ &\leqslant C(q)|I|\left\|(A-B)\varphi\right\|_{L^2(X)}. \end{aligned} \tag{2.14}$$

Thus, (2.10) is proved. As a consequence (2.8) and (2.10) give us (2.6).

We now prove the inhomogenous estimate (2.11). Using again (2.13) we have

$$S_A(t-s)f(s) - S_B(t-s)f(s) = \int_0^{t-s} S_B(t-s-\sigma)S_A(\sigma)(A-B)f(s)\,d\sigma.$$

We integrate this identity in the s variable. Applying Fubini's theorem on the triangle $\{(s,\sigma){:}\ 0 \leqslant s \leqslant t,\ 0 \leqslant \sigma \leqslant t-s\}$ and using that A and B commute, we get:

$$\begin{aligned} \Lambda f(t) :&= \int_0^t S_A(t-s)f(s)\,ds - \int_0^t S_B(t-s)f(s)\,ds \\ &= \int_0^t \int_0^{t-s} S_B(t-s-\sigma)S_A(\sigma)(A-B)f(s)\,d\sigma\,ds \\ &= \int_0^t \int_0^{t-\sigma} S_B(t-s-\sigma)S_A(\sigma)\,ds(A-B)f(s)\,d\sigma \\ &= \int_0^t S_A(\sigma) \int_0^{t-\sigma} S_B(t-s-\sigma)(A-B)f(s)\,ds\,d\sigma \end{aligned}$$

$$\overset{\sigma\to t-\sigma}{=} \int_0^t S_A(t-\sigma)\int_0^\sigma S_B(\sigma-s)(A-B)f(s)\,ds\,d\sigma$$

$$= \int_0^t S_A(t-\sigma)\Lambda_1(A-B)f(\sigma)\,d\sigma, \tag{2.15}$$

where

$$\Lambda_1 g(t) = \int_0^t S_B(t-\tau)g(\tau)\,d\tau.$$

Applying the inhomogeneous estimate (2.5) to the operator $S_A(\cdot)$ with $(\tilde{q}', \tilde{r}') = (1, 2)$ we obtain

$$\|\Lambda f\|_{L^q(I,L^r(X))} \leqslant C(q)\big\|\Lambda_1(A-B)f\big\|_{L^1(I,L^2(X))} \leqslant C(q)|I|\big\|\Lambda_1(A-B)f\big\|_{L^\infty(I,L^2(X))}. \tag{2.16}$$

Using again (2.5) for the semigroup $S_B(\cdot)$, $F = (A-B)f$ and $(q,r) = (\infty, 2)$ we get

$$\big\|\Lambda_1(A-B)f\big\|_{L^\infty(I,L^2(X))} \leqslant C(\tilde{q})\big\|(A-B)f\big\|_{L^{\tilde{q}'}(I,L^{\tilde{r}'}(X))}. \tag{2.17}$$

Combining (2.16) and (2.17) we deduce (2.11). Estimates (2.9) and (2.11) finish the proof. □

Remark 2.1. We point out that, in the proof of the following estimate:

$$\big\|S_A(t)\varphi - S_B(t)\varphi\big\|_{L^q(I,l^r(X))} \leqslant C(q)|I|\big\|(A-B)\varphi\big\|_{L^2(X)},$$

in view of (2.13) and (2.14), we do not need that the two operators $S_A(t)$ and $S_B(t)$ admit Strichartz estimates. Indeed, it is sufficient to assume that only one of the involved operators admits Strichartz estimates and the other one to be stable in $L^2(X)$.

2.2. *Spaces and notations*

In this section we introduce the spaces we will use along the paper. The computational mesh is $h\mathbb{Z} = \{jh\colon\ j \in \mathbb{Z}\}$ for some $h > 0$ and the $l^p(h\mathbb{Z})$ spaces are defined as follows:

$$l^p(h\mathbb{Z}) = \big\{\varphi : h\mathbb{Z} \to \mathbb{C}\colon\ \|\varphi\|_{l^p(h\mathbb{Z})} < \infty\big\}$$

where

$$\|\varphi\|_{l^p(h\mathbb{Z})} = \begin{cases} (h\sum_{j\in\mathbb{Z}}|u(jh)|^p)^{1/p}, & 1 \leqslant p < \infty, \\ \sup_{j\in\mathbb{Z}}|u(jh)|, & p = \infty. \end{cases}$$

On the Hilbert space $l^2(h\mathbb{Z})$ we will consider the following scalar product

$$(u, v)_h = \mathrm{Re}\bigg(h\sum_{j\in\mathbb{Z}} u(jh)\overline{v(jh)}\bigg).$$

When necessary, to simplify the presentation, we will write $(\varphi_j)_{j\in\mathbb{Z}}$ instead of $(\varphi(jh))_{j\in\mathbb{Z}}$.

For a discrete function $\{\varphi(jh)\}_{j\in\mathbb{Z}}$ we denote by $\hat{\varphi}$ its discrete Fourier transform:

$$\hat{\varphi}(\xi) = h\sum_{j\in\mathbb{Z}} e^{-ij\xi h}\varphi(jh). \tag{2.18}$$

For $s \geqslant 0$ and $1 < p < \infty$, $W^{s,p}(\mathbb{R})$ denotes the Sobolev space

$$W^{s,p}(\mathbb{R}) = \big\{\varphi \in \mathcal{S}'(\mathbb{R})\colon\ (I-\Delta)^{s/2}\varphi \in L^p(\mathbb{R})\big\}$$

with the norm

$$\|\varphi\|_{W^{s,p}(\mathbb{R})} = \big\|\big((1+|\xi|^2)^{s/2}\hat{\varphi}\big)^\vee\big\|_{L^p(\mathbb{R})},$$

and by $H^s(\mathbb{R})$ the Hilbert space $W^{s,2}(\mathbb{R})$.

The homogenous spaces $\dot{W}^{s,p}(\mathbb{R})$, $s \geqslant 0$ and $1 \leqslant p < \infty$, are given by

$$\dot{W}^{s,p}(\mathbb{R}) = \{\varphi \in \mathcal{S}'(\mathbb{R}):\ (-\Delta)^{s/2}\varphi \in L^p(\mathbb{R})\}$$

endowed with the semi-norm

$$\|\varphi\|_{\dot{W}^{s,p}(\mathbb{R})} = \big\|\big(|\xi|^s\hat{\varphi}\big)^\vee\big\|_{L^p(\mathbb{R})}.$$

If $p = 2$ we denote $\dot{H}^s(\mathbb{R}) = \dot{W}^{s,2}(\mathbb{R})$.

We will also use the Besov spaces both in the continuous and the discrete framework. It is convenient to consider a function $\eta_0 \in C_c(\mathbb{R})$ such that

$$\eta_0(\xi) = \begin{cases} 1 & \text{if } |\xi| \leqslant 1, \\ 0 & \text{if } |\xi| \geqslant 2, \end{cases}$$

and to define the sequence $(\eta_j)_{j\geqslant 1} \in \mathcal{S}(\mathbb{R})$ by

$$\eta_j = \eta_0\left(\frac{\xi}{2^j}\right) - \eta_0\left(\frac{\xi}{2^{j-1}}\right)$$

in order to define the Littlewood–Paley decomposition. For any $j \geqslant 0$ we set the cut-off projectors, $P_j\varphi$, as follows:

$$P_j\varphi = (\eta_j\hat{\varphi})^\vee. \tag{2.19}$$

We point out that these projectors can be defined both for functions of continuous and discrete variables by means of the classical and the semi-discrete Fourier transform.

Classical results on Fourier multipliers, namely Marcinkiewicz's multiplier theorem, (see Theorem 7.1) show the following uniform estimate on the projectors P_j: For all $p \in (1,\infty)$ there exists $c(p)$ such that

$$\|P_j\varphi\|_{L^p(\mathbb{R})} \leqslant c(p)\|\varphi\|_{L^p(\mathbb{R})}, \quad \forall \varphi \in L^p(\mathbb{R}). \tag{2.20}$$

We introduce the Besov spaces $B^s_{p,2}(\mathbb{R})$ for $1 \leqslant p \leqslant \infty$ by $B^s_{p,2} = \{u \in \mathcal{S}'(\mathbb{R}):\ \|u\|_{B^s_{p,2}(\mathbb{R})} < \infty\}$ with

$$\|u\|_{B^s_{p,2}(\mathbb{R})} = \|P_0u\|_{L^p(\mathbb{R})} + \left(\sum_{j=1}^{\infty} 2^{2sj}\|P_ju\|^2_{L^p(\mathbb{R})}\right)^{1/2}.$$

Their discrete counterpart $B^s_{p,2}(h\mathbb{Z})$ with $1 < p < \infty$ and $s \in \mathbb{R}$ is given by

$$B^s_{p,2}(h\mathbb{Z}) = \{u:\ \|u\|_{B^s_{p,2}(h\mathbb{Z})} < \infty\},$$

with

$$\|u\|_{B^s_{p,2}(h\mathbb{Z})} = \|P_0u\|_{l^p(h\mathbb{Z})} + \left(\sum_{j=1}^{\infty} 2^{2js}\|P_ju\|^2_{l^p(h\mathbb{Z})}\right)^{1/2}, \tag{2.21}$$

where P_ju given as in (2.19) are now defined by means of the discrete Fourier transform of the discrete function $u : h\mathbb{Z} \to \mathbb{C}$.

We will also adapt well-known results from harmonic analysis to the discrete framework. We recall now a result which goes back to Plancherel and Polya [25] (see also [31], Theorem 17, p. 96, and the comments on p. 182).

Lemma 2.1. *(See [25, p. 157].) For any $p \in (1,\infty)$ there exist two positive constants $A(p)$ and $B(p)$ such that the following holds for all functions f whose Fourier transform is supported on $[-\pi,\pi]$:*

$$A(p)\sum_{m\in\mathbb{Z}}\big|f(m)\big|^p \leqslant \int_{\mathbb{R}}\big|f(x)\big|^p\,dx \leqslant B(p)\sum_{m\in\mathbb{Z}}\big|f(m)\big|^p. \tag{2.22}$$

This result permits to show, by scaling, that, for all $h > 0$,

$$A(p)^{1/p}\|f\|_{l^p(h\mathbb{Z})} \leqslant \|f\|_{L^p(\mathbb{R})} \leqslant B(p)^{1/p}\|f\|_{l^p(h\mathbb{Z})} \tag{2.23}$$

holds for all functions f with their Fourier transform supported in $[-\pi/h, \pi/h]$.

For the sake of completeness we state now the discrete version of the well-known uniform L^p-estimate (2.20) for the cut-off projectors P_j.

Lemma 2.2. *For any $p \in (1,\infty)$ there exists a positive constant $c(p)$ such that*

$$\|P_j\varphi\|_{l^p(h\mathbb{Z})} \leqslant c(p)\|\varphi\|_{l^p(h\mathbb{Z})} \tag{2.24}$$

holds for all $\varphi \in l^p(h\mathbb{Z})$, $j \geqslant 0$, uniformly in $h > 0$.

Proof. For a given discrete function φ we consider its interpolator $\tilde{\varphi}$ defined as follows:

$$\tilde{\varphi}(x) = \int_{-\pi/h}^{\pi/h} e^{ix\xi}\hat{\varphi}(\xi)\,d\xi.$$

Thus, by (2.23) we obtain

$$\|P_j\varphi\|_{l^p(h\mathbb{Z})} \leqslant c(p)\big\|(P_j\varphi)\tilde{}\big\|_{L^p(\mathbb{R})} = c(p)\|P_j\tilde{\varphi}\|_{L^p(\mathbb{R})} \leqslant c(p)\|\tilde{\varphi}\|_{L^p(\mathbb{R})} \leqslant c(p)\|\varphi\|_{l^p(h\mathbb{Z})}. \qquad \square$$

We recall the following lemma which is a consequence of the Paley–Littlewood decomposition in the x variable and Minkowski's inequality in the time variable.

Lemma 2.3. *(See [26, Chapter 5, p. 113, Lemma 5.2].) Let $\eta \in C_c^\infty(\mathbb{R})$ and P_j be defined as in (2.19). Then*

$$\|\psi\|^2_{L^q(\mathbb{R},L^r(\mathbb{R}))} \lesssim \sum_{j\geqslant 0}\|P_j\psi\|^2_{L^q(\mathbb{R},L^r(\mathbb{R}))} \quad \text{if } 2 \leqslant r < \infty \text{ and } 2 \leqslant q \leqslant \infty \tag{2.25}$$

and

$$\sum_{j\geqslant 0}\|P_j\psi\|^2_{L^q(\mathbb{R},L^r(\mathbb{R}))} \lesssim \|\psi\|^2_{L^q(\mathbb{R},L^r(\mathbb{R}))} \quad \text{if } 1 \leqslant r < 2 \text{ and } 1 \leqslant q \leqslant 2 \tag{2.26}$$

hold for all $\psi \in L^q(\mathbb{R}, L^r(\mathbb{R}))$.

Applying the above result and Lemma 2.1 to functions with their Fourier transform supported in $[-\pi/h, \pi/h]$, as above, we can obtain a similar result in a discrete framework.

Lemma 2.4. *Let $\eta \in C_c^\infty(\mathbb{R})$ and P_j defined as in (2.19). Then*

$$\|\psi\|^2_{L^q(\mathbb{R},l^r(h\mathbb{Z}))} \lesssim \sum_{j\geqslant 0}\|P_j\psi\|^2_{L^q(\mathbb{R},l^r(h\mathbb{Z}))} \quad \text{if } 2 \leqslant r < \infty \text{ and } 2 \leqslant q \leqslant \infty \tag{2.27}$$

and

$$\sum_{j\geqslant 0}\|P_j\psi\|^2_{L^q(\mathbb{R},l^r(h\mathbb{Z}))} \lesssim \|\psi\|^2_{L^q(\mathbb{R},l^r(h\mathbb{Z}))} \quad \text{if } 1 \leqslant r < 2 \text{ and } 1 \leqslant q \leqslant 2 \tag{2.28}$$

hold for all $\psi \in L^q(\mathbb{R}, l^r(h\mathbb{Z}))$, uniformly in $h > 0$.

2.3. Operators on $l^p(h\mathbb{Z})$-spaces

In the following we apply the results of the previous section to the particular case $X = h\mathbb{Z}$. We consider operators A_h with symbol $a_h : [-\pi/h, \pi/h] \to \mathbb{C}$ such that

$$(A_h\varphi)_j = \int_{-\pi/h}^{\pi/h} e^{ij\xi h}a_h(\xi)\hat{\varphi}(\xi)\,d\xi, \quad j \in \mathbb{Z}.$$

Also we will consider the operator $|\nabla|^s$ acting on discrete spaces $l^2(h\mathbb{Z})$ whose symbol is given by $|\xi|^s$.

The numerical schemes we shall consider, associated to regular meshes, will enter in this frame by means of the Fourier representation formula of solutions.

Theorem 2.2. *Let $A_h, B_h : l^2(h\mathbb{Z}) \to l^2(h\mathbb{Z})$ be two operators whose symbols are a_h and b_h, ib_h being a real function, such that the semigroups they generate, $(S_{A_h}(t))_{t\geqslant 0}$ and $(S_{B_h}(t))_{t\geqslant 0}$, satisfy assumptions (2.1) and (2.2) with some constant C, independent of h. Finally, assume that for some functions $\{\mu(k,h)\}_{k\in F}$, with F a finite set, the following holds for all $\xi \in [-\pi/h, \pi/h]$:*

$$\left|a_h(\xi) - b_h(\xi)\right| \leqslant \sum_{k\in F} \mu(k,h)|\xi|^k. \tag{2.29}$$

For any $s > 0$, denoting

$$\varepsilon(s,h) = \sum_{k\in F} \mu(k,h)^{\min\{s/k,1\}}, \tag{2.30}$$

the following hold for all (q,r), $(\tilde{q},\tilde{r})$, α-admissible pairs:

(a) *There exists a positive constant $C(q)$ such that*

$$\left\| S_{A_h}(t)\varphi - S_{B_h}(t)\varphi \right\|_{L^q(I,l^r(h\mathbb{Z}))} \leqslant C(q)\varepsilon(s,h)\max\{1,|I|\}\|\varphi\|_{B^s_{2,2}(h\mathbb{Z})} \tag{2.31}$$

holds for all $\varphi \in B^s_{2,2}(h\mathbb{Z})$ uniformly in $h > 0$.

(b) *There exists a positive constant $C(s,q,\tilde{q})$ such that*

$$\begin{aligned} &\left\| \int_0^t S_{A_h}(t-\sigma)f(\sigma)\,d\sigma - \int_0^t S_{B_h}(t-\sigma)f(\sigma)\,d\sigma \right\|_{L^q(I,l^r(h\mathbb{Z}))} \\ &\quad \leqslant C(s,q,\tilde{q})\varepsilon(s,h)\max\{1,|I|\}\|f\|_{L^{\tilde{q}'}(I,B^s_{\tilde{r}',2}(h\mathbb{Z}))} \end{aligned} \tag{2.32}$$

holds for all $f \in L^{\tilde{q}'}(I, B^s_{\tilde{r}',2}(h\mathbb{Z}))$.

Remark 2.2. The assumption that the semigroups $(S_{A_h}(t))_{t\geqslant 0}$ and $(S_{B_h}(t))_{t\geqslant 0}$, satisfy (2.1) and (2.2) with some constant C, independent of h, means that both of them are $l^2(h\mathbb{Z})$-stable with constants that are independent of h and that the corresponding numerical schemes are dispersive.

Taking into account that both operators, A_h and B_h, commute in view that they are associated to their symbols, the hypotheses of Theorem 2.1 are fulfilled. They also commute with $|\nabla|$ and P_j which are also defined by a Fourier symbol.

Assumption (2.29) on the operators A_h and B_h implies

$$\left\|(A_h - B_h)\varphi\right\|_{l^2(h\mathbb{Z})} \lesssim \sum_{k\in F} a(k,h)\left\||\nabla|^k\varphi\right\|_{l^2(h\mathbb{Z})}.$$

However, this assumption is not sufficient to obtain a similar estimate in $l^r(h\mathbb{Z})$-norms, $r \neq 2$. As we will see this will be a drawback in obtaining (2.32) as a consequence of (2.7).

The requirement that ib_h is a real function is needed to assure that the semigroup generated by B_h, S_{B_h}, satisfies

$$S_{B_h}(t-\sigma) = S_{B_h}(t)S_{B_h}(-\sigma) = S_{B_h}(t)S_{B_h}(\sigma)^*,$$

identity which will be used in the proof.

In Section 3 we will give examples of operators A_h and B_h verifying these hypotheses. In all our estimates we will choose $b_h(\xi) = i\xi^2$, which is the symbol of the continuous Schrödinger semigroup.

Proof of Theorem 2.2. We divide the proof in two steps corresponding to the proof of (2.31) and (2.32) respectively.

Step I. Proof of (2.31). We apply inequality (2.25) to the difference $S_{A_h}(t)\varphi - S_{B_h}(t)\varphi$:

$$\left\|S_{A_h}(t)\varphi - S_{B_h}(t)\varphi\right\|_{L^q(I;l^r(h\mathbb{Z}))} \leqslant \left(\sum_{j\geqslant 0}\left\|P_jS_{A_h}(t)\varphi - P_jS_{B_h}(t)\varphi\right\|^2_{L^q(I,l^r(h\mathbb{Z}))}\right)^{1/2}.$$

Using that P_j commutes with $S_{A_h}(\cdot)$ and $S_{B_h}(\cdot)$ we get:

$$\left\| S_{A_h}(t)\varphi - S_{B_h}(t)\varphi \right\|_{L^q(I;l^r(h\mathbb{Z}))} \leqslant \left(\sum_{j\geqslant 0} \left\| \left(S_{A_h}(t) - S_{B_h}(t)\right) P_j\varphi \right\|^2_{L^q(I,l^r(h\mathbb{Z}))} \right)^{1/2}. \tag{2.33}$$

In order to evaluate each term in the right-hand side of (2.33) we apply estimate (2.6) to the difference $S_{A_h}(\cdot) - S_{B_h}(\cdot)$ when acting on each projection $P_j\varphi$. Thus, using hypothesis (2.29) we obtain:

$$\begin{aligned}
\left\| S_{A_h}(t)P_j\varphi - S_{B_h}(t)P_j\varphi \right\|_{L^q(I,l^r(h\mathbb{Z}))} &\leqslant C(q)\max\{|I|,1\}\min\left\{\|P_j\varphi\|_{l^2(h\mathbb{Z})}, \left\|(A_h - B_h)P_j\varphi\right\|_{l^2(h\mathbb{Z})}\right\} \\
&\leqslant C(q)\max\{|I|,1\}\min\left\{\|P_j\varphi\|_{l^2(h\mathbb{Z})}, \sum_{k\in F}\mu(k,h)\left\||\nabla|^k P_j\varphi\right\|_{l^2(h\mathbb{Z})}\right\} \\
&\leqslant C(q)\max\{|I|,1\}\sum_{k\in F}\min\left\{\|P_j\varphi\|_{l^2(h\mathbb{Z})}, \mu(k,h)2^{jk}\|P_j\varphi\|_{l^2(h\mathbb{Z})}\right\} \\
&\leqslant C(q)\max\{|I|,1\}\|P_j\varphi\|_{L^2(\mathbb{R})}\sum_{k\in F}\min\left\{1,\mu(k,h)2^{jk}\right\}.
\end{aligned} \tag{2.34}$$

Going back to estimate (2.33) we get

$$\left\| S_{A_h}(t)\varphi - S_{B_h}(t)\varphi \right\|_{L^q(I;l^r(h\mathbb{Z}))} \leqslant C(q)\max\{|I|,1\}\left(\sum_{j\geqslant 0}\|P_j\varphi\|^2_{L^2(\mathbb{R})}\sum_{k\in F}\min\left\{1,\mu^2(k,h)2^{2jk}\right\}\right)^{1/2}.$$

We claim that for any $j \geqslant 0$ the following holds:

$$\sum_{k\in F}\min\left\{1,\mu^2(k,h)2^{2jk}\right\} \leqslant \sum_{k\in F}\mu(k,h)^{\min\{2s/k,2\}}2^{2js} \tag{2.35}$$

for all $s > 0$.

