BOSTON UNIVERSITY

GRADUATE SCHOOL OF ARTS AND SCIENCES

Dissertation

INTEGRABLE SYSTEMS AND FEYNMAN DIAGRAMS

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M.Sc., University of Bucharest, 1998
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Submitted in partial fulfillment of the

requirements for the degree of

Doctor of Philosophy

2007

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Acknowledgments

I would like to express my deepest gratitude to my advisor Steven Rosenberg for sharing his ideas and time. None of this work would have been possible without his guidance. I would like to Dirk Kreimer for his great lectures and useful discussions.

I want to thank the Dissertation Committee Professors David Fried, Takashi Kimura, Dirk Kreimer, Steven Rosenberg and Matt Szczesny for their time reading the dissertation.

INTEGRABLE SYSTEMS AND FEYNMAN DIAGRAMS

(Order No.)

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Abstract

In the theory of integrable systems, a solution to a Lax pair equation associated to a coadjoint orbit of a semisimple Lie group is given by a Birkhoff factorization. By the work of Connes-Kreimer, there is a Birkhoff factorization of characters on the Kreimer Hopf algebra of Feynman diagrams. In this thesis, we reverse the usual procedure in integrable systems by producing a Lax pair equation $\frac{dL}{dt} = [M, L]$ whose solution is given precisely by the Connes-Kreimer Birkhoff factorization. The main technical issue, that the Lie algebra of infinitesimal characters is not semisimple, is overcome by passing to the double Lie algebra with the simplest possible Lie algebra structure. In particular, the Lax pair gives a flow for the character φ given by Feynman rules in dimensional regularization. We work out an explicit example of the theory on a finitely generated subalgebra of the Hopf algebra of Feynman diagrams.

Contents

List of Symbols

$\mathcal{A}$	The algebra of Laurent series
$\mathcal{A}_-$	The algebra of polynomials without free term in λ^{-1}
$\mathcal{A}_+$	The algebra of the holomorphic functions
$G_{\mathcal{A}}$	The group of characters with values in $\mathcal{A}$
$G = \mathrm{C}har(H)$	The group of characters with values in $\mathbb{C}$
$\mathfrak{g}_{\mathcal{A}}$	The Lie algebra of infinitesimal characters with values in $\mathcal{A}$
$\mathfrak{g} = \partial \mathrm{C}har(H)$	The Lie algebra of infinitesimal characters
$C^\infty(M)$	The set of smooth functions on M
X_φ	The hamiltonian vector field of φ
W	The Poisson bivector of a Poisson structure
δ	The double Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$
$L\delta$	The loop algebra (of Laurent polynomials) of δ
I	Identification map between $L\delta$ and $L\delta^*$

Chapter 1

Introduction

In the theory of integrable systems, writing a system in Lax pair form is very important for showing the complete integrability of the system. All known examples in the classical theory that can be written in Lax pair form are integrable. As the resulting integration by quadrature is not easy to perform in the most cases, one can alternatively use a Birkhoff decomposition to produce solutions of a system in Lax pair form. As a third approach for matrix systems the geometric data of the system is controlled by the spectral curve. In particular, the coefficients of the spectral curve give invariants of motion of the system, so one can hope to prove complete integrability from the spectral curve.

In [7, 8], Connes-Kreimer discovered a Birkhoff factorization of characters on Kreimer's Hopf algebra of Feynman diagrams. In this thesis, we reverse the usual procedure in integrable systems by producing a Lax pair equation whose solution is given precisely by the Connes-Kreimer Birkhoff factorization (Theorem 4.4.3). The main technical issue, that the Lie algebra of infinitesimal characters is not endowed with an ad-invariant nondegenerate symmetric product, is overcome by passing to the double Lie algebra with the simplest possible Lie algebra structure. In particular, the Lax pair gives a flow for the character φ given by Feynman rules in dimensional regularization. It would be very interesting to know if this flow has physical significance.

The thesis is organized as following: Chapters 2 and 3 are background material on the two main fields of the thesis: Hopf algebras in QFT and integrable systems associated to Lie algebras. In Chapter 2, we briefly present the notions of Hopf algebras, the Kreimer Hopf algebra of Feynman graphs, and its group of characters. At the end of the second chapter we introduce the Connes-Kreimer Birkhoff decomposition. In this chapter we follow the references [7, 8, 18, 19].