Assuming for the moment that the claim (2.35) is correct we deduce that

$$\begin{aligned}
\left\| S_{A_h}(t)\varphi - S_{B_h}(t)\varphi \right\|_{L^q(I;l^r(h\mathbb{Z}))} &\leqslant C(q)\max\{|I|,1\}\left(\sum_{k\in F}\sum_{j\geqslant 0}\mu(k,h)^{\min\{2s/k,2\}}2^{2js}\|P_j\varphi\|^2_{l^2(h\mathbb{Z})}\right)^{1/2} \\
&= C(q)\max\{|I|,1\}\left(\sum_{k\in F}\mu(k,h)^{\min\{2s/k,2\}}\sum_{j\geqslant 0}2^{2js}\|P_j\varphi\|^2_{l^2(h\mathbb{Z})}\right)^{1/2} \\
&\leqslant C(q,F)\max\{|I|,1\}\varepsilon(s,h)\|\varphi\|_{B^s_{2,2}(\mathbb{R})}.
\end{aligned}$$

We now prove (2.35) by showing that

$$\min\left\{1,\mu 2^{jk}\right\} \leqslant \mu^{\min\{s/k,1\}}2^{js} \tag{2.36}$$

holds for all $\mu \geqslant 0$ and $j \geqslant 1$. It is obvious when $\mu \geqslant 1$. It remains to prove it in the case $\mu \leqslant 1$. For any $|\xi| \geqslant 1$ we have the following inequalities:

$$\begin{aligned}
\min\left\{1,\mu|\xi|^k\right\} &\leqslant \min\left\{1,\mu|\xi|^k\right\}^{\min\{s/k,1\}} = \min\left\{1,\mu^{\min\{s/k,1\}}|\xi|^{k\min\{s/k,1\}}\right\} \\
&\leqslant \mu^{\min\{s/k,1\}}|\xi|^{k\min\{s/k,1\}} \leqslant \mu^{\min\{s/k,1\}}|\xi|^s.
\end{aligned}$$

Applying this inequality to $\xi = 2^j$, $j \geqslant 0$, we get (2.36) and thus (2.35). The proof of the first step is now complete.

Step II. Proof of (2.32). Let us denote by Λ_h the following operator:

$$\Lambda_h f(t) = \int_0^t S_{A_h}(t-\sigma)f(\sigma)\,d\sigma - \int_0^t S_{B_h}(t-\sigma)f(\sigma)\,d\sigma.$$

As in the case of the homogenous estimate (2.31), we use a Paley–Littlewood decomposition of the function f. Inequality (2.27) and the fact that Λ_h commutes with each projection P_j give us

$$\|\Lambda_h f\|^2_{L^q(I,l^r(h\mathbb{Z}))} \leqslant c(q)\sum_{j\geqslant 0}\big\|P_j(\Lambda_h f)\big\|^2_{L^q(I,l^r(h\mathbb{Z}))} = c(q)\sum_{j\geqslant 0}\big\|\Lambda_h(P_j f)\big\|^2_{L^q(I,l^r(h\mathbb{Z}))}. \tag{2.37}$$

We claim that each term $\Lambda(P_j f)$ in the right-hand side of (2.37) satisfies:

$$\begin{aligned}&\big\|\Lambda_h(P_j f)\big\|_{L^q(I,l^r(h\mathbb{Z}))}\\ &\quad\leqslant c(q,\tilde q)\max\{1,|I|\}\min\left\{\|P_j f\|_{L^{\tilde q'}(I,l^{\tilde r'}(h\mathbb{Z}))}, \sum_{k\in F}\mu(k,h)\big\||\nabla|^k P_j f\big\|_{L^{\tilde q'}(I,l^{\tilde r'}(h\mathbb{Z}))}\right\}.\end{aligned} \tag{2.38}$$

In view of (2.36), the above claim implies

$$\begin{aligned}\big\|\Lambda_h(P_j f)\big\|_{L^q(I,l^r(h\mathbb{Z}))} &\leqslant c(q,\tilde q)\max\{1,|I|\}\min\left\{\|P_j f\|_{L^{\tilde q'}(I,l^{\tilde r'}(h\mathbb{Z}))}, \sum_{k\in F}\mu(k,h)2^{jk}\|P_j f\|_{L^{\tilde q'}(I,l^{\tilde r'}(h\mathbb{Z}))}\right\}\\ &= c(q,\tilde q)\max\{1,|I|\}\|P_j f\|_{L^{\tilde q'}(I,l^{\tilde r'}(h\mathbb{Z}))}\sum_{k\in F}\min\{1,\mu(k,h)2^{jk}\}\\ &\leqslant c(q,\tilde q)\max\{1,|I|\}\|P_j f\|_{L^{\tilde q'}(I,l^{\tilde r'}(h\mathbb{Z}))}\sum_{k\in F}\mu(k,h)^{\min\{s/k,1\}}2^{js}\\ &\leqslant c(q,\tilde q)\max\{1,|I|\}\varepsilon(s,h)2^{js}\|P_j f\|_{L^{\tilde q'}(I,l^{\tilde r'}(h\mathbb{Z}))}.\end{aligned} \tag{2.39}$$

Estimates (2.37) and (2.39) give us

$$\|\Lambda_h f\|_{L^q(I,l^r(h\mathbb{Z}))} \leqslant c(q,\tilde q)\max\{1,|I|\}\varepsilon(s,h)\left(\sum_{j\geqslant 0}2^{2js}\|P_j f\|^2_{L^{\tilde q'}(I,l^{\tilde r'}(h\mathbb{Z}))}\right)^{1/2}. \tag{2.40}$$

Using that $\tilde q'\leqslant 2$, we can use the reverse Minkowski's inequality in $L^{\tilde q'/2}(I)$ to get

$$\begin{aligned}\sum_{j\geqslant 0}2^{2js}\|P_j f\|^2_{L^{\tilde q'}(I,l^{\tilde r'}(h\mathbb{Z}))} &= \sum_{j\geqslant 0}\big\|2^{2js}\|P_j f\|^2_{l^{\tilde r'}(h\mathbb{Z})}\big\|_{L^{\tilde q'/2}(I)} \leqslant \left\|\sum_{j\geqslant 0}2^{2js}\|P_j f\|^2_{l^{\tilde r'}(h\mathbb{Z})}\right\|_{L^{\tilde q'/2}(I)}\\ &\lesssim \left\|\left(\sum_{j\geqslant 0}2^{2js}\|P_j f\|^2_{l^{\tilde r'}(h\mathbb{Z})}\right)^{1/2}\right\|^2_{L^{\tilde q'}(I)} = \|f\|^2_{L^{\tilde q'}(I,B^s_{\tilde r,2}(h\mathbb{Z}))}.\end{aligned}$$

By (2.40) we get

$$\|\Lambda_h f\|_{L^q(I,l^r(h\mathbb{Z}))} \leqslant c(q,\tilde q)\max\{1,|I|\}\varepsilon(s,h)\|f\|_{L^{q'}(I,B^s_{\tilde r,2})}$$

which finishes the proof.

In the following we prove (2.38). Using that both operators S_{A_h} and S_{B_h} fulfill uniform Strichartz estimates, it is sufficient to prove that, under hypothesis (2.29), the following estimate holds for all functions $f\in L^{\tilde q'}(I,l^{\tilde r'}(h\mathbb{Z}))$:

$$\|\Lambda_h f\|_{L^q(I,l^r(h\mathbb{Z}))} \leqslant c(q,\tilde q)|I|\sum_{k\in F}a(k,h)\big\||\nabla|^k f\big\|_{L^{\tilde q'}(I,l^{\tilde r'}(h\mathbb{Z}))}. \tag{2.41}$$

We point out that, in general, this estimate is not a direct consequence of (2.7) since, under assumption (2.29), we cannot establish the following inequality (of course, in the particular case $\tilde r'=2$ this can be obtained by Plancherel's identity):

$$\big\|(A_h-B_h)f\big\|_{L^{\tilde q'}(I,l^{\tilde r'}(h\mathbb{Z}))} \lesssim \sum_{k\in F}a(k,h)\big\||\nabla|^k f\big\|_{L^{\tilde q'}(I,l^{\tilde r'}(h\mathbb{Z}))}.$$

Identity (2.15) gives us that

$$\Lambda_h f(t) = \int_0^t S_{A_h}(t-s)\Lambda_{1h}(A_h-B_h)f(s)\,ds,$$

where

$$\Lambda_{1h} g(t) = \int_0^t S_{B_h}(t-\sigma) g(\sigma)\, d\sigma.$$

The inhomogeneous estimate (2.5) with $(\tilde{q}', \tilde{r}') = (1, 2)$ shows that

$$\|\Lambda_h f\|_{L^q(I,l^r(h\mathbb{Z}))} \leqslant c(q) \big\| \Lambda_{1h}(A_h - B_h) f \big\|_{L^1(I,l^2(h\mathbb{Z}))}. \tag{2.42}$$

Using that B_h satisfies $S_{B_h}(t-\sigma) = S_{B_h}(t) S_{B_h}(-\sigma) = S_{B_h}(t) S_{B_h}(\sigma)^*$ and that it commutes with A_h we get

$$\Lambda_{1h}(A_h - B_h) f(t) = S_{B_h}(t)(A_h - B_h) \int_0^t S_{B_h}(\sigma)^* f(\sigma)\, d\sigma.$$

Thus, using the uniform stability property, with respect to h, of the operators S_{B_h}:

$$\big\| S_{B_h}(\cdot) \big\|_{l^2(h\mathbb{Z}) \to l^2(h\mathbb{Z})} \lesssim 1$$

and hypothesis (2.29) we get

$$\begin{aligned} \big\| \Lambda_{1h}(A_h - B_h) f \big\|_{L^1(I,l^2(h\mathbb{Z}))} &\leqslant \left\| (A_h - B_h) \int_0^t S_{B_h}(\sigma)^* f(\sigma)\, d\sigma \right\|_{L^1(I,l^2(h\mathbb{Z}))} \\ &\leqslant \sum_{k \in F} a(k,h) \left\| |\nabla|^k \int_0^t S_{B_h}(\sigma)^* f(\sigma)\, d\sigma \right\|_{L^1(I,l^2(h\mathbb{Z}))}. \end{aligned} \tag{2.43}$$

Using that B_h and $|\nabla|$ commute, estimate (2.4) with $U(\cdot) = S_{B_h}(\cdot)$ gives us that

$$\begin{aligned} \big\| \Lambda_{1h}(A_h - B_h) f \big\|_{L^1(I,l^2(h\mathbb{Z}))} &\leqslant |I| \sum_{k \in F} a(k,h) \left\| \int_0^t S_{B_h}(s)^* |\nabla|^k f(\sigma)\, d\sigma \right\|_{L^\infty(I,l^2(h\mathbb{Z}))} \\ &\leqslant |I| \sum_{k \in F} a(k,h) \sup_{J \subset I} \left\| \int_J S_{B_h}(\sigma)^* |\nabla|^k f(\sigma)\, d\sigma \right\|_{l^2(h\mathbb{Z})} \\ &\leqslant c(\tilde{q}) |I| \sum_{k \in F} a(k,h) \big\| |\nabla|^k f \big\|_{L^{\tilde{q}'}(I,l^{\tilde{r}'}(h\mathbb{Z}))}. \end{aligned}$$

Thus, by (2.42) we obtain (2.41) which finishes the proof. □

3. Dispersive schemes for the linear Schrödinger equation

In this section we obtain error estimates for the numerical approximations of the linear Schrödinger equation. We do this not only in the $l^2(h\mathbb{Z})$-norm but also in the auxiliary spaces that are needed in the analysis of the nonlinear Schrödinger equation.

3.1. *A general result*

The numerical schemes we shall consider can all be written in the abstract form

$$\begin{cases} i u_t^h(t) + A_h u^h = 0, & t > 0, \\ u^h(0) = \mathbf{T}_h \varphi. \end{cases} \tag{3.1}$$

We assume that the operator A_h is an approximation of the $1-d$ Laplacian. On the other hand, $\mathbf{T}_h\varphi$ is an approximation of the initial data φ, $\mathbf{T}_h$ being a map from $L^2(\mathbb{R})$ into $l^2(h\mathbb{Z})$ defined as follows:

$$(\mathbf{T}_h\varphi)(jh) = \int_{-\pi/h}^{\pi/h} e^{ijh\xi}\hat{\varphi}(\xi)\,d\xi. \tag{3.2}$$

Observe that this operator acts by truncating the continuous Fourier transform of φ on the interval $(-\pi/h, \pi/h)$ and then considering the discrete inverse Fourier transform on the grid points $h\mathbb{Z}$.

To estimate the error committed in the approximation of the LSE we assume that the operator A_h, approximating the continuous Laplacian, has a symbol a_h which satisfies

$$\left|a_h(\xi) - \xi^2\right| \leqslant \sum_{k\in F} a(k,h)|\xi|^k, \quad \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right], \tag{3.3}$$

for a finite set of indexes F. As we shall see, different approximation schemes enter in this class for different sets F and orders k.

This condition on the operator A_h suffices to analyze the rate of convergence in the $L^\infty(-T,T;l^2(h\mathbb{Z}))$ norm. However, one of our main objectives in this paper is to analyze this error in the auxiliary norms $L^q(-T,T;l^r(h\mathbb{Z}))$ which is necessary for addressing the NSE with rough initial data. More precisely, we need to identify classes of approximating operators A_h of the $1-d$ Laplacian so that the semi-discrete semigroup $\exp(itA_h)$ maps uniformly, with respect to parameter h, $l^2(h\mathbb{Z})$ into those spaces.

In the following we consider operators A_h generating dispersive schemes which are $l^2(h\mathbb{Z})$-stable

$$\left\|\exp(itA_h)\varphi\right\|_{l^2(h\mathbb{Z})} \leqslant C\|\varphi\|_{l^2(h\mathbb{Z})}, \quad \forall t \geqslant 0 \tag{3.4}$$

and satisfy the uniform $l^1(h\mathbb{Z}) - l^\infty(h\mathbb{Z})$ dispersive property:

$$\left\|\exp(itA_h)\varphi\right\|_{l^\infty(h\mathbb{Z})} \leqslant \frac{C}{|t|^{1/2}}\|\varphi\|_{l^1(h\mathbb{Z})}, \quad \forall t \geqslant 0, \tag{3.5}$$

for all $h>0$ and for all $\varphi \in l^1(h\mathbb{Z})$, where the above constant C is independent of h. We point out that (3.4) is the standard stability property while the second one, (3.5), holds only for well chosen numerical schemes.

Applying Theorem 2.2 to the operator B_h whose symbol is $-i\xi^2$ and to iA_h, A_h being the approximation of the Laplace operator with the symbol $a_h(\xi)$, we obtain the following result.

Theorem 3.1. *Let $s \geqslant 0$, A_h satisfying* (3.3), (3.4), (3.5)*, and (q,r) and $(\tilde{q},\tilde{r})$ be two $1/2$-admissible pairs. Denoting*

$$\varepsilon(s,h) = \sum_{k\in F} a(k,h)^{\min\{s/k,1\}}, \tag{3.6}$$

the following hold:

(a) *There exists a positive constant $C(q)$ such that*

$$\left\|\exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h\exp\left(it\partial_x^2\right)\varphi\right\|_{L^q(0,T;l^r(h\mathbb{Z}))} \leqslant \max\{1,T\}C(q)\varepsilon(s,h)\|\varphi\|_{H^s(\mathbb{R})} \tag{3.7}$$

for all $\varphi \in H^s(\mathbb{R})$, $T>0$ and $h>0$.

(b) *There exists a positive constant $C(q,\tilde{q})$ such that*

$$\left\|\int_0^t \exp\left(i(t-\sigma)A_h\right)\mathbf{T}_h f(\sigma)\,d\sigma - \int_0^t \mathbf{T}_h\exp\left(i(t-\sigma)\partial_x^2\right)f(\sigma)\,d\sigma\right\|_{L^q(0,T;l^r(h\mathbb{Z}))} \leqslant C(q,\tilde{q})\max\{1,T\}\varepsilon(s,h)\|f\|_{L^{\tilde{q}'}(0,T;B^s_{\tilde{r}',2}(\mathbb{R}))}, \tag{3.8}$$

for all $T>0$, $f \in L^{\tilde{q}'}(0,T;B^s_{\tilde{r}',2}(\mathbb{R}))$ and $h>0$.

Remark 3.1. In the particular case when $(q,r)=(\infty,2)$ and the set F of indices k entering in the definition (3.6) of $\varepsilon(s,h)$ is reduced to a simple element, the statements in this theorem are proved in [28, Theorem 10.1.2, p. 201]:

$$\left\|\exp(itA_h)\mathbf{T}_h\varphi-\mathbf{T}_h\exp\left(it\partial_x^2\right)\varphi\right\|_{L^\infty(0,T;l^2(h\mathbb{Z}))}\leqslant C(q)T\varepsilon(s,h)\|\varphi\|_{H^s(\mathbb{R})}. \tag{3.9}$$

Remark 3.2. Observe that for $s\geqslant s_0=\max\{k\colon k\in F\}$ the function $s\to\varepsilon(s,h)$ is independent of the s-variable:

$$\varepsilon(s,k)=\varepsilon(s_0,k)=\sum_{k\in F}a(k,h).$$

This means that imposing more than $H^{s_0}(\mathbb{R})$ regularity on the initial data does not improve the order of convergence in (3.7) and (3.8).

Remark 3.3. In the case $0\leqslant s\leqslant s_0$, with s_0 as above, the estimate $H^{s_0}(\mathbb{R})\to L^\infty(0,T;l^2(h\mathbb{Z}))$ in (3.7) and the one given by the stability of the scheme $L^2(\mathbb{R})\to L^\infty(0,T;l^2(h\mathbb{Z}))$, allow to obtain, using an interpolation argument, a weaker estimate:

$$\left\|\exp(itA_h)\mathbf{T}_h\varphi-\mathbf{T}_h\exp\left(it\partial_x^2\right)\varphi\right\|_{L^\infty(0,T;l^2(h\mathbb{Z}))}\leqslant C(T)\varepsilon(s_0,h)^{s/s_0}\|\varphi\|_{H^s(\mathbb{R})}.$$

If the set F has an unique element then this estimate is equivalent to (3.7). However, the improved estimates (3.7) and (3.8) cannot be proved without using Paley–Littlewood's decomposition, as in the proof of Theorem 2.2.

3.2. *Examples of operators A_h*

In this section we will analyze various operators A_h which approximate the $1-d$ Laplace operator ∂_x^2.

Example 1. The 3-point conservative approximation. The simplest example of approximation scheme for the Laplace operator ∂_x^2 is given by the classical finite difference approximation Δ_h

$$(\Delta_h u)_j=\frac{u_{j+1}+u_{j-1}-2u_j}{h^2}. \tag{3.10}$$

It satisfies hypothesis (3.3) with $F=\{4\}$ and $a(4,h)=h^2$. Thus, we are dealing with an approximation scheme of order two. Indeed, we have:

$$\left|\frac{4}{h^2}\sin^2\left(\frac{\xi h}{2}\right)-\xi^2\right|\lesssim h^2|\xi|^4,\quad \forall\xi\in\left[-\frac{\pi}{h},\frac{\pi}{h}\right].$$

However, this operator does not satisfy (3.5) with a constant C independent of the mesh size h (see [16, Theorem 1.1]), and Theorem 3.1 cannot be applied. This means that we cannot obtain the same estimate as for second order dispersive schemes:

$$\left\|\exp(itA_h)\mathbf{T}_h\varphi-\mathbf{T}_h\exp\left(it\partial_x^2\right)\varphi\right\|_{L^q(0,T;l^r(h\mathbb{Z}))}\leqslant C(q,T)\|\varphi\|_{H^s(\mathbb{R})}\begin{cases}h^{s/2}, & s\in(0,4),\\ h^2, & s>4.\end{cases} \tag{3.11}$$

However, using the ideas of Brenner on the order of convergence in the $l^r(h\mathbb{Z})$-norm, $r>2$ [3, Chapter 6, Theorems 3.2, 3.3 and Chapter 3, Corollary 5.1], we can get the following estimates:

$$\begin{aligned}\left\|\exp(itA_h)\mathbf{T}_h\varphi-\mathbf{T}_h\exp\left(it\partial_x^2\right)\varphi\right\|_{L^q(0,T;l^r(h\mathbb{Z}))}&\leqslant C(q,T)\|\varphi\|_{B^s_{r,\infty}(\mathbb{R})}\begin{cases}h^{\frac12(s-1+\frac2r)}, & s\in(0,4+1-\frac2r),\\ h^2, & s\geqslant 4+1-\frac2r,\end{cases}\\ &\leqslant C(q,T)\|\varphi\|_{H^{s+\frac12-\frac1r}(\mathbb{R})}\begin{cases}h^{\frac12(s-1+\frac2r)}, & s\in(0,4+1-\frac2r),\\ h^2, & s\geqslant 4+1-\frac2r,\end{cases}\end{aligned}$$

where we have used that $H^{s_0}(\mathbb{R})=B^{s_0}_{2,2}(\mathbb{R})\hookrightarrow B^s_{r,\infty}(\mathbb{R})$ when $s_0-1/2=s-1/r$.

Observe that in the case $s\in(0,4)$ the above estimate guarantees that

$$\left\|\exp(itA_h)\mathbf{T}_h\varphi-\mathbf{T}_h\exp\left(it\partial_x^2\right)\varphi\right\|_{L^q(0,T;l^r(h\mathbb{Z}))}\leqslant C(q,T)\|\varphi\|_{H^{s+\frac12-\frac1r}(\mathbb{R})}h^{\frac12(s+\frac12-\frac1r)}h^{-\frac32(\frac12-\frac1r)}. \tag{3.12}$$

Moreover for any $\sigma \in (1/2 - 1/r, 4 + 1/2 - 1/r)$ we can find $s \in (0,4)$ with $\sigma = s + 1/2 - 1/r$ and using (3.12) we obtain

$$\left\| \exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h \exp\left(it\partial_x^2\right)\varphi \right\|_{L^q(0,T;l^r(h\mathbb{Z}))} \leqslant C(q,T)\|\varphi\|_{H^\sigma(\mathbb{R})} h^{\frac{\sigma}{2}} h^{-\frac{3}{2}(\frac{1}{2}-\frac{1}{r})}. \tag{3.13}$$

In the case of an approximation of order two one could expect the error in the above estimate to be of order $h^{\sigma/2}$ as in the $L^\infty(0,T;l^2(h\mathbb{Z}))$ case. But, here we get an extra factor of order $h^{-3/2(1/2-1/r)}$ which diverges unless $r=2$, which corresponds to the classical energy estimate in $L^\infty(0,T;L^2(\mathbb{R}))$. This does not happen in the case of a second order dispersive approximation of the Schrödinger operator, where Theorem 3.1 gives us an order of error as in (3.11).

Note that, according to Theorem 3.1, this loss in the rate of convergence is due to the lack of dispersive properties of the scheme.

Also we point out that to obtain an error of order h^2 in (3.12) we need to consider initial data in $H^{4+1-2/r}(\mathbb{R})$. So we need to impose an extra regularity condition of $1-2/r$ derivatives on the initial data φ to assure the same order of convergence as the one in (3.11) for dispersive schemes.

Example 2. Fourier filtering of the 3-point conservative approximation. Another example is given by the spectral filtering $\Delta_{h,\gamma}$ defined by:

$$\Delta_{h,\gamma}\varphi = \Delta_h\left(\mathbf{1}_{(-\frac{\gamma\pi}{h},\frac{\gamma\pi}{h})}\hat{\varphi}\right)^\vee, \quad \gamma < \frac{1}{2}. \tag{3.14}$$

In other words, $\Delta_{h,\gamma}$ is a discrete operator whose action is as follows:

$$(\Delta_{h,\gamma}\varphi)_j = \int_{-\gamma\pi/h}^{\gamma\pi/h} \frac{4}{h^2}\sin^2\left(\frac{\xi h}{2}\right) e^{ijh\xi}\hat{\varphi}(\xi)\,d\xi, \quad j \in \mathbb{Z},$$

i.e. it has the symbol

$$a_{h,\gamma}(\xi) = \frac{4}{h^2}\sin^2\left(\frac{\xi h}{2}\right)\mathbf{1}_{(-\gamma\pi/h,\gamma\pi/h)}.$$

In this case

$$\left|a_{h,\gamma}(\xi) - \xi^2\right| \leqslant c(\gamma)\begin{cases} h^2\xi^4, & |\xi| \leqslant \pi\gamma/h \\ \xi^2, & |\xi| \geqslant \pi\gamma/h \end{cases} \leqslant c(\gamma)h^2\xi^4 \quad \text{for all } \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right].$$

Thus $\Delta_{h,\gamma}$ constitutes an approximation of the Laplace operator Δ of order two and the semigroup generated by $i\Delta_{h,\gamma}$ has uniform dispersive properties (see [17]). Theorem 3.1, which exploits the dispersive character of the numerical scheme, gives us

$$\left\| \exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h\exp(it\Delta)\varphi \right\|_{L^q(0,T;l^r(h\mathbb{Z}))} \leqslant C(q,T)\|\varphi\|_{H^s(\mathbb{R})}\begin{cases} h^{s/2}, & s \in (0,4), \\ h^2, & s > 4. \end{cases}$$

We note that using the same arguments based on $l^r(h\mathbb{Z})$-error estimates (given in [3]), as in Example 1, we can obtain the same result only if $r=2$ or assuming more regularity of the initial data φ.

This scheme, however, has a serious drawback to be implemented in nonlinear problems since it requires the Fourier filtering to be applied on the initial data and also on the nonlinearity, which is computationally expensive.

Example 3. Viscous approximation. To overcome the lack of uniform $L^q(I,l^r(h\mathbb{Z}))$ estimates, in [17] and [14] numerical schemes based in adding extra numerical viscosity have been introduced. The first possibility is to take $A_h = \Delta_h + ia(h)\Delta_h$ with $a(h) = h^{2-1/\alpha(h)}$ and $\alpha(h) \to 1/2$ such that $a(h) \to 0$. In this case (3.3) is satisfied as follows:

$$\left| \frac{4}{h^2}\sin^2\left(\frac{\xi h}{2}\right) + ia(h)\frac{4}{h^2}\sin^2\left(\frac{\xi h}{2}\right) - \xi^2 \right| \leqslant h^2\xi^4 + a(h)\xi^2. \tag{3.15}$$

This numerical approximation of the Schrödinger semigroup has been used in [17] and [19] to construct convergent numerical schemes for the NSE. However, the special choice of the function $a(h)$ that is required, shows that the error in the right-hand side of (3.15) goes to zero slower that any polynomial function of h and thus, at least theoretically, the convergence towards LSE, and, consequently to the NSE, will be very slow. Thus, we will not further analyze this scheme.

Example 4. A higher order viscous approximation. A possibility to overcome the drawbacks of the previous scheme, associated to the different behavior of the $l^1(h\mathbb{Z}) - l^\infty(h\mathbb{Z})$ decay rate of the solutions, is to choose higher order dissipative schemes as introduced in [14]:

$$A_h = \Delta_h - ih^{2(m-1)}(-\Delta_h)^m, \quad m \geqslant 2. \tag{3.16}$$

In this case, hypothesis (3.3) reads:

$$\left|\frac{4}{h^2}\sin^2\left(\frac{\xi h}{2}\right) + ih^{2(m-1)}\left(\frac{4}{h^2}\sin^2\left(\frac{\xi h}{2}\right)\right)^m - \xi^2\right| \leqslant h^2\xi^4 + h^{2(m-1)}\xi^{2m}. \tag{3.17}$$

Theorem 3.1 then guarantees that for any $0 \leqslant s \leqslant 4$ the following estimate holds:

$$\begin{aligned}\left\|\exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h\exp(it\Delta)\varphi\right\|_{L^q(0,T;l^r(h\mathbb{Z}))} &\leqslant \max\{1,T\}\left(h^{s/2} + h^{(m-1)s/m}\right)\|\varphi\|_{H^s(\mathbb{R})} \\ &\leqslant \max\{1,T\}h^{s/2}\|\varphi\|_{H^s(\mathbb{R})}.\end{aligned}$$

Thus we obtain the same order of error as for the discrete Laplacian $A_h = \Delta_h$ but this time not only in the $L^\infty(I; l^2(h\mathbb{Z}))$-norm but in all the auxiliary $L^q(I, l^r(h\mathbb{Z}))$-norms. We thus get the same optimal results as for the other dispersive scheme in Example 2 based on Fourier filtering.

4. A two-grid algorithm

In this section we analyze one further strategy introduced in [15,17] to recover the uniformity of the dispersive properties. It is based on the two-grid algorithm that we now describe. We consider the standard conservative 3-point approximation of the Laplacian: $A_h = \Delta_h$. But, this time, in order to avoid the lack of dispersive properties associated with the high frequency components, the scheme will be restricted to the class of slowly oscillatory data obtained by a two-grid algorithm. The main advantage of this filtering method with respect to the Fourier one is that the filtering can be realized in the physical space.

The method, inspired by [11], is roughly as follows. We consider two meshes: the coarse one of size $4h$, $h > 0$, $4h\mathbb{Z}$, and the finer one, the computational one, $h\mathbb{Z}$, of size $h > 0$. The method relies basically on solving the finite-difference semi-discretization on the fine mesh $h\mathbb{Z}$, but only for slowly oscillating data, interpolated from the coarse grid $4h\mathbb{Z}$. The $1/4$ ratio between the two meshes is important to guarantee the dispersive properties of the method. This particular structure of the data cancels the pathology of the discrete symbol at the points $\pm\pi/2h$.

To be more precise we introduce the extension operator $\boldsymbol{\Pi}_h^{4h}$ which associates to any function $\psi : 4h\mathbb{Z} \to \mathbb{C}$ a new function $\boldsymbol{\Pi}_h^{4h}\psi : h\mathbb{Z} \to \mathbb{C}$ obtained by an interpolation process:

$$\left(\boldsymbol{\Pi}_h^{4h}\psi\right)_j = \left(\mathbf{P}_{4h}^1\psi\right)(jh), \quad j \in \mathbb{Z},$$

where $\mathbf{P}_{4h}^1\psi$ is the piecewise linear interpolator of ψ.

The semi-discrete method we propose is the following:

$$\begin{cases} iu_t^h(t) + \Delta_h u^h = 0, & t > 0, \\ u^h(0) = \boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi. \end{cases} \tag{4.1}$$

The Fourier transform of the two-grid initial datum can be characterized as follows (see [17, Lemma 5.2]):

$$\left(\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi\right)^\wedge(\xi) = m(h\xi)\widetilde{\widehat{\mathbf{T}_{4h}\varphi}}(\xi), \quad \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right], \tag{4.2}$$

where $\widetilde{\widehat{\mathbf{T}_{4h}\varphi}}(\xi)$ is the extension by periodicity of the function $\widehat{\mathbf{T}_{4h}\varphi}$, initially defined on $[-\pi/4h, \pi/4h]$, to the interval $[-\pi/h, \pi/h]$, and

$$m(\xi) = \left(\frac{e^{4i\xi} - 1}{4(e^{i\xi} - 1)}\right)^2, \quad p \geqslant 2. \tag{4.3}$$

The following result, proved in [15], guarantees that system (4.1) is dispersive in the sense that the discrete version of the Strichartz inequalities hold, uniformly on $h > 0$.

Theorem 4.1. *Let (q, r), $(\tilde{q}, \tilde{r})$ be two $1/2$-admissible pairs. The following properties hold*:

(i) *There exists a positive constant $C(q)$ such that*
$$\big\|e^{it\Delta_h}\boldsymbol{\Pi}_h^{4h}\varphi\big\|_{L^q(\mathbb{R},l^r(h\mathbb{Z}))} \leqslant C(q)\big\|\boldsymbol{\Pi}_h^{4h}\varphi\big\|_{l^2(h\mathbb{Z})} \tag{4.4}$$
uniformly on $h > 0$.

(ii) *There exists a positive constant $C(d, r, \tilde{r})$ such that*
$$\bigg\|\int_{s<t} e^{i(t-s)\Delta_h}\boldsymbol{\Pi}_h^{4h} f(s)\,ds\bigg\|_{L^q(\mathbb{R},l^r(h\mathbb{Z}))} \leqslant C(q,\tilde{q})\big\|\boldsymbol{\Pi}_h^{4h} f\big\|_{L^{\tilde{q}'}(\mathbb{R},l^{\tilde{r}'}(h\mathbb{Z}))} \tag{4.5}$$
for all $f \in L^{\tilde{q}'}(\mathbb{R}, l^{\tilde{r}'}(4h\mathbb{Z}))$, uniformly in $h > 0$.

In the following lemma we estimate the error introduced by the two-grid algorithm.

Theorem 4.2. *Let $s \geqslant 0$ and (q, r), $(\tilde{q}, \tilde{r})$ be two admissible pairs.*

(a) *There exists a positive constant $C(q, s)$ such that*
$$\begin{aligned}&\big\|\exp(it\Delta_h)\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi - \mathbf{T}_h\exp\big(it\partial_x^2\big)\varphi\big\|_{L^q(I;l^r(h\mathbb{Z}))}\\ &\quad\leqslant C(q,s)\max\big\{1,|I|\big\}\big(h^{\min\{s/2,2\}} + h^{\min\{s,1\}}\big)\|\varphi\|_{H^s(\mathbb{R})},\end{aligned} \tag{4.6}$$
holds for all $\varphi \in H^s(\mathbb{R})$ and $h > 0$.

(b) *There exists a positive constant $C(q, \tilde{q}, s)$ such that*
$$\begin{aligned}&\bigg\|\int_{s<t}\exp\big(i(t-s)\Delta_h\big)\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h} f(s)\,ds - \int_{s<t}\mathbf{T}_h\exp\big(i(t-s)\partial_x^2\big)f(s)\,ds\bigg\|_{L^q(I;l^r(h\mathbb{Z}))}\\ &\quad\leqslant C(q,\tilde{q},s)\max\big\{1,|I|\big\}\big(h^{\min\{s/2,2\}} + h^{\min\{s,1\}}\big)\|f\|_{L^{\tilde{q}'}(I;B^s_{\tilde{r}',2}(\mathbb{R}))}.\end{aligned} \tag{4.7}$$

Remark 4.1. There are two error terms in the above estimates: $h^{\min\{s/2,2\}}$ and $h^{\min\{s,1\}}$. The first one comes from a second order numerical scheme generated by the approximation of the Laplacian ∂_x^2 with Δ_h and the second one from the use of a two-grid interpolator. Observe that for initial data $\varphi \in H^s(\mathbb{R})$, $s \in (0, 2)$ the results are the same as in the case of the second order schemes. Also, imposing more than $H^2(\mathbb{R})$ regularity on the initial data does not improve the order of convergence. This is a consequence of the fact that the two-grid interpolator appears. The multiplier $m(\xi)$ defined in (4.3) satisfies $m(\xi) - 1 \simeq \xi$ as $\xi \sim 0$ and then the following estimate, which occurs in the proof of Theorem 4.2,
$$\int_{-\pi/4h}^{\pi/4h} \big|m(h\xi) - 1\big|^2\big|\hat{\varphi}(\xi)\big|^2\,d\xi \lesssim \big(h\|\varphi\|_{H^1(\mathbb{R})}\big)^2,$$
cannot be improved by imposing more regularity on the function φ.

Proof of Theorem 4.2.

Case I. Proof of the homogenous estimate (4.6). Let us consider Δ^h acting on discrete functions as follows:
$$\big(\Delta^h\varphi\big)_j = \int_{-\pi/h}^{\pi/h} \xi^2 e^{ij\xi h}\hat{\varphi}(\xi)\,d\xi.$$

Note that Δ^h differs from the finite-difference approximation Δ_h on the fact that, in Δ^h, ξ^2 replaces the symbol $4/h^2\sin^2(\xi h/2)$ of Δ_h.

In view of the definition of Δ^h, we have

$$\exp\big(it\Delta^h\big)\mathbf{T}_h\varphi=\mathbf{T}_h\exp\big(it\partial_x^2\big)\varphi.$$

Using the last identity, we write

$$\begin{aligned}\exp(it\Delta_h)\boldsymbol{\Pi}_h^{4h}\mathbf{T}_h\varphi-\mathbf{T}_h\exp\big(it\partial_x^2\big)\varphi&=\exp(it\Delta_h)\boldsymbol{\Pi}_h^{4h}\mathbf{T}_h\varphi-\exp\big(it\Delta^h\big)\mathbf{T}_h\varphi\\&=I_1(t)+I_2(t)\end{aligned}$$

where:

$$I_1(t)=\exp(it\Delta_h)\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi-\exp\big(it\Delta^h\big)\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi$$

and

$$I_2(t)=\exp\big(it\Delta^h\big)\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi-\exp\big(it\Delta^h\big)\mathbf{T}_h\varphi.$$

In the following we estimate each of them.

Applying Theorem 2.2 to operators Δ_h and Δ^h we get

$$\|I_1\|_{L^q(0,T;l^r(h\mathbb{Z}))}\leqslant h^{\min\{s/2,2\}}\max\{1,T\}\big\|\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi\big\|_{B^s_{2,2}(h\mathbb{Z})}\leqslant h^{\min\{s/2,2\}}\max\{1,T\}\|\varphi\|_{H^s(\mathbb{R})}.$$

In the case of I_2 we claim that for any $s\geqslant 0$

$$\|I_2\|_{L^q(0,T;l^r(h\mathbb{Z}))}\leqslant h^{\min\{s,1\}}\|\varphi\|_{H^s(\mathbb{R})}.\tag{4.8}$$

To prove this claim, we remark that the operator $\exp(it\Delta^h)$ satisfies (2.1) and (2.2). Thus Proposition 2.1 guarantees that $\exp(it\Delta^h)$ has uniform Strichartz estimates and

$$\|I_2\|_{L^q(0,T;l^r(h\mathbb{Z}))}\leqslant\big\|\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi-\mathbf{T}_h\varphi\big\|_{l^2(h\mathbb{Z})}.\tag{4.9}$$

It is then sufficient to prove that

$$\big\|\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi-\mathbf{T}_h\varphi\big\|_{l^2(h\mathbb{Z})}\leqslant h^{\min\{s,1\}}\|\varphi\|_{H^s(\mathbb{R})}\tag{4.10}$$

holds for any $s\geqslant 0$. Actually it suffices to prove it for $0\leqslant s\leqslant 1$. Also the cases $s\in(0,1)$ follow by interpolation between the cases $s=0$ and $s=1$. We will consider now these two cases.