Chapter 3 covers background material on integrable system on Lie algebras. In particular, we introduce Poisson structures, integrable systems, Poisson-Lie structures and Lie bialgebras, and give the equivalence between the category of Lie bialgebras and connected and simply connected Poisson-Lie groups. We also discuss a specific example of the Toda lattice and discuss the associated spectral curve. This chapter uses the references [2, 4, 20, 23].

In Chapter 4, we introduce a method to produce a Lax pair on any Lie algebra from the equations of motion on the double Lie algebra. In Section 4.4, we apply this method to the particular case of the Lie algebra of infinitesimal characters of the Hopf algebra of Feynman diagrams, and produce a Lax pair equation whose Birkhoff factorization coincides with the Connes-Kreimer factorization. The main result of Connes-Kreimer factorization and Lax pair equations is Theorem 4.4.3. According to this theorem, we can start with any infinitesimal character and use the Connes-Kreimer factorization to give a flow of infinitesimal characters. By adjusting the initial condition, we can find the Connes-Kreimer factorization of a specific infinitesimal character as part of this flow. We also discuss the flow of the beta function associated to the flow of (exponentiated infinitesimal) characters.

In Chapter 5, we work out an explicit example of the theory on a finitely generated subalgebra of the Hopf algebra of Feynman diagrams. We discuss how this example can be generalized to many other finitely generated Hopf algebras; the only constraint is the amount of available computing power. We also discuss the spectral curve technique for our example.

It is natural to look for invariants of Lax pair equations by spectral curve techniques, and to linearize the flow on the Jacobian of the spectral curve. Unfortunately, in the worked example of Chapter 5, the spectral curve is highly reducible, and the only invariants we find are trivial. We hope to find examples with nontrivial invariants in the future.

Chapters 4 and 5 contain the results from [3].

Appendices A, B, C, D and E contain the Mathematica files with some comments to support the results from Chapter 5.

Chapter 2

A Hopf algebra of Feynman diagrams

In this chapter we recall some basic definitions and notation and present a Hopf algebra of Feynman diagrams. Closely following the presentation from [19] and the ideas from [7, 8], we introduce the Birkhoff decomposition for a connected filtered Hopf algebra $\mathcal{H}$ of a character $\varphi : \mathcal{H} \to \mathcal{A}$ where $\mathcal{A}$ is a unital algebra that admits a renormalization scheme, i.e. a splitting into two subalgebras $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$ with $1 \in \mathcal{A}_+$.

2.1 Hopf algebras

Let k be a field. A k-vector space H with an associative bilinear map $\mu : H \otimes H \to H$ is called a *k-algebra*. Associativity is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} H & \xleftarrow[\mu]{} & H \otimes H \\ {\scriptstyle \mu}\uparrow & & \uparrow{\scriptstyle \mathrm{id}\otimes\mu} \\ H \otimes H & \xleftarrow[\mu\otimes\mathrm{id}]{} & H \otimes H \otimes H \end{array}$$

The k-algebra is called *unital* if it has a unit 1. We denote by $\eta : k \to H$ the map given by $\eta(c) = c \cdot 1$ for any $c \in k$. We shall assume that all our algebras are unital and morphisms between two unital algebras are unital.

Definition 2.1.1. A coalgebra is a triple (H, Δ, ε), where H is a vector space and $\varepsilon : H \to k$ and $\Delta : H \to H \otimes H$ are linear maps, such that the following diagrams commute.

Coassociativity

$$\begin{array}{ccc} H & \xrightarrow{\Delta} & H\otimes H \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes\Delta} \\ H\otimes H & \xrightarrow{\Delta\otimes \mathrm{id}} & H\otimes H\otimes H \end{array}$$

Counity

$$\begin{array}{ccccc} k\otimes H & \xleftarrow{\varepsilon\otimes\mathrm{id}} & H\otimes H & \xrightarrow{\mathrm{id}\otimes\varepsilon} & H\otimes k \\ & {\scriptstyle\simeq}\nwarrow & \uparrow{\scriptstyle\Delta} & \nearrow{\scriptstyle\simeq} & \\ & & H & & \end{array}$$

Definition 2.1.2. $(H, \mu, \eta, \Delta, \varepsilon)$ is called a bialgebra if

i) (H, μ, η) is an algebra,

ii) (H, Δ, ε) is a coalgebra and

iii) Δ and ε are morphisms of algebras.

We recall Sweedler's sigma notation:

$$\Delta(x) = \sum_{(x)} x' \otimes x''.$$

Let $(H, \mu, \eta, \Delta, \varepsilon)$ be a bialgebra. We introduce the following convolution on $\mathcal{L}(H)$, the set of linear maps from H to $\mathbb{C}$:

$$(f \star g)(x) = \sum_{(x)} f(x')g(x''),$$

for $f, g \in \mathcal{L}(H)$.

Definition 2.1.3. Let $(H, \mu, \eta, \delta, \varepsilon)$ be a bialgebra. A linear map $S : H \to H$ is called an *antipode* of H if

$$S \star \mathrm{id}_H = \mathrm{id}_H \star S = \eta \circ \varepsilon.$$

A bialgebra $(H, \mu, \eta, \delta, \varepsilon)$ endowed with an antipode is called a *Hopf algebra.*

We introduce the notion of graded bialgebra.

Definition 2.1.4. Let k be a field of zero characteristic. A *graded bialgebra* on k is a graded k-vector space

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$$

which is endowed with a bialgebra structure and satisfies the following:

$$\mathcal{H}_p \cdot \mathcal{H}_q \subset \mathcal{H}_{p+q}$$

$$\Delta(\mathcal{H}_n) \subset \bigoplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q.$$

A *graded Hopf algebra* is a graded bialgebra $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ endowed with an antipode $S : \mathcal{H} \to \mathcal{H}$ such that

$$S(\mathcal{H}_n) \subset \mathcal{H}_n.$$

For a graded bialgebra one can construct a filtration

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \cdots \subset \mathcal{H}^n \subset \mathcal{H}^{n+1} \subset \cdots ,$$

by setting

$$\mathcal{H}^n = \bigoplus_{p=0}^{n} \mathcal{H}_p.$$

Definition 2.1.5. A graded bialgebra is called *connected* if $\mathcal{H}_0$ is one-dimensional.

Proposition 2.1.6. *Any connected filtered bialgebra $\mathcal{H}$ is a filtrated Hopf algebra. The antipode is given by S(1)=1 and the recursive formula*

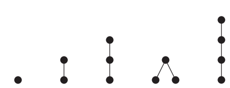
$$S(x) = -x - \sum_{(x)} S(x')x''.$$

2.2 The Hopf algebra of rooted trees

Definition 2.2.1. A *rooted (non-planar) tree* t is a connected and simply connected one dimensional simplicial complex with a point base *(t), which is called the root of t. We denote by V(t) the set of vertices and by E(t) the set of edges, each of which will be assumed oriented.

The convention for drawing the rooted trees is to put the root in the uppermost position.

Example 1. The following are examples of rooted trees:



Definition 2.2.2. The set of trees is denoted by $\mathcal{T}$. The empty tree is consider a tree and will be denoted by $1_{\mathcal{T}}$. Any finite subset of $\mathcal{T}$ is called a *forest.* The set of all forests is denoted by $\mathcal{F}(\mathcal{T})$.

In what follows we shall introduce a Hopf algebra structure on the set on $\mathcal{F}(\mathcal{T})$. First we define the multiplication and the unit.

Definition 2.2.3.

$$m : \mathcal{F}(\mathcal{T}) \otimes \mathcal{F}(\mathcal{T}) \to \mathcal{F}(\mathcal{T}),$$

$$m(t_{i_1} \cdots t_{i_n} \otimes t_{j_1} \cdots t_{j_m}) = t_{i_1} \cdots t_{i_n} t_{j_1} \cdots t_{j_m}$$

where $t_{i_1}\cdots t_{i_n}, t_{j_1}\cdots t_{j_m} \in \mathcal{F}(\mathcal{T})$. $1_{\mathcal{T}}$ is defined to be the unit element in $\mathcal{F}(\mathcal{T})$.

To define a grading on $\mathcal{F}(\mathcal{T})$ we set the degree $\deg(t)$ of a tree t to be the number of vertices and the degree of a forest $t_1\cdots t_n$ is given by

$$\deg(t_1\cdots t_n) = \sum_1^n \deg(t_i),$$

the number of vertices of the forest.

The algebra generated by $\mathcal{F}(\mathcal{T})$ with the natural multiplication given above is also denoted by $\mathcal{F}(\mathcal{T})$. $\mathcal{F}(\mathcal{T})$ is a graded commutative algebra. Notice that $\mathcal{F}(\mathcal{T})$ is an algebra freely generated by $\mathcal{T}$.