The case $s=0$ easily follows since

$$\big\|\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi\big\|_{l^2(h\mathbb{Z})}\lesssim\|\mathbf{T}_{4h}\varphi\|_{l^2(4h\mathbb{Z})}\lesssim\|\varphi\|_{L^2(\mathbb{R})}$$

and

$$\|\mathbf{T}_h\varphi\|_{l^2(h\mathbb{Z})}\lesssim\|\varphi\|_{L^2(\mathbb{R})}.$$

We now prove (4.10) in the case $s=1$:

$$\big\|\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi-\mathbf{T}_h\varphi\big\|_{l^2(h\mathbb{Z})}\lesssim h\|\varphi\|_{H^1(\mathbb{R})}.\tag{4.11}$$

Using that

$$\|\mathbf{T}_{4h}\varphi-\mathbf{T}_h\varphi\|_{l^2(h\mathbb{Z})}\leqslant\bigg(\int_{|\xi|\geqslant\pi/4h}\big|\hat\varphi(\xi)\big|^2\,d\xi\bigg)^{1/2}\lesssim h\|\varphi\|_{H^1(\mathbb{R})},$$

it is sufficient to prove the following estimate

$$\big\|\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi-\mathbf{T}_{4h}\varphi\big\|_{l^2(h\mathbb{Z})}\lesssim h\|\varphi\|_{H^1(\mathbb{R})}.\tag{4.12}$$

The representation formula (4.2) gives us that

$$\big\|\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h}\varphi-\mathbf{T}_{4h}\varphi\big\|^2_{l^2(h\mathbb{Z})}\leqslant\int_{-\pi/4h}^{\pi/4h}\big|m(h\xi)-1\big|^2\big|\hat\varphi(\xi)\big|^2\,d\xi+\int_{\pi/4h\leqslant|\xi|\leqslant\pi/h}\big|m(h\xi)\big|^2\big|\widetilde{\mathbf{T}_{4h}\varphi}(\xi)\big|^2\,d\xi.\tag{4.13}$$

Using that $|m(\xi)-1| \leqslant |\xi|$ for $\xi \in [-\pi/4, \pi/4]$ we obtain

$$\int_{-\pi/4h}^{\pi/4h} \left|m(h\xi)-1\right|^2 \left|\hat{\varphi}(\xi)\right|^2 d\xi \lesssim \left(h\|\varphi\|_{H^1(\mathbb{R})}\right)^2. \tag{4.14}$$

Previous results on the Fourier analysis of the two-grid method (see [18, Appendix B]) and the periodicity with period $\pi/2h$ of the function $\widetilde{\mathbf{T}_{4h}\varphi}(\xi)$ give us that

$$\begin{aligned}
\int_{\pi/4h \leqslant |\xi| \leqslant \pi/h} \left|m(h\xi)\right|^2 \left|\widetilde{\mathbf{T}_{4h}\varphi}(\xi)\right|^2 d\xi &= \int_{\pi/4h \leqslant |\xi| \leqslant \pi/h} \left|\frac{e^{4i\xi h}-1}{4(e^{i\xi h}-1)}\right|^4 \left|\widetilde{\mathbf{T}_{4h}\varphi}(\xi)\right|^2 \\
&\lesssim \int_{\pi/4h \leqslant |\xi| \leqslant \pi/h} \left|e^{4i\xi h}-1\right|^4 \left|\widetilde{\mathbf{T}_{4h}\varphi}(\xi)\right|^2 \\
&\lesssim \int_{-\pi/4h \leqslant \xi \leqslant \pi/4h} \left|e^{4i\xi h}-1\right|^4 \left|\widetilde{\mathbf{T}_{4h}\varphi}(\xi)\right|^2 \\
&\lesssim \int_{-\pi/4h \leqslant \xi \leqslant \pi/4h} |\xi h|^4 \left|\widetilde{\mathbf{T}_{4h}\varphi}(\xi)\right|^2 d\xi \lesssim \left(h\|\varphi\|_{H^1(\mathbb{R})}\right)^2.
\end{aligned}$$

We obtain that (4.12) holds and, consequently, (4.11) too. Thus (4.8) is satisfied for any positive s.

Observe that the main term in the right-hand side of (4.13) is given by (4.14), and this estimate cannot be improved by imposing more than $H^1(\mathbb{R})$ smoothness on φ.

Case II. Proof of the inhomogeneous estimate (4.7). We proceed as in the previous case by splitting the difference we want to evaluate as

$$\int_{s<t} \exp\left(i(t-s)\Delta_h\right)\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h} f(s)\,ds - \int_{s<t} \mathbf{T}_h \exp\left(i(t-s)\partial_x^2\right) f(s)\,ds = I_1 + I_2$$

where

$$I_1 = \int_{s<t} \left(\exp\left(i(t-s)\Delta_h\right) - \exp\left(i(t-s)\Delta^h\right)\right)\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h} f(s)\,ds,$$

and

$$I_2 = \int_{s<t} \exp\left(i(t-s)\Delta^h\right)\left(\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h} f(s) - \mathbf{T}_h f(s)\right) ds.$$

In the case of I_1, applying Theorem 2.2 to operators Δ_h and Δ^h, we get

$$\|I_1\|_{L^q(0,T;l^r(h\mathbb{Z}))} \leqslant h^{\min\{s/2,2\}} \max\{1,T\} \left\|\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h} f\right\|_{L^{\tilde{q}'}(0,T;B^s_{\tilde{r}',2}(h\mathbb{Z}))}.$$

Applying Theorem 7.1 below to the multiplier m given by (4.3), for any $s>0$ we obtain that

$$\left\|\boldsymbol{\Pi}_h^{4h}\mathbf{T}_{4h} f\right\|_{L^{\tilde{q}'}(0,T;B^s_{\tilde{r}',2}(h\mathbb{Z}))} \leqslant \|f\|_{L^{\tilde{q}'}(0,T;B^s_{\tilde{r}',2}(\mathbb{R}))}$$

and then I_1 satisfies:

$$\|I_1\|_{L^q(0,T;l^r(h\mathbb{Z}))} \leqslant h^{\min\{s/2,2\}} \max\{1,T\} \|f\|_{L^{\tilde{q}'}(0,T;B^s_{\tilde{r}',2}(\mathbb{R}))}. \tag{4.15}$$

In the case of I_2 we claim that

$$\|I_2\|_{L^q(0,T;l^r(h\mathbb{Z}))} \leqslant h^{\min\{s,1\}} \|f\|_{L^{\tilde{q}'}(0,T;B^s_{\tilde{r}',2}(\mathbb{R}))}. \tag{4.16}$$

To prove this claim we consider the cases $s=0$ and $s=1$. When $s\in(0,1)$ we use interpolation between the previous ones. Also the case $s>1$ follows by using the embedding $B^s_{\tilde r',2}(\mathbb{R})\hookrightarrow B^1_{\tilde r',2}(\mathbb{R})$.

The case $s=0$ follows from Proposition 2.1 applied to the operators $U_h(t)=\mathbf{T}_h\exp(it\partial_x^2)$.

We now consider the case $s=1$. Using Strichartz estimates given by Proposition 2.1 to the operator $\exp(it\Delta^h)$ we get:

$$\|I_2\|_{L^q(0,T;l^r(h\mathbb{Z}))}\leqslant \big\|\Pi_h^{4h}\mathbf{T}_{4h}f-\mathbf{T}_h f\big\|_{L^{\tilde q'}(0,T;l^{\tilde r'}(h\mathbb{Z}))}.$$

Theorem 7.1 applied to the multiplier m gives us

$$\big\|\Pi_h^{4h}\mathbf{T}_{4h}f-\mathbf{T}_{4h} f\big\|_{L^{\tilde q'}(0,T;l^{\tilde r'}(h\mathbb{Z}))}\leqslant h\|f\|_{L^{\tilde q'}(0,T;B^1_{\tilde r',2}(\mathbb{R}))}$$

and

$$\|\mathbf{T}_{4h}f-\mathbf{T}_h f\|_{L^{\tilde q'}(0,T;l^{\tilde r'}(h\mathbb{Z}))}\leqslant h\|f\|_{L^{\tilde q'}(0,T;B^1_{\tilde r',2}(\mathbb{R}))}.$$

Thus (4.16) holds for $s=1$, and in view of the above comments, for all $s\geqslant 0$.

Putting together (4.15) and (4.15) we obtain the inhomogeneous estimate (4.7).

The proof is now complete. □

5. Convergence of the dispersive method for the NSE

In this section we introduce numerical schemes for the NSE based on dispersive approximations of the LSE. We first present some classical results on well-posedness and regularity of solutions of the NSE. Secondly we obtain the order of convergence for the approximations of the NSE described above.

5.1. Classical facts on NSE

We consider the NSE with nonlinearity $f(u)=|u|^pu$ and $\varphi\in H^s(\mathbb{R})$. We are interested in the case of $H^s(\mathbb{R})$ initial data with $s\leqslant 1$. The following well-posedness result is known.

Theorem 5.1. *Let $f(u)=|u|^pu$ with $p\in(0,4)$. Then*

(i) (*Global existence and uniqueness* [5, *Theorem* 4.6.1, *Chapter* 4, *p.* 109]) *For any $\varphi\in L^2(\mathbb{R})$, there exists a unique global solution u of* (1.2) *in the class*

$$u\in C\big(\mathbb{R},L^2(\mathbb{R})\big)\cap L^q_{\mathrm{loc}}\big(\mathbb{R},L^r(\mathbb{R})\big)$$

for all $1/2$-admissible pairs (q,r) such that

$$\big\|u(t)\big\|_{L^2(\mathbb{R})}=\|\varphi\|_{L^2(\mathbb{R})},\quad \forall t\in\mathbb{R}.$$

(ii) (*Stability* [5, *Theorem* 4.6.1, *Chapter* 4, *p.* 109]) *Let φ and ψ be two $L^2(\mathbb{R})$ functions, and u and v the corresponding solutions of the NSE. Then for any $T>0$ there exists a positive constant $C(T,\|\varphi\|_{L^2(\mathbb{R})},\|\psi\|_{L^2(\mathbb{R})})$ such that the following holds*:

$$\|u-v\|_{L^\infty(0,T;L^2(\mathbb{R}))}\leqslant C\big(T,\|\varphi\|_{L^2(\mathbb{R})},\|\psi\|_{L^2(\mathbb{R})}\big)\|\varphi-\psi\|_{L^2(\mathbb{R})}. \tag{5.1}$$

(iii) (*Regularity*) *Moreover if $\varphi\in H^s(\mathbb{R})$, $s\in(0,1/2)$ then* [5, *Theorem* 5.1.1, *Chapter* 5, *p.* 147],

$$u\in C\big(\mathbb{R},H^s(\mathbb{R})\big)\cap L^q_{\mathrm{loc}}\big(\mathbb{R},B^s_{r,2}(\mathbb{R})\big)$$

for every admissible pairs (q,r).

Also if $\varphi\in H^1(\mathbb{R})$ then $u\in C(\mathbb{R},H^1(\mathbb{R}))$ [5, *Theorem* 5.2.1, *Chapter* 5, *p.* 149].

Remark 5.1. The embedding $B^s_{r,2}(\mathbb{R})\hookrightarrow W^{s,r}(\mathbb{R})$, $r\geqslant 2$ (see [5, Remark 1.4.3, p. 14]), guarantees that, in particular, $u\in L^q_{\mathrm{loc}}(\mathbb{R},W^{s,r}(\mathbb{R}))$. Moreover, $f(u)\in L^{q'}_{\mathrm{loc}}(\mathbb{R},B^s_{r',2}(\mathbb{R}))$ and for any $0<s\leqslant 1$ (see [5, formula (4.9.20), p. 128]),

$$\big\|f(u)\big\|_{L^{q'}(I,B^s_{r',2}(\mathbb{R}))}\lesssim |I|^{\frac{4-p(1-2s)}{4}}\|u\|^{p+1}_{L^q(I,B^s_{r,2}(\mathbb{R}))}. \tag{5.2}$$

The fixed point argument used to prove the existence and uniqueness result in Theorem 5.1 gives us also quantitative information of the solutions of NSE in terms of the $L^2(\mathbb{R})$-norm of the initial data. The following holds:

Lemma 5.1. *Let $\varphi \in L^2(\mathbb{R})$ and u be the solution of the NSE with initial data φ and nonlinearity $f(u) = |u|^p u$, $p \in (0,4)$, as in Theorem 5.1. There exists $c(p) > 0$ and $T_0 = c(p)\|\varphi\|_{L^2(\mathbb{R})}^{-4p/(4-p)}$ such that for any $1/2$-admissible pairs (q,r), there exists a positive constant $C(p,q)$ such that*

$$\|u\|_{L^q(I;L^r(\mathbb{R}))} \leqslant C(p,q)\|\varphi\|_{L^2(\mathbb{R})} \tag{5.3}$$

holds for all intervals I with $|I| \leqslant T_0$.

Proof. Let us fix an admissible pair (q,r). The fixed point argument used in the proof of Theorem 5.1 (see [4, Theorem 5.5.1, p. 15]) gives us the existence of a time T_0,

$$T_0 = c(p)\|\varphi\|_{L^2(\mathbb{R})}^{-\frac{4p}{4-p}},$$

such that

$$\|u\|_{L^q(0,T_0;L^r(\mathbb{R}))} \leqslant C(p,q)\|\varphi\|_{L^2(\mathbb{R})}.$$

The same argument applied to the interval $[(k-1)T_0, kT_0]$, $k \geqslant 1$, and the conservation of the $L^2(\mathbb{R})$-norm of the solution u of the NSE gives us that

$$\|u\|_{L^q((k-1)T_0,kT_0;L^r(\mathbb{R}))} \leqslant C(p,q)\big\|u\big((k-1)T_0\big)\big\|_{L^2(\mathbb{R})} = C(p,q)\|\varphi\|_{L^2(\mathbb{R})}.$$

This proves (5.3) and finishes the proof of Lemma 5.1. □

5.2. Approximation of the NSE by dispersive numerical schemes

In this section we consider a numerical scheme for the NSE based on approximations of the LSE that has uniform dispersive properties of Strichartz type. Examples of such schemes have been given in Sections 3 and 4.

To be more precise, we deal with the following numerical schemes:

- Consider

$$\begin{cases} iu_t^h + A_h u^h = f\big(u^h\big), & t > 0, \\ u^h(0) = \varphi^h, \end{cases} \tag{5.4}$$

where A_h is an approximation of Δ such that $\exp(itA_h)$ has uniform dispersive properties of Strichartz type. We also assume that A_h satisfies $\mathrm{Re}(iA_h\varphi,\varphi)_h \leqslant 0$, Re being the real part, and has a symbol $a_h(\xi)$ which verifies

$$\big|a_h(\xi) - \xi^2\big| \leqslant \sum_{k\in F} a(k,h)|\xi|^k, \quad \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]. \tag{5.5}$$

- The two-grid scheme. The two-grid scheme can be adapted to the nonlinear frame as follows. Consider the equation

$$\begin{cases} iu_t^{0,h} + \Delta_h u^{0,h} = \boldsymbol{\Pi}_h^{4h} f\big(\big(\boldsymbol{\Pi}_h^{4h}\big)^* u^{0,h}\big), & t > 0, \\ u^{0,h}(0) = \boldsymbol{\Pi}_h^{4h}\varphi^h, \end{cases} \tag{5.6}$$

where $(\boldsymbol{\Pi}_h^{4h})^* : l^2(h\mathbb{Z}) \to l^2(4h\mathbb{Z})$ is the adjoint of $\boldsymbol{\Pi}_h^{4h} : l^2(4h\mathbb{Z}) \to l^2(h\mathbb{Z})$ and φ^h is an approximation of φ. By [15, Theorem 4.1], *for any $p \in (0,4)$ there exists of a positive time $T_0 = T_0(\|\varphi\|_{L^2(\mathbb{R})})$ and a unique solution $u^{h,0} \in C(0,T_0;l^2(h\mathbb{Z}^d)) \cap L^q(0,T_0;l^{p+2}(h\mathbb{Z}^d))$, $q = 4(p+2)/p$, of the system (5.6). Moreover, $u^{h,0}$ satisfies*

$$\big\|u^h\big\|_{L^\infty(\mathbb{R},l^2(h\mathbb{Z}^d))} \leqslant \big\|\boldsymbol{\Pi}_h^{4h}\varphi^h\big\|_{l^2(h\mathbb{Z}^d)} \tag{5.7}$$

and

$$\big\|u^h\big\|_{L^q(0,T_0;l^{p+2}(h\mathbb{Z}^d))} \leqslant c(T_0)\big\|\boldsymbol{\Pi}_h^{4h}\varphi^h\big\|_{l^2(h\mathbb{Z}^d)}, \tag{5.8}$$

where the above constant is independent of h.

With T_0 obtained above, for any $k \geqslant 1$ we consider $u^{k,h}:[kT_0,(k+1)T_0] \to \mathbb{C}$ the solution of the following system:

$$\begin{cases} iu_t^{k,h} + \Delta_h u^{k,h} = \boldsymbol{\Pi}_h^{4h} f\big(\big(\boldsymbol{\Pi}_h^{4h}\big)^* u^{k,h}\big), & t \in \big[kT_0,(k+1)T_0\big], \\ u^{k,h}(kT_0) = \boldsymbol{\Pi}_h^{4h} u^{k-1,h}(kT_0). \end{cases} \tag{5.9}$$

Once $u^{k,h}$ are computed the approximation u^h of NSE is defined as

$$u^h(t) = u^{k,h}(t), \quad t \in \big[kT_0,(k+1)T_0\big). \tag{5.10}$$

We point out that systems (5.6) and (5.9) have always a global solution in the class $C(\mathbb{R}, l^2(h\mathbb{Z}))$ (use the embedding $l^2(h\mathbb{Z}) \subset l^\infty(h\mathbb{Z})$, a classical fix point argument and the conservation of the $l^2(h\mathbb{Z})$-norm). However, estimates in the $L^q(0,T;l^r(h\mathbb{Z}))$-norm, uniformly with respect to the mesh-size parameter $h>0$, cannot be proved without using Strichartz estimates given by Theorem 4.1. Thus we need to take initial data obtained through a two-grid process. Since the two-grid class of functions is not invariant under the flow of system (5.6) we need to update the solution at some time-step T_0 which depends only on $L^2(\mathbb{R})$-norm of the initial data φ.

The following theorems give us the existence and uniqueness of solutions for the above systems as well as quantitative dispersive estimates of solutions u^h, similar to those obtained in Lemma 5.1 for the continuous NSE, uniformly on the mesh-size parameter $h>0$.

Theorem 5.2. *Let $p \in (0,4)$, $f(u) = |u|^p u$ and A_h be such that $\operatorname{Re}(iA_h\varphi,\varphi)_h \leqslant 0$ and (3.5) holds. Then for every $\varphi^h \in l^2(h\mathbb{Z})$, there exists a unique global solution $u^h \in C(\mathbb{R}, l^2(h\mathbb{Z}))$ of (5.4) which satisfies*

$$\|u^h\|_{L^\infty(\mathbb{R},l^2(h\mathbb{Z}))} \leqslant \|\varphi^h\|_{l^2(h\mathbb{Z})}. \tag{5.11}$$

Moreover, there exist $c(p)>0$ and $C(p,q)>0$ such that for any finite interval I with $|I| \leqslant T_0 = c(p)\|\varphi^h\|_{l^2(h\mathbb{Z})}^{-4p/(4-p)}$

$$\|u^h\|_{L^q(I,l^r(h\mathbb{Z}))} \leqslant C(p,q)\|\varphi^h\|_{l^2(h\mathbb{Z})}, \tag{5.12}$$

where (q,r) is a $1/2$-admissible pair and the above constant is independent of h.

Proof. Condition $\operatorname{Re}(iA_h\varphi,\varphi)_h \leqslant 0$ implies the $l^2(h\mathbb{Z})$ stability property (3.4). Then local existence is obtained by using Strichartz estimates given by Proposition 2.1 applied to the operator $\exp(itA_h)$ and a classical fix point argument in a suitable Banach space (see [17] and [19] for more details). The global existence of solutions and estimate (5.11) are guaranteed by the properties $\operatorname{Re}(iA_h\varphi,\varphi)_h \leqslant 0$, $\operatorname{Re}(if(u_h),u_h)_h = 0$ and the energy identity:

$$\frac{d}{dt}\|u^h(t)\|_{l^2(h\mathbb{Z})}^2 = 2\operatorname{Re}\big(iA_h u^h, u^h\big)_h + 2\operatorname{Re}\big(if\big(u^h\big),u^h\big)_h \leqslant 0. \tag{5.13}$$

Once the global existence is proved, estimate (5.12) is obtained in a similar manner as Lemma 5.1 and we will omit its proof. □

Theorem 5.3. *Let $p \in (0,4)$ and $q = 4(p+2)/p$. Then for all $h>0$ and for every $\varphi^h \in l^2(4h\mathbb{Z})$, there exists a unique global solution $u^h \in C(\mathbb{R}, l^2(h\mathbb{Z})) \cap L^q_{\mathrm{loc}}(\mathbb{R}, l^{p+2}(h\mathbb{Z}^d))$ of (5.6)–(5.10) which satisfies*

$$\|u^h\|_{L^\infty(\mathbb{R},l^2(h\mathbb{Z}))} \leqslant \|\boldsymbol{\Pi}_h^{4h}\varphi^h\|_{l^2(h\mathbb{Z})}. \tag{5.14}$$

Moreover, there exist $c(p)>0$ and $C(p,q)>0$ such that for any finite interval I with $|I| \leqslant T_0 = c(p)\|\varphi^h\|_{l^2(h\mathbb{Z})}^{-4p/(4-p)}$

$$\|u^h\|_{L^q(I,l^{p+2}(h\mathbb{Z}))} \leqslant C(p,q)\|\boldsymbol{\Pi}_h^{4h}\varphi^h\|_{l^2(h\mathbb{Z})}, \tag{5.15}$$

where (q,r) is a $1/2$-admissible pair and the above constant is independent of h.