To define the comultiplication on $\mathcal{F}(\mathcal{T})$, we need to define the notion of an admissible cut of a tree.

Definition 2.2.4. Let t be a tree. An *admissible cut* is a subset c of $E(t)$ with at least one element and such that any unique path from the root to any vertex of t one has at most one element in c. Removing the elements of c from $E(t)$, we split t into several several connected components. The connected component containing the root will be denoted by $R_c(t)$, while the set of all the other connected components not containing the root will be denoted by $P_c(t)$. Notice that $P_c(t)$ is in general a forest, while $R_c(t)$ is a tree. Let C_t be the set of all admissible cuts of t. Notice that the empty and the full cuts are excluded, (i.e. we don't allow $P_c(t) = t$ or $R_c(t) = \emptyset$).

Now, we define the coalgebra structure on $\mathcal{F}(\mathcal{T})$.

Definition 2.2.5. We consider the following maps:

1) the counity: $\varepsilon : \mathcal{F}(\mathcal{T}) \to k$, $\varepsilon(x) = 0$ for any $x \in \mathcal{F}(\mathcal{T}) \setminus \{1_{\mathcal{T}}\}$ and $\varepsilon(1_{\mathcal{T}}) = 1$.

2) the comultiplication: $\Delta : \mathcal{F}(\mathcal{T}) \to \mathcal{F}(\mathcal{T}) \otimes \mathcal{F}(\mathcal{T})$ is given on generators by

$$\Delta(1_{\mathcal{T}}) = 1_{\mathcal{T}} \otimes 1_{\mathcal{T}}, \quad \Delta(t) = t \otimes 1_{\mathcal{T}} + 1_{\mathcal{T}} \otimes t + \sum_{c \in C_t} P_c(t) \otimes R_c(t),$$

Proposition 2.2.6. *$(\mathcal{F}(\mathcal{T}), m, 1_{\mathcal{T}}, \Delta, \varepsilon)$ is a bialgebra.*

Notice that

$$\deg(t) = \deg(\mathrm{P_c(t)}) + \deg(\mathrm{R_c(t)}),$$

and this implies that the bialgebra $(\mathcal{F}(\mathcal{T}), m, 1_{\mathcal{T}}, \Delta, \varepsilon)$ is graded.

By Proposition 2.1.6, the grading of the bialgebra gives recursively the antipode:

$$S(t) = -t - \sum_{c \in C_t} S(P_c(t)) R_c(t)).$$

Notice that the only tree of degree zero is $1_{\mathcal{T}}$, so the bialgebra of rooted trees is connected. Summarizing, we state the following result.

Theorem 2.2.7. *$(\mathcal{F}(\mathcal{T}), m, 1_{\mathcal{T}}, \Delta, \varepsilon)$ is a graded connected Hopf algebra.*

Now we given some computations for Δ and S.

Example 2.

$$\Delta(\bullet) = \bullet \otimes 1_{\mathcal{T}} + 1_{\mathcal{T}} \otimes \bullet, \qquad S(\bullet) = -\bullet$$

$$\Delta(\overset{\bullet}{\underset{\bullet}{|}}) = \overset{\bullet}{\underset{\bullet}{|}} \otimes 1_{\mathcal{T}} + 1_{\mathcal{T}} \otimes \overset{\bullet}{\underset{\bullet}{|}} + \bullet \otimes \bullet, \qquad S(\overset{\bullet}{\underset{\bullet}{|}}) = -\overset{\bullet}{\underset{\bullet}{|}} + \bullet\bullet$$

$$\Delta(\overset{\bullet}{\bullet\wedge\bullet}) = \overset{\bullet}{\bullet\wedge\bullet} \otimes 1_{\mathcal{T}} + 1_{\mathcal{T}} \otimes \overset{\bullet}{\bullet\wedge\bullet} + 2\bullet \otimes \overset{\bullet}{\underset{\bullet}{|}} + \bullet\bullet \otimes \bullet, \qquad S(\overset{\bullet}{\bullet\wedge\bullet}) = -\overset{\bullet}{\bullet\wedge\bullet} + 2\bullet\overset{\bullet}{\underset{\bullet}{|}} - \bullet\bullet\bullet$$

$$\Delta(\bullet\bullet) = \Delta(\bullet)\Delta(\bullet) = \bullet\bullet \otimes 1_{\mathcal{T}} + 1_{\mathcal{T}} \otimes \bullet\bullet + 2\bullet \otimes \bullet$$

The construction in this section can be extended to the Hopf algebra of decorated rooted trees. In the next section we introduce the Hopf algebra of 1PI Feynman graphs. Since any Feynman diagram has an associated decorated rooted tree the Feynman graph Hopf algebra can be considered as a Hopf algebra of decorated rooted trees.

2.3 The Hopf algebra of Feynman graphs

In this section we recall the construction of Hopf algebras of Feynman graphs, based on [18, 19].

Definition 2.3.1. A *Feynman graph (diagram)* is a non-oriented, non-planar graph with a finite number of vertices and edges. An internal edge is an edge connected to both ends to a vertex. An external edge is an edge with one open end and with the other end connected to a vertex.

To construct a Hopf algebra of Feynman graphs we consider the set of 1PI graphs.

Definition 2.3.2. A *one-particle irreducible graph* (1PI graph) consists of edges and vertices, without self-loops, such that the graph remains connected upon removal of any one edge. Its set of vertices is denoted by $\Gamma^{[0]}$ and set of edges by $\Gamma^{[1]}$. The set of internal edges is denoted by $\Gamma^{[1]}_{int}$ and the set of external edges by $\Gamma^{[1]}_{ext}$.

Definition 2.3.3. A *Feynman subgraph* of a 1PI graph Γ is defined to be a graph γ with $\gamma^{[1]} \subset \Gamma^{[1]}$ and containing all vertices adjacent to $\gamma^{[1]}$.

Definition 2.3.4. The *residue* of a connected graph is the graph obtained by shrinking all internal edges and vertices to a point (i.e. the set $\Gamma^{[0]} \cup \Gamma^{[1]}_{int}$ is replaced by a point).

For any Feynman subgraph γ of Γ, we define Γ/γ to be the contracted graph obtained by replacing all connected components of γ with their residues inside Γ.

Definition 2.3.5. For any connected graph Γ with $V(\Gamma)$ vertices and with $I(\Gamma)$ internal edges, the *loop number* is given by

$$L(\Gamma) = I(\Gamma) - V(\Gamma) + 1.$$

Definition 2.3.6 ([21, p. 309])**.** The *superficial degree of divergence* of graph Γ is $dL - 2I$, where d is the dimension of the configuration space, L is the loop number of Γ and I is the number of internal edges of Γ. We say Γ is *superficially divergent* if the superficial degree of divergence is positive.

Definition 2.3.7. Let $\mathcal{H}$ be the algebra generated by 1PI graphs. The multiplication of two elements in $\mathcal{H}$ is given by the disjoint union, the unit element 1 is the empty set $\emptyset$, and the sum is formal addition. Notice that $\mathcal{H}$ is a commutative algebra.

Let $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ be the comultiplication given on any generator Γ with $L(\Gamma) > 0$ by

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma$$

where the sum is over all unions of superficially divergent 1PI proper subgraphs γ, and $\Delta(1) = 1 \otimes 1$.

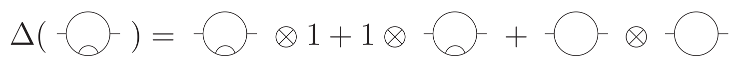
Let $\varepsilon : \mathcal{H} \to k$ be the linear map given by $\varepsilon(\Gamma) = 0$ for any $\Gamma \neq 1$ and $\varepsilon(1) = 1$.

Proposition 2.3.8. *$\mathcal{H}$ defined above is a connected commutative Hopf algebra. The antipode $S : \mathcal{H} \to \mathcal{H}$ is given by $S(1) = 1$ and*

$$S(\Gamma) = -\Gamma - \sum_{\gamma} S(\gamma)\Gamma/\gamma$$

for any $\Gamma \neq 1$. Here the sum is over all unions of 1PI superficially divergent proper subgraphs γ.

Example 3.