Proof. The existence in the interval $(0,T_0)$, $T_0 = T_0(\|\varphi^h\|_{l^2(h\mathbb{Z})})$ for system (5.4) is obtained by using the Strichartz estimates given by Theorem 4.1 and a classical fix point argument in a suitable Banach space (see [17] and [19] for more details).

For any $k \geqslant 1$ the same arguments guarantee the local existence for systems (5.9). To prove that each system has solutions on an interval of length T_0 we have to prove a priori that the $l^2(h\mathbb{Z})$-norm of u^h does not increase. The particular approximation we have introduced of the nonlinear term in (5.6)–(5.9) gives us (after multiplying these equations by $u^{k,h}$ and taking the $l^2(h\mathbb{Z})$-norm) that for any $t \in [kT_0, (k+1)T_0]$

$$\|u^{k,h}(t)\|_{l^2(h\mathbb{Z})} = \|u^{k,h}(kT_0)\|_{l^2(h\mathbb{Z})} \leqslant \|u^{k-1,h}(kT_0)\|_{l^2(h\mathbb{Z})}$$

and then

$$\|u^{k,h}(t)\|_{l^2(h\mathbb{Z})} \leqslant \|u^{0,h}(0)\|_{l^2(h\mathbb{Z})} = \|\boldsymbol{\Pi}_h^{4h}\varphi^h\|_{l^2(h\mathbb{Z})}.$$

This proves (5.14) and the fact that for any $k \geqslant 1$ system (5.9) has a solution on the whole interval $[kT_0, (k+1)T_0]$. Estimate (5.15) is obtained locally on each interval $[kT_0, (k+1)T_0]$ together with the local existence result. □

Let us consider u^h the solution of the semi-discrete problem (5.4) and u of the continuous one (1.2). In the following theorem we evaluate the difference between u^h and $\mathbf{T}_h u$.

Theorem 5.4. *Let $p \in (0,4)$, $s \in (0,1/2)$, $f(u) = |u|^p u$ and A_h be as in Theorem 5.2 satisfying (5.5). For any $\varphi \in H^s(\mathbb{R})$, we consider u^h and $u \in L^\infty(\mathbb{R}, H^s(\mathbb{R})) \cap L^{q_0}_{loc}(\mathbb{R}, B^s_{p+2,2}(\mathbb{R}))$, $q_0 = 4(p+2)/p$ solutions of problems (5.4) and (1.2), respectively. Then for any $T > 0$ there exists a positive constant $C(T, \|\varphi\|_{L^2(\mathbb{R})})$ such that*

$$\begin{aligned} &\|u^h - \mathbf{T}_h u\|_{L^{q_0}(0,T;l^{p+2}(h\mathbb{Z}))} + \|u^h - \mathbf{T}_h u\|_{L^\infty(0,T;l^2(h\mathbb{Z}))} \\ &\quad \leqslant C(T, \|\varphi\|_{L^2(\mathbb{R})}, p)\big[\varepsilon(s,h)\|u\|_{L^\infty(0,T;H^s(\mathbb{R}))} + \big(h^s + \varepsilon(s,h)\big)\|u\|^{p+1}_{L^{q_0}(0,T;B^s_{p+2,2}(\mathbb{R}))}\big] \end{aligned} \tag{5.16}$$

holds for all $h > 0$.

In the case of the two-grid method, the solution u^h of system (5.6) approximates the solution u of the NSE (1.2) and the error committed is given by the following theorem.

Theorem 5.5. *Let $p \in (0,4)$, $s \in (0,1/2)$, $f(u) = |u|^p u$. For any $\varphi \in H^s(\mathbb{R})$, we consider u^h and $u \in L^\infty(\mathbb{R}, H^s(\mathbb{R})) \cap L^{q_0}_{loc}(\mathbb{R}, B^s_{p+2,2}(\mathbb{R}))$, $q_0 = 4(p+2)/p$, solutions of problems (5.6)–(5.10) and (1.2), respectively. Then for any $T > 0$ there exists a positive constant $C(T, \|\varphi\|_{L^2(\mathbb{R})})$ such that*

$$\begin{aligned} &\|u^h - \mathbf{T}_h u\|_{L^{q_0}(0,T;l^{p+2}(h\mathbb{Z}))} + \|u^h - \mathbf{T}_h u\|_{L^\infty(0,T;l^2(h\mathbb{Z}))} \\ &\quad \leqslant C(T, \|\varphi\|_{L^2(\mathbb{R})}, p)\big[h^{s/2}\|u\|_{L^\infty(0,T;H^s(\mathbb{R}))} + \big(h^s + h^{s/2}\big)\|u\|^{p+1}_{L^{q_0}(0,T;B^s_{p+2,2}(\mathbb{R}))}\big] \end{aligned} \tag{5.17}$$

holds for all $h > 0$.

Remark 5.2. Using classical results on the solutions of the NSE (see for example [4, Theorem 5.1.1, Chapter 5, p. 147]) we can state the above result in a more compact way: For any $T > 0$ there exists a positive constant $C(T, \|\varphi\|_{H^s(\mathbb{R})})$ such that

$$\|u^h - \mathbf{T}_h u\|_{L^{q_0}(0,T;l^{p+2}(h\mathbb{Z}))} + \|u^h - \mathbf{T}_h u\|_{L^\infty(0,T;l^2(h\mathbb{Z}))} \leqslant C\big(T, \|\varphi\|_{H^s(\mathbb{R})}\big)h^{s/2} \tag{5.18}$$

holds for all $h > 0$.

Theorem 5.4 shows that if $h^s \leqslant \varepsilon(s,h)$ then the error committed to approximate the nonlinear problem is the same as for the linear problem with the same initial data. As we proved in Section 3.2, for the higher order dissipative scheme $A_h = \Delta_h - ih^{2(m-1)}(-\Delta_h)^m$, $m \geqslant 2$, and for the two-grid method, $\varepsilon(s,h) = h^{s/2} \geqslant h^s$. So these schemes enter in this framework. It is also remarkable that the use of dispersive schemes allows to prove the convergence for the NSE and to obtain the convergence rate for $H^s(\mathbb{R})$ initial data with $0 < s < 1/2$. We point out that the energy method does not provide any error estimate in this case, the minimal smoothing required for the energy method being $H^s(\mathbb{R})$, with $s > 1/2$ (see Section 6 for all the details).

In the following we prove Theorem 5.4, the proof of Theorem 5.5 being similar since the estimates in any interval $(0,T)$ are obtained reiterating the argument in each interval $(kT_0,(k+1)T_0)$, $k\geqslant 0$, for some $T_0=T_0(\|\varphi\|_{L^2(\mathbb{R})})$ in view of the structure of the scheme.

Proof of Theorem 5.4. The idea of the proof is that there exists a time T_1 depending on the $L^2(\mathbb{R})$-norm of the initial data:

$$T_1\simeq\min\bigl\{1,\|\varphi\|_{L^2(\mathbb{R})}^{-4p/(4-p)}\bigr\},$$

such that the error in the approximation of the nonlinear problem

$$\mathrm{err}_h(t)=u^h(t)-\mathbf{T}_h u(t),$$

when considered in the $L^{q_0}(0,T_1;l^{p+2}(h\mathbb{Z}))\cap L^\infty(0,T_1;l^2(h\mathbb{Z}))$-norm is controlled by the error produced in the linear part

$$\mathrm{err}_h^{\mathrm{lin}}(t)=\exp(itA_h)\mathbf{T}_h\varphi-\mathbf{T}_h\exp\bigl(it\partial_x^2\bigr)\varphi.$$

In the following we denote by (q,r) one of the admissible pairs $(\infty,2)$ or $(q_0,p+2)$. We now write the two solutions in the semigroup formulation given by systems (5.4) and (1.2):

$$u^h(t)=\exp(itA_h)\mathbf{T}_h\varphi+i\int_0^t\exp\bigl(i(t-s)A_h\bigr)f\bigl(u^h(s)\bigr)\,ds,$$

and, respectively,

$$\mathbf{T}_h u(t)=\mathbf{T}_h\exp\bigl(it\partial_x^2\bigr)\varphi+i\int_0^t\mathbf{T}_h\exp\bigl(i(t-s)\partial_x^2\bigr)f\bigl(u(s)\bigr)\,ds.$$

Thus

$$\|\mathrm{err}_h\|_{L^q(0,T;l^r(h\mathbb{Z}))}\leqslant\bigl\|\mathrm{err}_h^{\mathrm{lin}}\bigr\|_{L^q(0,T;l^r(h\mathbb{Z}))}+\bigl\|\mathrm{err}_h^{\mathrm{non}}\bigr\|_{L^q(0,T;l^r(h\mathbb{Z}))}\tag{5.19}$$

where, by definition,

$$\mathrm{err}_h^{\mathrm{non}}(t)=\int_0^t\exp\bigl(i(t-s)A_h\bigr)f\bigl(u^h(s)\bigr)\,ds-\int_0^t\mathbf{T}_h\exp\bigl(i(t-s)\partial_x^2\bigr)f\bigl(u(s)\bigr)\,ds.$$

For the linear part the error is estimated in Theorem 3.1:

$$\bigl\|\mathrm{err}_h^{\mathrm{lin}}\bigr\|_{L^q(0,T;l^r(h\mathbb{Z}))}\leqslant C(q)\varepsilon(s,h)\max\{T,1\}\|\varphi\|_{H^s(\mathbb{R})}.\tag{5.20}$$

In the following we will estimate $\mathrm{err}_h^{\mathrm{non}}$. We write $\mathrm{err}_h^{\mathrm{non}}(t)=I_2^h(t)+I_3^h(t)$, where

$$I_2^h(t)=\int_0^t\exp\bigl(i(t-s)A_h\bigr)\bigl(f\bigl(u^h(s)\bigr)-\mathbf{T}_h f\bigl(u(s)\bigr)\bigr)\,ds$$

and

$$I_3^h(t)=\int_0^t\bigl(\exp\bigl(i(t-s)A_h\bigr)\mathbf{T}_h f\bigl(u(s)\bigr)-\mathbf{T}_h\exp\bigl(i(t-s)\partial_x^2\bigr)f\bigl(u(s)\bigr)\bigr)\,ds.$$

Step I. Estimate of I_3^h. For the last term, the inhomogeneous estimate (3.8) in Theorem 3.1 and estimate (5.2) give us that

$$\begin{aligned}\bigl\|I_3^h(t)\bigr\|_{L^q(0,T;l^r(h\mathbb{Z}))}&\leqslant C(q)\varepsilon(s,h)\max\{1,T\}\bigl\|f(u)\bigr\|_{L^{q_0'}(0,T;B^s_{(p+2)',2}(\mathbb{R}))}\\&\leqslant C(q)\varepsilon(s,h)\max\{1,T\}T^{\frac{4-p(1-2s)}{4}}\|u\|^{p+1}_{L^q(0,T;B^s_{p+2,2}(\mathbb{R}))}.\end{aligned}\tag{5.21}$$

Step II. Estimate of I_2^h. We now prove the existence of a time T_0 such that for all $T < T_0$, I_2^h satisfies

$$\left\| I_2(t) \right\|_{L^q(0,T;l^r(h\mathbb{Z}))} \leqslant C(p) T^{1-\frac{p}{4}} \|\mathrm{err}_h\|_{L^{q_0}(0,T;l^{p+2}(h\mathbb{Z}))} \|\varphi\|^p_{L^2(\mathbb{R})} + h^s T^{1-\frac{p}{4}} \|u\|^{p+1}_{L^{q_0}(0,T;B^s_{p+2,2}(\mathbb{R}))}. \tag{5.22}$$

The inhomogeneous Strichartz's estimate (2.5) applied to the operators $(\exp(itA_h))_{t\geqslant 0}$ shows that

$$\begin{aligned} \left\| I_2^h(t) \right\|_{L^q(0,T;l^r(h\mathbb{Z}))} &\leqslant C(q) \left\| f\left(u^h\right) - \mathbf{T}_h f(u) \right\|_{L^{q_0'}(0,T;l^{(p+2)'}(h\mathbb{Z}))} \\ &\leqslant C(q) \left\| f\left(u^h\right) - f(\mathbf{T}_h u) \right\|_{L^{q_0'}(0,T;l^{(p+2)'}(h\mathbb{Z}))} \\ &\quad + C(q) \left\| f(\mathbf{T}_h u) - \mathbf{T}_h f(u) \right\|_{L^{q_0'}(0,T;l^{(p+2)'}(h\mathbb{Z}))}. \end{aligned} \tag{5.23}$$

We evaluate each term in the right-hand side of (2.31). In the case of the first one, applying Hölder's inequality in time we get

$$\begin{aligned} &\left\| f\left(u^h\right) - f(\mathbf{T}_h u) \right\|_{L^{q_0'}(0,T;l^{(p+2)'}(h\mathbb{Z}))} \\ &\quad \leqslant T^{1-\frac{p}{4}} \left\| u^h - \mathbf{T}_h u \right\|_{L^{q_0}(0,T;l^{p+2}(h\mathbb{Z}))} \left(\left\| u^h \right\|^p_{L^{q_0}(0,T;l^{p+2}(h\mathbb{Z}))} + \|\mathbf{T}_h u\|^p_{L^{q_0}(0,T;l^{p+2}(h\mathbb{Z}))} \right). \end{aligned}$$

Let us now set T_0 as it is given by Lemma 5.1 and Theorem 5.2:

$$T_0 \simeq \|\varphi\|_{L^2(\mathbb{R})}^{-\frac{4p}{4-p}}.$$

Thus, by Theorem 5.1, Lemma 5.1 and Theorem 5.3 both u^h and $\mathbf{T}_h u$ have their $L^q(0,T;l^r(h\mathbb{Z}))$-norm controlled by the L^2-norm of the initial data:

$$\left\| u^h \right\|_{L^{q_0}(0,T_0;l^{p+2}(h\mathbb{Z}))} \leqslant C(p) \|\varphi\|_{L^2(\mathbb{R})}$$

and

$$\|\mathbf{T}_h u\|_{L^{q_0}(0,T_0;l^{p+2}(h\mathbb{Z}))} \leqslant C(p) \|u\|_{L^{q_0}(0,T_0;L^{p+2}(\mathbb{R}))} \leqslant C(p) \|\varphi\|_{L^2(\mathbb{R})}.$$

These estimates show that for any $T < T_0$ the following holds:

$$\left\| f\left(u^h\right) - f(\mathbf{T}_h u) \right\|_{L^{q_0'}(0,T;l^{(p+2)'}(h\mathbb{Z}))} \leqslant C(p) T^{1-\frac{p}{4}} \left\| u^h - \mathbf{T}_h u \right\|_{L^{q_0}(0,T;l^{p+2}(h\mathbb{Z}))} \|\varphi\|^p_{L^2(\mathbb{R})}. \tag{5.24}$$

It remains to estimate the second term in the right-hand side of (5.23). We will use now the following result which will be proved in Section 7.

Lemma 5.2. *Let $s \in [0,1]$, $p \geqslant 0$ and $f(u) = |u|^p u$. Then there exists a positive constant $c(p,s)$ such that*

$$\left\| f(\mathbf{T}_h u) - \mathbf{T}_h f(u) \right\|_{l^{(p+2)'}(h\mathbb{Z})} \leqslant c(p,s) h^s \|u\|^{p+1}_{W^{s,p+2}(\mathbb{R})} \tag{5.25}$$

holds for all $u \in W^{s,p+2}(\mathbb{R})$ and $h > 0$.

Using this lemma, Hölder inequality in time and the embedding $B^s_{p+2,2}(\mathbb{R}) \hookrightarrow W^{s,p+2}(\mathbb{R})$ [5, Remark 1.4.3], we obtain:

$$\begin{aligned} \left\| f(\mathbf{T}_h u) - \mathbf{T}_h f(u) \right\|_{L^{q_0'}(0,T;l^{(p+2)'}(h\mathbb{Z}))} &\leqslant c(p,s) h^s T^{1-\frac{p}{4}} \|u\|^{p+1}_{L^{q_0}(0,T;W^{s,p+2}(\mathbb{R}))} \\ &\leqslant c(p,s) h^s T^{1-\frac{p}{4}} \|u\|^{p+1}_{L^{q_0}(0,T;B^s_{p+2,2}(\mathbb{R}))}. \end{aligned} \tag{5.26}$$

Both (5.24) and (5.26) show that $I_2(t)$ satisfies (5.22).

Step III. Estimate of $\mathbf{err}_h$. Collecting estimates (5.20), (5.21) and (5.22) for both $(q,r) = (q_0, p+2)$ and $(q,r) = (\infty, 2)$ we obtain that for any $T < T_0$ the error err_h satisfies:

$$\begin{aligned}&\|\mathrm{err}_h\|_{L^{q_0}(0,T;l^{p+2}(h\mathbb{Z}))}+\|\mathrm{err}_h\|_{L^\infty(0,T;l^2(h\mathbb{Z}))}\\&\quad\leqslant C(p)\max\{1,T\}\varepsilon(s,h)\|\varphi\|_{H^s(\mathbb{R})}+C(p)\|\mathrm{err}_h\|_{L^{q_0}(0,T;l^{p+2}(h\mathbb{Z}))}T^{1-p/4}\|\varphi\|^p_{L^2(\mathbb{R})}\\&\qquad+h^sT^{1-\frac{p}{4}}\|u\|^{p+1}_{L^{q_0}(0,T;B^s_{p+2,2}(\mathbb{R}))}+\varepsilon(s,h)\max\{1,T\}T^{\frac{4-p(1-2s)}{4}}\|u\|^{p+1}_{L^{q_0}(0,T;B^s_{p+2,2}(\mathbb{R}))}.\end{aligned}\tag{5.27}$$

Now, let us set $T_1\leqslant\min\{1,T_0\}$ such that

$$C(p)T_1^{1-p/4}\|\varphi\|^p_{L^2(\mathbb{R})}\leqslant\frac{1}{2}.$$

Then the error term err_h in the right-hand side of (5.27) is absorbed in the left-hand side:

$$\begin{aligned}&\|\mathrm{err}_h\|_{L^{q_0}(0,T_1;l^{p+2}(h\mathbb{Z}))}+\|\mathrm{err}_h\|_{L^\infty(0,T_1;l^2(h\mathbb{Z}))}\\&\quad\leqslant C(p)\varepsilon(s,h)\|\varphi\|_{H^s(\mathbb{R})}+C(p)\|u\|^{p+1}_{L^{q_0}(0,T;B^s_{p+2,2}(\mathbb{R}))}\big(h^s+\varepsilon(s,h)\big).\end{aligned}$$

We now obtain the same estimate in any interval $(0,T)$. Using that the $L^2(\mathbb{R})$-norm of the solution u is conserved in time we can apply the same argument in the interval $[kT_1,(k+1)T_1]$:

$$\begin{aligned}&\|\mathrm{err}_h\|_{L^{q_0}(kT_1,(k+1)T_1;l^{p+2}(h\mathbb{Z}))}+\|\mathrm{err}_h\|_{L^\infty(kT_1,(k+1)T_1;l^2(h\mathbb{Z}))}\\&\quad\leqslant C(p)\varepsilon(s,h)\big\|u(kT_1)\big\|_{H^s(\mathbb{R})}+C(p)\big(h^s+\varepsilon(s,h)\big)\|u\|^{p+1}_{L^{q_0}(kT_1,(k+1)T_1;B^s_{p+2,2}(\mathbb{R}))}.\end{aligned}$$

Let us choose $T>0$ and $N\geqslant 1$ an integer such that $(N-1)T_1\leqslant T<NT_1$. Thus

$$\begin{aligned}&\|\mathrm{err}_h\|_{L^{q_0}(0,T;l^{p+2}(h\mathbb{Z}))}+\|\mathrm{err}_h\|_{L^\infty(0,T;l^2(h\mathbb{Z}))}\\&\quad\leqslant\sum_{k=0}^{N-1}\big(\|\mathrm{err}_h\|_{L^{q_0}(kT_1,(k+1)T_1;l^{p+2}(h\mathbb{Z}))}+\|\mathrm{err}_h\|_{L^\infty(kT_1,(k+1)T_1;l^2(h\mathbb{Z}))}\big)\\&\quad\leqslant C(p)\varepsilon(s,h)\sum_{k=0}^{N-1}\big\|u(kT_1)\big\|_{H^s(\mathbb{R})}+C(p)\big(h^s+\varepsilon(s,h)\big)\sum_{k=0}^{N-1}\|u\|^{p+1}_{L^{q_0}(kT_1,(k+1)T_1;B^s_{p+2,2}(\mathbb{R}))}.\end{aligned}$$

Using that $(p+1)/q_0<1$ we have by the discrete Hölder's inequality that

$$\sum_{k=0}^{N-1}\|u\|^{p+1}_{L^{q_0}(kT_1,(k+1)T_1;B^s_{p+2,2}(\mathbb{R}))}\leqslant N^{1-\frac{p+1}{q_0}}\|u\|^{p+1}_{L^{q_0}(0,T;B^s_{p+2,2}(\mathbb{R}))}.$$

Thus the error satisfies:

$$\begin{aligned}&\|\mathrm{err}_h\|_{L^{q_0}(0,T;l^{p+2}(h\mathbb{Z}))}+\|\mathrm{err}_h\|_{L^\infty(0,T;l^2(h\mathbb{Z}))}\\&\quad\leqslant N\varepsilon(s,h)\|u\|_{L^\infty(0,T;H^s(\mathbb{R}))}+\big(h^s+\varepsilon(s,h)\big)N^{1-\frac{p+1}{q_0}}\|u\|^{p+1}_{L^{q_0}(0,T;B^s_{p+2,2}(\mathbb{R}))}\\&\quad\leqslant N\big[\varepsilon(s,h)\|u\|_{L^\infty(0,T;H^s(\mathbb{R}))}+\big(h^s+\varepsilon(s,h)\big)\|u\|^{p+1}_{L^{q_0}(0,T;B^s_{p+2,2}(\mathbb{R}))}\big]\\&\quad\leqslant C\big(T,\|\varphi\|_{L^2(\mathbb{R})}\big)\big[\varepsilon(s,h)\|u\|_{L^\infty(0,T;H^s(\mathbb{R}))}+\big(h^s+\varepsilon(s,h)\big)\|u\|^{p+1}_{L^{q_0}(0,T;B^s_{p+2,2}(\mathbb{R}))}\big].\end{aligned}$$

This finishes the proof of Theorem 5.4. □

6. Non-dispersive methods

In this section we will consider a numerical scheme for which the operator A_h has no uniform (with respect to the mesh size h) dispersive properties of Strichartz type. Accordingly we may not use $L^q_tL^r_x$ estimates for the linear semigroup $\exp(itA_h)$ and all the possible convergence estimates need to be based on the fact that the solution u of the continuous problem is uniformly bounded in space and time: $u\in L^\infty((0,T);L^\infty(\mathbb{R}))$. Thus, the only estimates we can use are those that the L^2-theory may yield. When working with $H^s(\mathbb{R})$-data with $s>1/2$, using $L^\infty(\mathbb{R};H^s(\mathbb{R}))$ estimates on solutions and Sobolev's embedding we can get L^2-estimates.

There is a classical argument that works whenever the nonlinearity f satisfies

$$\left|f(u)-f(v)\right| \leqslant C\left(|u|^p+|v|^p\right)|u-v|. \tag{6.1}$$

Standard error estimates (see Theorem 3.1 with the particular case $(q,r)=(\infty,2)$ or [28, Theorem 10.1.2, p. 201]) and Gronwall's inequality yield when $0\leqslant t\leqslant T$:

$$\left\|u^h(t)-\mathbf{T}_h u(t)\right\|_{l^2(h\mathbb{Z})} \leqslant h^{1/2}C(T)\left(\|\varphi\|_{H^1(\mathbb{R})}+\|u\|^{p+1}_{L^\infty(0,T:\,H^1(\mathbb{R}))}\right)\exp\left(T\|u\|^p_{L^\infty(0,T;H^1(\mathbb{R}))}\right), \tag{6.2}$$

for the conservative semi-discrete finite-difference scheme. For the sake of completeness we will prove this estimate in Section 6.1.

We emphasize that in order to obtain estimate (6.2) we need to use that the solution u, which we want to approximate, belongs to the space $L^\infty(\mathbb{R})$, condition which is guaranteed by assuming that the initial data is smooth enough. However, obviously, in general, solutions of the NSE do not belong to $L^\infty(\mathbb{R})$ and therefore these estimates cannot be applied. One can overcome this drawback assuming that the initial data belong to $H^1(\mathbb{R})$ or even to $H^s(\mathbb{R})$ with $s>1/2$ since in this case $H^s(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$. Using H^1-energy estimates and Sobolev's embedding we can deduce L^∞-bounds on solutions allowing to apply (6.2). We emphasize that this standard approach fails to provide any error estimate for initial data in $H^s(\mathbb{R})$ with $s<1/2$.

However, this type of error estimate can also be used for $H^s(\mathbb{R})$-initial data with $s<1/2$ (or even for $L^2(\mathbb{R})$-initial data), by a density argument. Indeed, given $\varphi\in H^s(\mathbb{R})$ with $0\leqslant s<1/2$, for any $\delta>0$ we may choose $\varphi_\delta\in H^1(\mathbb{R})$ such that

$$\|\varphi-\varphi_\delta\|_{H^s(\mathbb{R})}\leqslant\delta.$$

Let u_δ be the solution of NSE corresponding to φ_δ. Obviously, φ_δ being $H^1(\mathbb{R})$-smooth, we can apply standard results as (6.2) to u_δ. On the other hand, stability results for NSE allow us to prove the proximity of u and u_δ in $H^s(\mathbb{R})$. This allows showing the convergence of numerical approximations of u_δ, that we may denote by $u_{\delta,h}$, towards the solution u associated to φ as both $\delta\to 0$ and $h\to 0$. But for this to be true h needs to be exponentially small of the order of $\exp(-1/\delta)$ which is much smaller than the typical mesh-size needed to apply the results of the previous sections on dispersive schemes that required h to be of the order of $\delta^{2/s}$.

6.1. A classical argument for smooth initial data

In this section we present the technical details of the error estimates in the case of $H^1(\mathbb{R})$-initial data. In this case we do not require the numerical scheme to be dispersive, the only ingredient being the Sobolev's embedding $H^1(\mathbb{R})\hookrightarrow L^\infty(\mathbb{R})$.

Theorem 6.1. *Let $f(u)=|u|^p u$ with $p\in(0,4)$ and $u\in C(\mathbb{R},H^1(\mathbb{R}))$ be solution of (1.2) with initial data $\varphi\in H^1(\mathbb{R})$. Also assume that A_h is an approximation of order two of the Laplace operator ∂_x^2 and u^h is the solution of the following system*

$$\begin{cases} iu^h_t+A_h u^h=f\left(u^h\right), & t>0,\\ u^h(0)=\mathbf{T}_h\varphi, \end{cases} \tag{6.3}$$

satisfying $\|u^h\|_{L^\infty((0,T)\times h\mathbb{Z})}\leqslant C(T,\|\varphi\|_{H^1(\mathbb{R})})$.

Then for all $T>0$ and $h>0$

$$\left\|u^h(t)-\mathbf{T}_h u(t)\right\|_{l^2(h\mathbb{Z})} \leqslant h^{1/2}\max\left\{T,T^2\right\}\left(\|\varphi\|_{H^1(\mathbb{R})}+\|u\|^{p+1}_{L^\infty(0,T:\,H^1(\mathbb{R}))}\right)\exp\left(T\|u\|^p_{L^\infty(0,T;H^1(\mathbb{R}))}\right). \tag{6.4}$$

We now give an example where the hypotheses of the above theorem are verified. We consider the following NSE:

$$\begin{cases} iu_t+\partial_x^2 u=|u|^p u, & x\in\mathbb{R},\ t>0,\\ u(0,x)=\varphi(x), & x\in\mathbb{R}, \end{cases} \tag{6.5}$$

and its numerical approximation

$$\begin{cases} iu^h_t+\Delta_h u^h=\left|u^h\right|^p u^h, & t>0,\\ u^h(0)=\varphi^h. \end{cases} \tag{6.6}$$

In the case of the continuous problem we have the following conservation laws (see [5, Corollary 4.3.4, p. 93]):

$$\|u(t)\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})},$$

and

$$\frac{d}{dt}\left(\frac{1}{2}\int_{\mathbb{R}}|u_x(t,x)|^2\,dx + \frac{1}{p+2}\int_{\mathbb{R}}|u(t,x)|^{p+2}\,dx\right) = 0.$$

The same identities apply in the semi-discrete case (it suffices to multiply Eq. (6.6) by $\bar{u}^h$, respectively $\bar{u}^h_t$, to sum over the integers and to take the real part of the resulting identity):

$$\|u^h(t)\|_{l^2(h\mathbb{Z})} = \|\varphi^h\|_{l^2(h\mathbb{Z})},$$

and

$$\frac{d}{dt}\left(\frac{h}{2}\sum_{j\in\mathbb{Z}}\left|\frac{u^h_{j+1}(t) - u^h_j(t)}{h}\right|^2 + \frac{h}{p+2}\sum_{j\in\mathbb{Z}}|u^h_j(t)|^{p+2}\right) = 0.$$

In view of the above identities, the hypotheses of Theorem 6.1 are verified.

Proof of Theorem 6.1. Using the variations of constants formula we get

$$\mathbf{T}_h u(t) = \mathbf{T}_h \exp\bigl(it\partial_x^2\bigr)\varphi + \int_0^t \mathbf{T}_h \exp\bigl(i(t-\sigma)\partial_x^2\bigr) f\bigl(u(\sigma)\bigr)\,d\sigma$$

and

$$u^h(t) = \exp(itA_h)\mathbf{T}_h\varphi + \int_0^t \exp\bigl(i(t-\sigma)A_h\bigr) f\bigl(u^h(\sigma)\bigr)\,d\sigma.$$

Then

$$\begin{aligned}
\mathrm{err}_h(t) &:= \|u^h(t) - \mathbf{T}_h u(t)\|_{l^2(h\mathbb{Z})} \\
&\leqslant \|\exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h\exp\bigl(it\partial_x^2\bigr)\varphi\|_{l^2(h\mathbb{Z})} \\
&\quad + \int_0^t \|\exp\bigl(i(t-\sigma)A_h\bigr)\bigl(f\bigl(u^h(\sigma)\bigr) - \mathbf{T}_h f\bigl(u(\sigma)\bigr)\bigr)\,d\sigma\|_{l^2(h\mathbb{Z})}\,d\sigma \\
&\quad + \int_0^t \|\exp\bigl(i(t-\sigma)A_h\bigr)\mathbf{T}_h f\bigl(u(\sigma)\bigr) - \mathbf{T}_h\exp\bigl((t-\sigma)\partial_x^2\bigr) f\bigl(u(\sigma)\bigr)\|_{l^2(h\mathbb{Z})}\,d\sigma.
\end{aligned} \tag{6.7}$$

Now, applying the error estimates for the linear terms as in (3.9) with $\varepsilon(1,h) = h^{1/2}$, we get

$$\|\exp(itA_h)\mathbf{T}_h\varphi - \mathbf{T}_h\exp\bigl(it\partial_x^2\bigr)\varphi\|_{l^2(h\mathbb{Z})} \leqslant Th^{1/2}\|\varphi\|_{H^1(\mathbb{R})}. \tag{6.8}$$

Also, using that $f(u) = |u|^p u$ we have that $\|f(u)\|_{H^1(\mathbb{R})} \leqslant C\|u\|^p_{H^1(\mathbb{R})}$ and then by (3.9) we get

$$\begin{aligned}
&\int_0^t \|\exp\bigl(i(t-\sigma)A_h\bigr)\mathbf{T}_h f\bigl(u(\sigma)\bigr) - \mathbf{T}_h\exp\bigl(i(t-\sigma)\partial_x^2\bigr) f\bigl(u(\sigma)\bigr)\|_{l^2(h\mathbb{Z})}\,d\sigma \\
&\quad \leqslant CTh^{1/2}\|f(u)\|_{L^1(0,T;H^1(\mathbb{R}))} \\
&\quad \leqslant CT^2h^{1/2}\|u\|^{p+1}_{L^\infty(0,T;H^1(\mathbb{R}))}.
\end{aligned} \tag{6.9}$$

Using the $l^2(h\mathbb{Z})$-stability of $\exp(itA_h)$, (6.7), (6.8) and (6.9) we obtain

$$\mathrm{err}_h(t) \leqslant Th^{1/2}\|\varphi\|_{H^1(\mathbb{R})} + CT^2h^{1/2}\|u\|^{p+1}_{L^\infty(0,T;H^1(\mathbb{R}))} + \int_0^t \big\| f\big(u^h(\sigma)\big) - \mathbf{T}_h f\big(u(\sigma)\big)\big\|_{l^2(h\mathbb{Z})}.$$

Now we write $f(u^h(s)) - \mathbf{T}_h f(u(s)) = I_1^h(s) + I_2^h(s)$ where

$$I_1^h(s) = f\big(u^h(s)\big) - f\big(\mathbf{T}_h u(s)\big), \qquad I_2^h(s) = f\big(\mathbf{T}_h u(s)\big) - \mathbf{T}_h f\big(u(s)\big).$$

In the case of I_1^h we use that f satisfies (6.1) to get

$$\begin{aligned}\big\|I_1^h(s)\big\|_{l^2(h\mathbb{Z})} &\leqslant C\big(\big\|u^h(s)\big\|^p_{l^\infty(h\mathbb{Z})} + \big\|\mathbf{T}_h u(s)\big\|^p_{l^\infty(h\mathbb{Z})}\big)\big\|u^h(s) - \mathbf{T}_h u(s)\big\|_{l^2(h\mathbb{Z})} \\ &\leqslant C\big(\big\|u^h\big\|^p_{L^\infty((0,T)\times h\mathbb{Z})} + \|u\|^p_{L^\infty((0,T)\times\mathbb{R})}\big)\big\|u^h(s) - \mathbf{T}_h u(s)\big\|_{l^2(h\mathbb{Z})} \\ &\leqslant C\|u\|^p_{L^\infty(0,T;H^1(\mathbb{R}))}\mathrm{err}_h(s).\end{aligned}$$

Using the same arguments as in Lemma 5.2 we obtain that

$$\big\|I_2^h(s)\big\|_{l^2(h\mathbb{Z})} \leqslant h\big\|u(s)\big\|^{p+1}_{H^1(\mathbb{R})}.$$

Putting together all the above estimates, for any $0 \leqslant t \leqslant T$ we obtain:

$$\begin{aligned}\mathrm{err}_h(t) &\leqslant h^{1/2}T\|\varphi\|_{H^1(\mathbb{R})} + \|u\|^p_{L^\infty(0,T;H^1(\mathbb{R}))}\int_0^t \mathrm{err}_h(\sigma)\,d\sigma \\ &\quad + hT\|u\|^{p+1}_{L^\infty(0,T;H^1(\mathbb{R}))} + T^2h^{1/2}\|u\|^{p+1}_{L^\infty(0,T:\,H^1(\mathbb{R}))} \\ &\leqslant h^{1/2}\max\{T,T^2\}\big(\|\varphi\|_{H^1(\mathbb{R})} + \|u\|^{p+1}_{L^\infty(0,T:\,H^1(\mathbb{R}))}\big) + \|u\|^p_{L^\infty(0,T;H^1(\mathbb{R}))}\int_0^t \mathrm{err}_h(s)\,ds.\end{aligned}$$

Applying Gronwall's Lemma we obtain

$$\mathrm{err}_h(t) \lesssim h^{1/2}\max\{T,T^2\}\big(\|\varphi\|_{H^1(\mathbb{R})} + \|u\|^{p+1}_{L^\infty(0,T:\,H^1(\mathbb{R}))}\big)\exp\big(T\|u\|^p_{L^\infty(0,T;H^1(\mathbb{R}))}\big). \tag{6.10}$$

The proof is now finished. □

6.2. Approximating $H^s(\mathbb{R})$, $s < 1/2$, solutions by smooth ones

Given $\varphi \in H^s(\mathbb{R})$ we choose an approximation $\tilde\varphi \in H^1(\mathbb{R})$ such that $\|\varphi - \tilde\varphi\|_{H^s(\mathbb{R})}$ is small (a similar analysis can be done by considering $\varphi_\delta \in H^{s_1}$ with $s_1 > 1/2$). For $\tilde\varphi$ we consider the following approximation of $\tilde u$ solution of the NSE (1.2) with initial data $\tilde\varphi$:

$$\begin{cases} i\partial_t\tilde u_h(t) + A_h\tilde u_h = f(\tilde u_h), & t > 0, \\ \tilde u_h(0) = \mathbf{T}_h\tilde\varphi, \end{cases} \tag{6.11}$$

where the operator A_h is a second order approximation of the Laplace operator. We do not require the linear scheme associated to the operator A_h to satisfy uniform dispersive estimates.

Solving (6.11) we obtain an approximation $\tilde u_h$ of the solutions $\tilde u$ of NSE with initial datum $\tilde\varphi$, which itself is an approximation of the solution u of the NSE with initial datum φ.

In the following theorem we give an explicit estimate of the distance between $\tilde u_h$ and u.

Theorem 6.2. *Let $0 \leqslant s < 1/2$, $\varphi \in H^s(\mathbb{R})$, and $u \in C(\mathbb{R}; H^s(\mathbb{R}))$ be the solution of NSE with initial datum φ given by Theorem 5.1. For any $T > 0$ there exists a positive constant $C(T, \|\varphi\|_{L^2(\mathbb{R})})$ such that the following holds*

$$\|\mathbf{T}_h u - \tilde u_h\|_{L^\infty(0,T;l^2(h\mathbb{Z}))} \leqslant C\big(T, p, \|\varphi\|_{L^2(\mathbb{R})}\big)\|\varphi - \tilde\varphi\|_{L^2(\mathbb{R})} + h^{1/2}\exp\big(T\|\tilde u\|^p_{L^\infty(0,T;H^1(\mathbb{R}))}\big) \tag{6.12}$$

for all $h > 0$ and $\delta > 0$.

In the following we show that the above method of regularizing the initial data $\varphi \in H^s(\mathbb{R})$ and then applying the $H^1(\mathbb{R})$ theory for that approximation does not give the same rate of convergence $h^{s/2}$ obtained in the case of a dispersive method of order two (see (5.18)). This occurs since for $\|\varphi-\tilde{\varphi}\|_{L^2(\mathbb{R})}$ to be small, $\|\tilde{\varphi}\|_{H^1(\mathbb{R})}$ needs to be large and $\|\tilde{u}\|_{L^\infty(0,T;H^1(\mathbb{R}))}$ too.

To simplify the presentation we will consider the case $p=2$.

Theorem 6.3. *Let $p=2$, $0<s<1/2$, $\varphi \in H^s(\mathbb{R})$ and $u \in C(\mathbb{R}, H^s(\mathbb{R}))$ be solution of NSE with initial data φ given by Theorem 5.1 and u_h^* be the best approximation with $H^1(\mathbb{R})$-initial data as given by (6.11) with the conservative approximation $A_h=\Delta_h$. Then for any time T, there exists a constant $C(\|\varphi\|_{H^s(\mathbb{R})}, T, s)$ such that*

$$\big\|\mathbf{T}_h u - u_h^*\big\|_{L^\infty(0,T;l^2(h\mathbb{Z}))} \leqslant C\big(\|\varphi\|_{H^s(\mathbb{R})}, T, s\big)|\log h|^{-\frac{s}{1-s}}. \tag{6.13}$$

To prove this result we will use in an essential manner the following lemma.

Lemma 6.1. *Let $0<s<1$ and $h \in (0,1)$. Then for any $\varphi \in H^s(\mathbb{R})$ the functional $J_{h,\varphi}$ defined by*

$$J_{h,\varphi}(g) = \frac{1}{2}\|\varphi - g\|^2_{L^2(\mathbb{R})} + \frac{h}{2}\exp\big(\|g\|^2_{H^1(\mathbb{R})}\big) \tag{6.14}$$

satisfies:

$$\min_{g \in H^1(\mathbb{R})} J_{h,\varphi}(g) \leqslant C\big(\|\varphi\|_{H^s(\mathbb{R})}, s\big)|\log h|^{-s/(1-s)}. \tag{6.15}$$

Moreover, the above estimate is optimal in the sense that the power of the $|\log h|$ term cannot be improved: *for any $0<\epsilon<1-s$ there exists $\varphi_\epsilon \in H^s(\mathbb{R})$ such that*

$$\liminf_{h\to 0} \frac{\min_{g\in H^1(\mathbb{R})} J_{h,\varphi_\varepsilon}(g)}{|\log h|^{-(s+\varepsilon)/(1-s-\varepsilon)}} > 0. \tag{6.16}$$

Remark 6.1. We point out that, to obtain (6.15) and (6.16), we will use in an essential manner that $s<1$. In fact in the case $s=1$ the minimum of J_h over $H^1(\mathbb{R})$ is of order h. This can be seen by choosing $g=\varphi$ and observing that $J_h(\varphi) = h\exp(\|\varphi\|_{H^1(\mathbb{R})})$. This choice cannot be done if $\varphi \in H^s(\mathbb{R})\backslash H^1(\mathbb{R})$.

Proof of Theorem 6.3. Let us choose $\tilde{\varphi} \in H^1(\mathbb{R})$ which approximates φ in $H^s(\mathbb{R})$. Then by Theorem 6.2 we get

$$\begin{aligned}\|\mathbf{T}_h u - \tilde{u}_h\|^2_{L^\infty(0,T;l^2(h\mathbb{Z}))} &\leqslant C\big(T, \|\varphi\|_{L^2(\mathbb{R})}\big)\|\varphi-\tilde{\varphi}\|^2_{L^2(\mathbb{R})} + h\exp\big(2T\|\tilde{\varphi}\|^2_{H^1(\mathbb{R})}\big) \\ &\leqslant C\big(T, \|\varphi\|_{L^2(\mathbb{R})}\big)J_{h,\sqrt{2T}\varphi}(\sqrt{2T}\tilde{\varphi}),\end{aligned} \tag{6.17}$$

where $\tilde{u}_h$ is the solution of (6.11) with initial data $\mathbf{T}_h\tilde{\varphi}$.

For each h fixed, in order to obtain the best approximation u_h^* of $\mathbf{T}_h u$, we have to choose in the right-hand side of the above inequality the function φ^* which minimizes the functional $J_{h,\sqrt{2T}\varphi}(\cdot)$ defined by (6.14) over $H^1(\mathbb{R})$. Using estimate (6.15) from Lemma 6.1 we obtain the desired result:

$$\begin{aligned}\big\|\mathbf{T}_h u - u_h^*\big\|_{L^\infty(0,T;l^2(h\mathbb{Z}))} &\leqslant C\big(\|\varphi\|_{H^s(\mathbb{R})}, T, s\big)\min_{\tilde{\varphi}\in H^1(\mathbb{R})} J_{h,\sqrt{2T}\varphi}\big(\sqrt{2T}\tilde{\varphi}\big) \\ &\leqslant C\big(\|\varphi\|_{H^s(\mathbb{R})}, T, s\big)|\log h|^{-\frac{s}{1-s}},\end{aligned}$$

where u_h^* is the solution of (6.11) with initial data $\mathbf{T}_h\varphi^*$. □

Proof of Lemma 6.1. The functional $J_{h,\varphi}$ is convex and its minimizer, g_h, is unique. The function g_h satisfies the following equation:

$$-\varphi + g_h + h\exp\big(\|g_h\|^2_{H^1(\mathbb{R})}\big)(-\Delta g_h + g_h) = 0, \tag{6.18}$$

and so

$$\big[I + h\exp\big(\|g_h\|^2_{H^1(\mathbb{R})}\big)(I-\Delta)\big]g_h = \varphi.$$

Thus $c_h = \|g_h\|_{H^1(\mathbb{R})}$ is the unique solution of

$$c_h = \big\|(I-\Delta)^{1/2}\big[I + h\exp\big(c_h^2\big)(I-\Delta)\big]^{-1}\varphi\big\|_{L^2(\mathbb{R})}. \tag{6.19}$$

Step I. A useful auxiliary function. Let us consider the function $q_h(x) = hx^\beta \exp(x) - c$ for some positive constants β and c. We prove that there exist two constants $a_1(c)$ and $a_2(c)$ such that the solution x_h of the equation $q_h(x) = 0$ satisfies

$$|\log h| - \beta\log|\log h| + a_1(c) \leqslant x_h \leqslant |\log h| - \beta\log|\log h| + a_2(c). \tag{6.20}$$

Let us choose a real number a. Using that $h = \exp(-|\log h|)$ we get:

$$\begin{aligned} q_h\big(\big(|\log h| - \beta\log|\log h| + a\big)\big) &= \big(|\log h| - \beta\log|\log h| + a\big)^\beta \exp\big(-\beta\log|\log h| + a\big) - c \\ &= \left(1 - \beta\frac{\log|\log h|}{|\log h|} + \frac{a}{|\log h|}\right)^\beta \exp(a) - c \\ &\underset{h\to 0}{\longrightarrow} \exp(a) - c. \end{aligned}$$

Choosing now two constants a_1 and a_2 such that $\exp(a_1) < c < \exp(a_2)$ and using that the function q_h is increasing we obtain that, for h small enough, x_h, solution of $q_h(x) = 0$, satisfies (6.20).

Step II. Upper bounds on c_h. Using that $\varphi \in H^s(\mathbb{R})$, identity (6.19) gives us

$$\begin{aligned} c_h &= \big\|(I-\Delta)^{1/2}\big[I + h\exp\big(c_h^2\big)(I-\Delta)\big]^{-1}\varphi\big\|_{L^2(\mathbb{R})} \\ &= \big\|(I-\Delta)^{(1-s)/2}\big[I + h\exp\big(c_h^2\big)(I-\Delta)\big]^{-1}(I-\Delta)^{s/2}\varphi\big\|_{L^2(\mathbb{R})} \\ &= \big(he^{c_h^2}\big)^{(s-1)/2}\big\|\big[he^{c_h^2}(I-\Delta)\big]^{(1-s)/2}\big[I + he^{c_h^2}(I-\Delta)\big]^{-1}(I-\Delta)^{s/2}\varphi\big\|_{L^2(\mathbb{R})} \\ &\leqslant \big(h\exp\big(c_h^2\big)\big)^{(s-1)/2}\|\varphi\|_{H^s(\mathbb{R})}, \end{aligned}$$

since, when $s \in [0,1]$, the symbol in the Fourier variable of the operator

$$\big[he^{c_h^2}(I-\Delta)\big]^{(1-s)/2}\big[I + he^{c_h^2}(I-\Delta)\big]^{-1}$$

is less than one.

Then $c_h^2(h\exp(c_h^2))^{1-s} \leqslant \|\varphi\|^2_{H^s(\mathbb{R})}$ and

$$\big(c_h^2\big)^{1/(1-s)}he^{c_h^2} \leqslant \|\varphi\|^{2/(1-s)}_{H^s(\mathbb{R})}. \tag{6.21}$$

Applying the result of Step I to $\beta = 1/(1-s)$ and $c = \|\varphi\|^{2/(1-s)}_{H^s(\mathbb{R})}$ we obtain that c_h satisfies:

$$c_h^2 \leqslant |\log h| - \frac{1}{1-s}\log|\log h| + a_2, \tag{6.22}$$

for some constant $a_2 = a_2(\|\varphi\|^{2/(1-s)}_{H^s(\mathbb{R})})$. In particular, when $s < 1$,

$$h\exp\big(c_h^2\big) = \exp\big(c_h^2 - |\log h|\big) \leqslant \exp\left(-\frac{1}{1-s}\log|\log h| + a_2\right) \to 0,$$

as $h \to 0$.

Step III. Estimates on $J_h(g_h)$. Using that the minimizer g_h satisfies Eq. (6.18) and $c_h = \|g_h\|_{H^1(\mathbb{R})}$, we get

$$\begin{aligned} 2\min_{g\in H^1(\mathbb{R})} J_h(g) = 2J_h(g_h) &= \|\varphi - g_h\|_{L^2(\mathbb{R})} + h\exp\big(\|g_h\|^2_{H^1(\mathbb{R})}\big) \\ &= \big(h\exp\big(c_h^2\big)\big)^2\big\|(I-\Delta)g_h\big\|^2_{L^2(\mathbb{R})} + h\exp\big(c_h^2\big) \\ &= \big(h\exp\big(c_h^2\big)\big)^2\big\|(I-\Delta)\big[I + h\exp\big(c_h^2\big)(I-\Delta)\big]^{-1}\varphi\big\|^2_{L^2(\mathbb{R})} + h\exp\big(c_h^2\big) \end{aligned}$$

$$
\begin{aligned}
&= \left(he^{c_h^2}\right)^s \left\| \left[he^{c_h^2}(I-\Delta)\right]^{1-s/2}\left[I+he^{c_h^2}(I-\Delta)\right]^{-1}(I-\Delta)^{s/2}\varphi\right\|_{L^2(\mathbb{R})}^2 + he^{c_h^2} \\
&\leqslant \left(he^{c_h^2}\right)^s \|\varphi\|_{H^s(\mathbb{R})}^2 + he^{c_h^2} \leqslant \left(he^{c_h^2}\right)^s \left(\|\varphi\|_{H^s(\mathbb{R})}^2 + \left(he^{c_h^2}\right)^{1-s}\right) \\
&\leqslant c\left(s, \|\varphi\|_{H^s(\mathbb{R})}\right)\left(h\exp\left(c_h^2\right)\right)^s,
\end{aligned}
$$

where in the last inequality we used that $s \leqslant 1$ and $h\exp(c_h^2) \to 0$ as $h \to 0$.

Thus, by (6.22) we obtain that

$$
\min_{g\in H^1(\mathbb{R})} J_h(g) \leqslant c\left(s, \|\varphi\|_{H^s(\mathbb{R})}\right)\left(h\exp\left(c_h^2\right)\right)^s \leqslant c\left(s, \|\varphi\|_{H^s(\mathbb{R})}\right)|\log h|^{-\frac{s}{1-s}}. \tag{6.23}
$$

Step IV. A particular function φ. Let us choose $\varepsilon > 0$ and φ_ε be defined by means of its Fourier transform

$$
\hat{\varphi}_\varepsilon^2(\xi) = \frac{1}{(1+\xi^2)^{s+\frac{1}{2}+\varepsilon}}.
$$

Thus, for any $\varepsilon > 0$, $\varphi_\varepsilon \in H^s(\mathbb{R})$. We will prove that, in this case, the solution $c_{\varepsilon,h}$ of (6.19) satisfies

$$
c_{\varepsilon,h}^2 \geqslant |\log h| - \frac{1}{1-s-\varepsilon}\log|\log h| + a_1, \tag{6.24}
$$

and

$$
\min_{g\in H^1(\mathbb{R})} J_{h,\varphi_\varepsilon}(g) \geqslant \left(h\exp\left(c_{\varepsilon,h}^2\right)\right)^{s+\varepsilon} \geqslant |\log h|^{-(s+\varepsilon)/(1-s-\varepsilon)}, \tag{6.25}
$$

for some constant a_1.

To prove (6.24) and (6.25) we claim that for any $\gamma \in (-1/2, 2)$ and x large enough the following holds:

$$
\int_{\mathbb{R}} \frac{(1+\xi^2)^\gamma}{(x+1+\xi^2)^2}\,d\xi \geqslant \frac{c(\gamma)}{x^{3/2-\gamma}}. \tag{6.26}
$$

Using that $c_{\varepsilon,h}$ is solution of (6.19) and estimate (6.26) with $\gamma = 1/2 - s - \varepsilon$ and $x = (h\exp c_{\varepsilon,h}^2)^{-1}$ we obtain

$$
\begin{aligned}
c_{\varepsilon,h}^2 &= \int_{\mathbb{R}} \frac{(1+\xi^2)\hat{\varphi}_\varepsilon^2(\xi)}{(1+h\exp(c_{\varepsilon,h}^2)(1+\xi^2))^2}\,d\xi = \frac{1}{(h\exp(c_{\varepsilon,h}^2))^2}\int_{\mathbb{R}} \frac{(1+\xi^2)^{\frac{1}{2}-s-\varepsilon}}{((h\exp(c_{\varepsilon,h}^2))^{-1}+(1+\xi^2))^2}\,d\xi \\
&\geqslant \frac{1}{(h\exp(c_{\varepsilon,h}^2))^{1-s-\varepsilon}},
\end{aligned}
$$

and

$$
h\exp\left(c_{\varepsilon,h}^2\right)\left(c_{\varepsilon,h}^2\right)^{1/(1-s-\varepsilon)} - 1 \geqslant 0.
$$

Applying Step I to the function $q_h = hx^{1/(1-s-\varepsilon)}\exp(x) - 1$ we find that

$$
c_{\varepsilon,h}^2 \geqslant |\log h| - \frac{1}{1-s-\varepsilon}\log|\log h| + a_1, \tag{6.27}
$$

for some constant a_1.

This concludes the proof of (6.24).

We now prove (6.25). In view of (6.18) the minimizer $g_{\varepsilon,h}$ satisfies

$$
-\varphi_\varepsilon + g_{\varepsilon,h} + h\exp\left(\|g_{\varepsilon,h}\|_{H^1(\mathbb{R})}^2\right)(-\Delta g_{\varepsilon,h} + g_{\varepsilon,h}) = 0, \tag{6.28}
$$

and

$$
g_{\varepsilon,h} = \left[I + h\exp\left(c_{\varepsilon,h}^2\right)(I-\Delta)\right]^{-1}\varphi_\varepsilon, \tag{6.29}
$$

where $c_{\varepsilon,h} = \|g_{\varepsilon,h}\|_{H^1(\mathbb{R})}$.

Thus

$$\begin{aligned}2J_{h,\varphi_\varepsilon}(g_{\varepsilon,h}) &= \|\varphi_\varepsilon - g_{\varepsilon,h}\|^2_{L^2(\mathbb{R})} + h\exp\bigl(\|g_{\varepsilon,h}\|^2_{H^1(\mathbb{R})}\bigr)\\ &= \bigl(h\exp\bigl(c^2_{\varepsilon,h}\bigr)\bigr)^2\bigl\|(I-\Delta)g_{\varepsilon,h}\bigr\|^2_{L^2(\mathbb{R})} + h\exp\bigl(c^2_{\varepsilon,h}\bigr)\\ &= \bigl(h\exp\bigl(c^2_{\varepsilon,h}\bigr)\bigr)^2\bigl\|(I-\Delta)\bigl[I+h\exp\bigl(c^2_{\varepsilon,h}\bigr)(I-\Delta)\bigr]^{-1}\varphi_\varepsilon\bigr\|^2_{L^2(\mathbb{R})} + h\exp\bigl(c^2_{\varepsilon,h}\bigr).\end{aligned}$$

Writing the last term in Fourier variable we get

$$\begin{aligned}2\min_{g\in H^1(\mathbb{R})} J_{h,\varphi_\varepsilon}(g) &= \bigl(h\exp\bigl(c^2_{\varepsilon,h}\bigr)\bigr)^2\int_{\mathbb{R}}\frac{(1+\xi^2)\hat{\varphi}^2_{\varepsilon,h}(\xi)}{(1+h\exp(c^2_{\varepsilon,h})(1+\xi^2))^2} + h\exp\bigl(c^2_{\varepsilon,h}\bigr)\\ &= \int_{\mathbb{R}}\frac{(1+\xi^2)^{\frac{3}{2}-s-\varepsilon}}{((h\exp(c^2_{\varepsilon,h}))^{-1}+1+\xi^2)^2}\,d\xi + h\exp\bigl(c^2_{\varepsilon,h}\bigr).\end{aligned}$$

The same arguments as in Step II give us that $h\exp(c^2_{\varepsilon,h}) \to 0$ as $h \to 0$. Then for small enough h, x_h defined by $x_h = (h\exp(c^2_{\varepsilon,h}))^{-1}$ is sufficiently large to apply inequality (6.26) with $\gamma = 3/2 - s - \varepsilon$. We get

$$2\min_{g\in H^1(\mathbb{R})} J_{h,\varphi_\varepsilon}(g) \geqslant \bigl(h\exp\bigl(c^2_{\varepsilon,h}\bigr)\bigr)^{s+\varepsilon} + h\exp\bigl(c^2_{\varepsilon,h}\bigr) \geqslant \bigl(h\exp\bigl(c^2_{\varepsilon,h}\bigr)\bigr)^{s+\varepsilon}.$$

Using now (6.27) we obtain

$$\min_{g\in H^1(\mathbb{R})} J_{h,\varphi_\varepsilon}(g) \gtrsim |\log h|^{-\frac{s+\varepsilon}{1-s-\varepsilon}}$$

which proves (6.16).

To finish the proof it remains to prove (6.26). For $|x| \to \infty$, using changes of variables we get

$$\begin{aligned}\int_0^\infty \frac{(1+\xi^2)^\gamma}{(x+1+\xi^2)^2}\,d\xi &\gtrsim \int_{\sqrt{3}}^\infty \frac{(1+\xi^2)^\gamma}{x^2+(1+\xi^2)^2}\,d\xi + O\Bigl(\frac{1}{x^2}\Bigr)\\ &\underset{\xi=(\mu^2-1)^{1/2}}{=} \int_2^\infty \frac{\mu^{2\gamma}}{x^2+\mu^4}\frac{\mu}{(\mu^2-1)^{1/2}}\,d\mu + O\Bigl(\frac{1}{x^2}\Bigr)\\ &\gtrsim \int_2^\infty \frac{\mu^{2\gamma}}{x^2+\mu^4}\,d\mu + O\Bigl(\frac{1}{x^2}\Bigr)\\ &\underset{\mu=x^{1/2}\xi}{=} \frac{1}{x^{3/2-\gamma}}\int_{\xi\geqslant x^{-1/2}} \frac{\xi^{2\gamma}}{1+\xi^4}\,d\xi + O\Bigl(\frac{1}{x^2}\Bigr)\\ &\gtrsim \frac{1}{x^{3/2-\gamma}} + O\Bigl(\frac{1}{x^2}\Bigr) \gtrsim \frac{1}{x^{3/2-\gamma}}\Bigl(1+\frac{1}{x^{1/2+\gamma}}\Bigr)\\ &\gtrsim \frac{1}{x^{3/2-\gamma}}\end{aligned}$$

which proves (6.26). □

Proof of Theorem 6.2. Using the stability result (5.1) for the NSE we obtain

$$\begin{aligned}\|u-\tilde{u}\|_{L^\infty(0,T;L^2(\mathbb{R}))} &\leqslant C\bigl(T,p,\|\varphi\|_{L^2(\mathbb{R})},\|\tilde{\varphi}\|_{L^2(\mathbb{R})}\bigr)\|\varphi-\tilde{\varphi}\|_{L^2(\mathbb{R})}\\ &\leqslant C\bigl(T,p,\|\varphi\|_{L^2(\mathbb{R})}\bigr)\|\varphi-\tilde{\varphi}\|_{L^2(\mathbb{R})}.\end{aligned}$$

Now using the classical results for smooth initial data presented in Section 6.1, by (6.10) we get

$$\big\|\mathbf{T}_h\tilde{u} - \tilde{u}_h^h\big\|_{L^\infty(0,T;l^2(h\mathbb{Z}))} \leqslant Ch^{1/2}\exp\big(T\|\tilde{u}\|^p_{L^\infty(0,T;H^1(\mathbb{R}))}\big).$$

Thus

$$\begin{aligned}
\|\mathbf{T}_h u - \tilde{u}_h\|_{L^\infty(0,T;l^2(h\mathbb{Z}))} &\leqslant \|\mathbf{T}_h u - \mathbf{T}_h\tilde{u}\|_{L^\infty(0,T;l^2(h\mathbb{Z}))} + \|\mathbf{T}_h\tilde{u} - \tilde{u}_h\|_{L^\infty(0,T;l^2(h\mathbb{Z}))} \\
&\leqslant \|u - \tilde{u}\|_{L^\infty(0,T;L^2(\mathbb{R}))} + \|\mathbf{T}_h\tilde{u} - \tilde{u}_h\|_{L^\infty(0,T;l^2(h\mathbb{Z}))} \\
&\leqslant C\big(T, p, \|\varphi\|_{L^2(\mathbb{R})}\big)\|\varphi - \tilde{\varphi}\|_{L^2(\mathbb{R})} + h^{1/2}\exp\big(T\|\tilde{u}\|^p_{L^\infty(0,T;H^1(\mathbb{R}))}\big).
\end{aligned}$$

This yields (6.12). □

7. Technical lemmas

In this section we prove some technical results that have been used along the paper. The main aim of this section is to obtain estimates on the difference $f(\mathbf{T}_h u) - \mathbf{T}_h f(u)$ in auxiliary norms $L^q(I, l^r(h\mathbb{Z}))$.

In the case of smooth enough functions u, the pointwise projection operator

$$(\mathbf{E}_h u)(jh) = u(jh) \tag{7.1}$$

makes sense. More precisely it is well defined in $H^s(\mathbb{R})$, $s > 1/2$. In these cases the use of the operator $\mathbf{E}_h$ has the advantage of commuting with the nonlinearity $f(\mathbf{E}_h u) = \mathbf{E}_h f(u)$.

The key ingredient is the following theorem:

Theorem 7.1 *(Marcinkiewicz multiplier theorem). (See [12, Theorem 5.2.2, p. 356].) Let $m:\mathbb{R}\to\mathbb{R}$ be a bounded function which is C^1 in every dyadic set $(2^j, 2^{j+1}) \cup (-2^{j+1}, -2^j)$ for $j \in \mathbb{Z}$. Assume that the derivative m' of m satisfies*

$$\sup_{j\in\mathbb{Z}}\Bigg[\int_{-2^{j+1}}^{-2^j} \big|m'(\xi)\big|\,d\xi + \int_{2^j}^{2^{j+1}} \big|m'(\xi)\big|\,d\xi\Bigg] \leqslant A < \infty. \tag{7.2}$$

Then there exists a positive constant C such that for all $1 < q < \infty$ the following holds:

$$\big\|(\hat{f}m)^\vee\big\|_{L^q(\mathbb{R})} \leqslant C\max\big\{q, (q-1)^{-1}\big\}^6\big(A + \|m\|_{L^\infty(\mathbb{R})}\big)\|f\|_{L^q(\mathbb{R})}.$$

Remark 7.1. Using a change of variables in the Fourier space the above dyadic intervals can be replaced by any other one of the form $(c^j, c^{j+1}) \cup (-c^{j+1}, -c^j)$, $j \in \mathbb{Z}$ and $c > 1$. In the following applications, the constant c will be chosen to be $c = \pi$.

For any function $u \in L^2(\mathbb{R})$ we define the new function $\tilde{u}_h$ by truncating the Fourier transform as follows:

$$\widehat{\tilde{u}_h}(\xi) = \hat{u}(\xi)\mathbf{1}_{(-\pi/h,\pi/h)}(\xi). \tag{7.3}$$

For $h = 1$, Theorem 7.1, applied with $m(\xi) = \mathbf{1}_{(-\pi,\pi)}$ which is C^1 in every dyadic interval, shows that for any $1 < q < \infty$, the $L^q(\mathbb{R})$-norm of $\tilde{u}_1$ can be controlled by the one of u:

$$\|\tilde{u}_1\|_{L^q(\mathbb{R})} \leqslant C(q)\|u\|_{L^q(\mathbb{R})}. \tag{7.4}$$

A scaling argument shows us that the above inequality also holds for all $h > 0$ with a constant $C(q)$ independent of h.

Using Theorem 7.1 we can refine this estimate as follows:

Lemma 7.1. *For any $s \geqslant 0$ and $q \in (1, \infty)$ the following hold:*

(a) *There exists a positive constant $c(s, q)$ such that*

$$\|u - \tilde{u}_h\|_{L^q(\mathbb{R})} \leqslant c(s, q)h^s\|u\|_{\dot{W}^{s,q}(\mathbb{R})} \tag{7.5}$$

for all $u \in \dot{W}^{s,q}(\mathbb{R})$ and $h > 0$.

(b) *Assuming* $s \in [0,1]$, *there exists a positive constant* $c(s,q)$ *such that*

$$h\|\tilde{u}_h\|_{\dot{W}^{1,q}(\mathbb{R})} \leqslant c(s,q)h^s\|u\|_{\dot{W}^{s,q}(\mathbb{R})} \tag{7.6}$$

for all $u \in \dot{W}^{s,q}(\mathbb{R})$ *and* $h > 0$.

Proof. We divide the proof in two steps corresponding to (7.5) and (7.6).

Step I. Proof of (7.5). Let us consider the following operator

$$M_h u := u - \tilde{u}_h = \left(\mathbf{1}_{\{|\xi| \geqslant \pi/h\}}\hat{u}\right)^\vee.$$

A change of variables gives us that

$$(M_h u)(x) = M_1\big(u(h\cdot)\big)\left(\frac{x}{h}\right).$$

Using this property the following identities hold:

$$\|M_h u\|_{L^q(\mathbb{R})} = h^{1/q}\big\|M_1\big(u(h\cdot)\big)\big\|_{L^q(\mathbb{R})},$$

and

$$\big\|u(h\cdot)\big\|_{\dot{W}^{s,q}(\mathbb{R})} = \big\||\nabla|^s\big[u(h\cdot)\big]\big\|_{L^q(\mathbb{R})} = h^s\big\|\big(|\nabla|^s u\big)(h\cdot)\big\|_{L^q(\mathbb{R})} = h^s h^{1/q}\big\||\nabla|^s u\big\|_{L^q(\mathbb{R})}.$$

Thus, it is sufficient to consider the case $h = 1$ and to prove that

$$\|M_1 v\|_{L^q(\mathbb{R})} \leqslant c(s,q)\big\||\nabla|^s v\big\|_{L^q(\mathbb{R})} \tag{7.7}$$

holds for all $v \in \dot{W}^{s,q}(\mathbb{R})$.

With the notation

$$m_s(\xi) := |\xi|^{-s}\mathbf{1}_{\{|\xi| \geqslant \pi\}}(\xi),$$

estimate (7.7) holds if $m_s(\xi)$ satisfies the hypothesis of Theorem 7.1. Using that $m_s(\xi) \in L^\infty(\mathbb{R})$ and that

$$\big|m_s'(\xi)\big| \leqslant \frac{c(s)}{|\xi|^{s+1}}\mathbf{1}_{\{|\xi| \geqslant \pi\}}(\xi), \quad \xi \in \mathbb{R},$$

by Theorem 7.1 we obtain (7.7).

Step II. Proof of (7.6). A similar argument as in the previous case reduces estimate (7.6) to the case $h = 1$:

$$\big\|\big(\hat{u}(\xi)\mathbf{1}_{(-\pi,\pi)}|\xi|\big)^\vee\big\|_{L^q(\mathbb{R})} \leqslant c(s,q)\big\|\big(\hat{u}(\xi)|\xi|^s\big)^\vee\big\|_{L^q(\mathbb{R})}.$$

Denoting $v = (\hat{u}(\xi)|\xi|^s)^\vee$, it remains to prove that

$$\big\|\big(\hat{v}(\xi)\mathbf{1}_{(-\pi,\pi)}|\xi|^{1-s}\big)^\vee\big\|_{L^q(\mathbb{R})} \leqslant c(s,q)\|v\|_{L^q(\mathbb{R})}. \tag{7.8}$$

In other words, it is sufficient to apply Theorem 7.1 to the multiplier $m_s(\xi)$ given by

$$m_s(\xi) = |\xi|^{1-s}\mathbf{1}_{(-\pi,\pi)}(\xi).$$

Using that $m_s(\xi) \in L^\infty(\mathbb{R})$ satisfies

$$\big|m_s'(\xi)\big| \leqslant c(s)|\xi|^{-s}\mathbf{1}_{(-\pi,\pi)}(\xi), \quad \xi \in \mathbb{R} \setminus \{0\},$$

we fit in the hypothesis of Theorem 7.1 and then (7.8) holds. This finishes the proof. □

In the following we obtain error estimates for the difference between the two interpolators $\mathbf{T}_h$ and $\mathbf{E}_h$ when applied to functions u and $f(u)$, where $\mathbf{T}_h$ and $\mathbf{E}_h$ are defined by (3.2) and (7.1) respectively.

Lemma 7.2. *Let* $s > 1/2$ *and* $q \in (1,\infty)$. *Then there exists a positive constant* $c(s,q)$ *such that*

$$\|\mathbf{T}_h u - \mathbf{E}_h u\|_{l^q(h\mathbb{Z})} \leqslant c(s,q)h^s\|u\|_{\dot{W}^{s,q}(\mathbb{R})} \tag{7.9}$$

holds for all $u \in W^{s,q}(\mathbb{R})$ *and* $h > 0$.

Remark 7.2. This lemma generalizes Theorem 10.1.3 of [28, p. 205], which addresses the case $q = 2$, $s > 1/2$. In this case using Plancherel's identity in the discrete setting it is easy to obtain

$$\|\mathbf{T}_h u - \mathbf{E}_h u\|_{l^2(h\mathbb{Z})} \leqslant c(s)h^s \|u\|_{\dot{H}^s(\mathbb{R})}. \tag{7.10}$$

Remark 7.3. Using the above results, we will be able to obtain estimates of the difference $\mathbf{T}_h f(u) - f(\mathbf{T}_h u)$, $f(u) = |u|^p u$, $p \geqslant 0$, given by Lemma 5.2.

Proof of Lemma 7.2. Estimate (7.10) provides the desired estimate $\dot{W}^{s,2}(\mathbb{R}) \to l^2(h\mathbb{Z})$ in the case $q = 2$. We will also prove the estimate $\dot{W}^{s,q} \to l^q(h\mathbb{Z})$ in the case $s > 1$. Using these two estimates the general case will be a consequence of an interpolation argument.

Case 1: $s > 1$, $q \in (1, \infty)$. We claim that

$$\|\mathbf{T}_h u - \mathbf{E}_h u\|_{l^p(h\mathbb{Z})} \leqslant c(p, s)h^s \big\||\nabla|^s u\big\|_{L^p(\mathbb{R})}. \tag{7.11}$$

By rescaling all the above quantities we can assume $h = 1$.

We have the following:

$$(\mathbf{T}_1 u - \mathbf{E}_1 u)(j) = \int_{|\xi|\geqslant\pi} e^{ij\xi}\hat{u}(\xi) = \int_{-\pi}^{\pi} e^{ij\xi} \sum_{l\neq 0} \hat{u}(\xi + 2\pi l).$$

Denoting by v the function whose Fourier transform is given by

$$\hat{v}(\xi) = \mathbf{1}_{(-\pi,\pi)} \sum_{l\neq 0} \hat{u}(\xi + 2\pi l), \tag{7.12}$$

we get

$$(\mathbf{T}_1 u - \mathbf{E}_1 u)(j) = \int_{-\pi}^{\pi} e^{ij\xi}\hat{v}(\xi)\, d\xi.$$

Classical results on band-limited functions (see Plancherel and Pólya [25]) give us that

$$\|\mathbf{T}_1 u - \mathbf{E}_1 u\|_{l^p(\mathbb{Z})} \leqslant \|v\|_{L^p(\mathbb{R})},$$

provided that the right-hand side term of the above inequality makes sense. It is then sufficient to prove that the function v defined by (7.12) satisfies:

$$\|v\|_{L^p(\mathbb{R})} \leqslant c(p, s)\big\||\nabla|^s u\big\|_{L^p(\mathbb{R})}. \tag{7.13}$$

Using the properties of the Fourier transform we get:

$$v(x) = \sum_{l\neq 0} e^{2i\pi lx}\big(\mathbf{1}_{((2l-1)\pi,(2l+1)\pi)}\hat{u}\big)^{\vee}.$$

It is sufficient to prove that

$$\Big\|\sum_{l\neq 0} e^{2i\pi lx}\big(\mathbf{1}_{((2l-1)\pi,(2l+1)\pi)}\hat{u}\big)^{\vee}\Big\|_{L^p(\mathbb{R})} \leqslant \big\||\nabla|^s u\big\|_{L^p(\mathbb{R})}$$

or equivalently

$$\Big\|\sum_{l\neq 0} e^{2i\pi lx}\big(|\xi|^{-s}\mathbf{1}_{((2l-1)\pi,(2l+1)\pi)}\hat{u}\big)^{\vee}\Big\|_{L^p(\mathbb{R})} \leqslant \|u\|_{L^p(\mathbb{R})}.$$

Minkowski's inequality gives us

$$\Big\|\sum_{l\neq 0} e^{2i\pi lx}\big(|\xi|^{-s}\mathbf{1}_{((2l-1)\pi,(2l+1)\pi)}\hat{u}\big)^{\vee}\Big\|_{L^p(\mathbb{R})} \leqslant \sum_{l\neq 0}\big\|\big(|\xi|^{-s}\mathbf{1}_{((2l-1)\pi,(2l+1)\pi)}\hat{u}\big)^{\vee}\big\|_{L^p(\mathbb{R})}.$$

We claim that for any $l \neq 0$:

$$\left\| \left(|\xi|^{-s} \mathbf{1}_{((2l-1)\pi,(2l+1)\pi)} \hat{u} \right)^{\vee} \right\|_{L^p} \leqslant \frac{c(s)}{|l|^s} \|u\|_{L^p(\mathbb{R})}. \tag{7.14}$$

Thus, summing all the above inequalities for $l \neq 0$ we obtain the desired estimate.

A translation in (7.14) reduces its proof to show that $m_{s,l}$, defined by

$$m_{s,l}(\xi) = |\xi - 2l\pi|^{-s} \mathbf{1}_{(-\pi,\pi)}(\xi), \quad l \neq 0,$$

verify the hypothesis of Theorem 7.1. Observe that

$$\left| m_{s,l}(\xi) \right| \leqslant \frac{c(s)}{|l|^s}, \quad \xi \in \mathbb{R}, \ l \neq 0,$$

and

$$\left| m'_{s,l}(\xi) \right| \leqslant \frac{c(s)}{|l|^s |\xi|} \mathbf{1}_{(-\pi,\pi)}(\xi), \quad \xi \in \mathbb{R} \setminus \{0\}, \ l \neq 0.$$

Applying Theorem 7.1 to each multiplier $m_{s,l}$ we get (7.14) and the proof of this case is finished.

Case 2: $s > 1/2$, $q \in (1, \infty)$. We set $U_h = \mathbf{T}_h - \mathbf{E}_h$. Using the estimates of the previous case we deduce that the operator U_h satisfies:

$$U_h : \dot{W}^{s_1,q_1}(\mathbb{R}) \to l^{q_1}(h\mathbb{Z}), \quad s_1 > 1, \ 1 < q_1 < \infty,$$

and by (7.10):

$$U_h : \dot{W}^{s_2,2}(\mathbb{R}) \to l^2(h\mathbb{Z}), \quad s_2 > 1/2.$$

Then for any $\theta \in (0, 1)$,

$$U_h : \left[\dot{W}^{s_1,q_1}(\mathbb{R}), \dot{W}^{s_2,2}(\mathbb{R})\right]_{[\theta]} \to \left[l^{q_1}(h\mathbb{Z}), l^2(h\mathbb{Z})\right]_{[\theta]}$$

with a norm that satisfies:

$$\|U_h\|_{[\dot{W}^{s_1,q_1}(\mathbb{R}),\dot{W}^{s_2,2}(\mathbb{R})]_{[\theta]}-[l^{q_1}(h\mathbb{Z}),l^2(h\mathbb{Z})]_{[\theta]}} \leqslant \|U_h\|^{\theta}_{\dot{W}^{s_1,q_1}(\mathbb{R})-l^{q_1}(h\mathbb{Z})} \|U_h\|^{1-\theta}_{\dot{W}^{s_2,2}(\mathbb{R})-l^2(h\mathbb{Z})}.$$

Classical results on interpolation theory [2, Theorem 6.4.5, p. 153] give us that

$$\left[\dot{W}^{s_1,q_1}(\mathbb{R}), \dot{W}^{s_2,2}(\mathbb{R})\right]_{[\theta]} = \dot{W}^{s,q}(\mathbb{R}),$$

and

$$\left[l^{q_1}(h\mathbb{Z}), l^2(h\mathbb{Z})\right]_{[\theta]} = l^q(h\mathbb{Z}),$$

where s and q are given by

$$\begin{cases} s = f_\theta(s_1, s_2) = s_1\theta + s_2(1-\theta), \\ \dfrac{1}{q} = g_\theta(q_1) = \dfrac{\theta}{q_1} + \dfrac{1-\theta}{2}. \end{cases} \tag{7.15}$$

Using that the ranks of functions f_θ and g_θ satisfy

$$\operatorname{Im}(f_\theta) = \left(\frac{1+\theta}{2}, \infty\right), \qquad \operatorname{Im}(g_\theta) = \left(\frac{1-\theta}{2}, \frac{1+\theta}{2}\right),$$

we obtain that for any $s > 1/2$ and $0 < q < 1$ we can find $s_1 > 1$, $s_2 > 1/2$, $q_1 > 1$ and $\theta \in (0, 1)$ such that (7.15) holds and

$$\|A_h\|_{\dot{W}^{s,q}(\mathbb{R})-l^q(h\mathbb{Z})} \leqslant h^{s_1\theta} h^{s_2(1-\theta)} \leqslant h^s. \tag{7.16}$$

The proof is now finished. □

Proof of Lemma 5.2. We first recall that the following inequality holds for all $u, v \in L^{p+2}(\mathbb{R})$:

$$\big\| f(u) - f(v) \big\|_{L^{(p+2)'}(\mathbb{R})} \leqslant C(p)\big(\|u\|^p_{L^{p+2}} + \|v\|^p_{L^{p+2}}\big)\|u - v\|_{L^{p+2}(\mathbb{R})}. \tag{7.17}$$

We set $\tilde{u}_h$ defined by $\widehat{\tilde{u}_h}(\xi) = \hat{u}(\xi)\mathbf{1}_{(-\pi/h,\pi/h)}(\xi)$. The difference $\mathbf{T}_h f(u) - f(\mathbf{T}_h u)$ in (5.25) satisfies:

$$\big\|\mathbf{T}_h f(u) - f(\mathbf{T}_h u)\big\|_{l^{(p+2)'}(h\mathbb{Z})} \leqslant \big\|\mathbf{T}_h f(u) - \mathbf{T}_h f(\tilde{u}_h)\big\|_{l^{(p+2)'}(h\mathbb{Z})} + \big\|\mathbf{T}_h f(\tilde{u}_h) - f(\mathbf{T}_h u)\big\|_{l^{(p+2)'}(h\mathbb{Z})}.$$

Using (7.17), (7.4) and Lemma 7.1, the first term in the right-hand side satisfies:

$$\begin{aligned}\big\|\mathbf{T}_h f(u) - \mathbf{T}_h f(\tilde{u}_h)\big\|_{l^{(p+2)'}(h\mathbb{Z})} &\leqslant c(p)\big\| f(u) - f(\tilde{u}_h)\big\|_{L^{(p+2)'}(\mathbb{R})} \\ &\leqslant c(p)\big(\|u\|^p_{L^{p+2}(\mathbb{R})} + \|\tilde{u}_h\|^p_{L^{p+2}(\mathbb{R})}\big)\|u - \tilde{u}_h\|_{L^{p+2}(\mathbb{R})} \\ &\leqslant c(p)h^s\|u\|^p_{L^{p+2}(\mathbb{R})}\|u\|_{\dot{W}^{s,p+2}(\mathbb{R})} \leqslant c(p)h^s\|u\|^{p+1}_{W^{s,p+2}(\mathbb{R})}.\end{aligned}$$

For the second term, using that on the grid $h\mathbb{Z}$, $\mathbf{T}_h u = \mathbf{E}_h \tilde{u}_h$, by Lemma 7.2 we get:

$$\begin{aligned}\big\|\mathbf{T}_h f(\tilde{u}_h) - f(\mathbf{T}_h u)\big\|_{l^{(p+2)'}(h\mathbb{Z})} &= \big\|\mathbf{T}_h f(\tilde{u}_h) - f(\mathbf{E}_h \tilde{u}_h)\big\|_{l^{(p+2)'}(h\mathbb{Z})} \\ &= \big\|\mathbf{T}_h f(\tilde{u}_h) - \mathbf{E}_h f(\tilde{u}_h)\big\|_{l^{(p+2)'}(h\mathbb{Z})} \\ &\leqslant h\big\| f(\tilde{u}_h)\big\|_{\dot{W}^{1,(p+2)'}(\mathbb{R})} \leqslant h\big\|\tilde{u}_h^p \partial_x \tilde{u}_h\big\|_{L^{(p+2)'}(\mathbb{R})}.\end{aligned} \tag{7.18}$$

Using that $s \in [0,1]$ we apply Young's inequality and (7.6) to obtain:

$$\begin{aligned}\big\|\tilde{u}_h^p \partial_x \tilde{u}_h\big\|_{L^{(p+2)/(p+1)}(\mathbb{R})} &= \bigg(\int_{\mathbb{R}} |\tilde{u}_h|^{p(p+2)/(p+1)}|\partial_x \tilde{u}_h|^{(p+2)/(p+1)}\bigg)^{(p+1)/(p+2)} \\ &\leqslant \Big(\big\||\tilde{u}_h|^{p(p+2)/(p+1)}\big\|_{(p+1)/p}\big\||\partial_x \tilde{u}_h|^{(p+2)/(p+1)}\big\|_{p+1}\Big)^{(p+1)/(p+2)} \\ &= \|\tilde{u}_h\|^p_{L^{p+2}(\mathbb{R})}\|\partial_x \tilde{u}_h\|_{L^{p+2}(\mathbb{R})} \leqslant \|u\|^p_{L^{p+2}(\mathbb{R})}\|\tilde{u}_h\|_{\dot{W}^{1,p+2}} \\ &\lesssim \|u\|^p_{L^{p+2}(\mathbb{R})}h^{s-1}\|u\|_{\dot{W}^{s,p+2}(\mathbb{R})} \leqslant h^{s-1}\|u\|^{p+1}_{W^{s,p+2}(\mathbb{R})}.\end{aligned} \tag{7.19}$$

Thus by (7.18) and (7.19) we obtain

$$\big\|\mathbf{T}_h f(\tilde{u}_h) - f(\mathbf{T}_h u)\big\|_{l^{(p+2)'}(h\mathbb{Z})} \leqslant h^s\|u\|^{p+1}_{W^{s,p+2}(\mathbb{R})}$$

which finishes the proof. □

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