Following [7], we shall discuss various gradings and filtrations of $\mathcal{H}$. To construct gradings on $\mathcal{H}$, we start by associating to any 1PI graph Γ, an integer $n(\Gamma)$ and then naturally extend the function n to the entire Hopf algebra $\mathcal{H}$ is given by

$$\deg(\Gamma_1 \dots \Gamma_l) = \sum_{i=1}^{l} n(\Gamma_i), \quad \deg(1) = 0.$$

We are particularly interested in a grading that is compatible with the coproduct, i.e.

$$\deg(\gamma) + \deg(\Gamma/\gamma) = \deg(\Gamma). \tag{2.3.1}$$

This will give a grading and in consequence a filtration on the Hopf algebra.

Proposition 2.3.9. *The following three gradings satisfy the compatibility condition* (2.3.1)*:*

$$I(\Gamma) = \textit{number of internal edges of } \Gamma,$$

$$v(\Gamma) = V(\Gamma) - 1 = \textit{number of vertices of } \Gamma - 1,$$

$$L(\Gamma) = I(\Gamma) - v(\Gamma) = I(\Gamma) - V(\Gamma) + 1,$$

$L(\Gamma)$ *is the loop number.*

In all following chapters, we shall consider the Hopf algebra of 1PI Feynman graphs $\mathcal{H}$ graded with respect to the loop number $L(\Gamma)$.

Proposition 2.3.10. $\mathcal{H}$ *defined above is a connected filtrated commutative Hopf algebra.*

We have the following property, which is also valid for any graded connected Hopf algebra:

Proposition 2.3.11. *If $\mathcal{H}$ is the Hopf algebra of 1PI Feynman graphs then for any homogeneous element $\Gamma \in \mathcal{H}_n$ we have*

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{(\Gamma)} \Gamma' \otimes \Gamma'',$$

where Γ', Γ'' are homogeneous elements of degree less than n.

2.4 The group of characters and the Connes-Kreimer Birkhoff decomposition

Let $H = (H, 1, \mu, \Delta, \varepsilon, S)$ be a Hopf algebra over $\mathbb{C}$ with unit element 1.

Definition 2.4.1. The *character group* $G = \mathrm{Char}(H)$ of a Hopf algebra H is given by

$$G = \{\phi : H \to \mathbb{C} \mid \phi \text{ is a linear map}, \phi(1) = 1, \phi(xy) = \phi(x)\phi(y) \text{ for any } x, y \in H\}.$$

The group law is given by the convolution product and the unit element is ε:

$$(\psi_1 \star \psi_2)(h) = \langle \psi_1 \otimes \psi_2, \Delta h \rangle,$$

$$\varepsilon(1) = 1,\ \varepsilon(h) = 0 \text{ for any } h \in H \setminus \{0\}.$$

The inverse of an element $\varphi \in \mathrm{Char}(H)$ is given by

$$\varphi^{-1}(x) \quad = \quad \varphi(S(x)) = \varphi(-x - \sum_{(x), x' \neq x} S(x')x'')$$

$$= \quad -\varphi(x) - \sum_{(x),x'\neq x} \varphi^{-1}(x')\varphi(x'').$$

Definition 2.4.2. An **infinitesimal character** of a Hopf algebra H is a $\mathbb{C}$-linear map $Z : H \to \mathbb{C}$ satisfying

$$\langle Z, hk\rangle = \langle Z, h\rangle\varepsilon(k) + \varepsilon(h)\langle Z, k\rangle.$$

The set of infinitesimal characters is denoted by $\partial\mathrm{Char}(H)$ and is endowed with a Lie algebra bracket:

$$[Z, Z'] = Z \star Z' - Z' \star Z, \quad \text{for } Z,\ Z' \in \partial\mathrm{Char}(H).$$

Note that $\mathfrak{g} = \partial\mathrm{Char}(H)$ is the Lie algebra of $\mathrm{Char}(H)$ and that for any infinitesimal character Z we have $Z(1) = 0$.

Let $\mathcal{A}$ be an algebra that admits a renormalization scheme i.e. a splitting into two subalgebras:

$$\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+,$$

with $1 \in \mathcal{A}_+$.

Example 4. We can take $\mathcal{A}$ to be the algebra of Laurent series over $\mathbb{C}$

$$\mathcal{A} = \{\sum_{i=m}^{\infty} c_i\lambda^i \mid m \in \mathbb{Z},\ c_i \in \mathbb{C}\},$$

$$\mathcal{A}_- = \{\sum_{i=m}^{-1} c_i\lambda^i \mid m \in \mathbb{Z},\ c_i \in \mathbb{C}\},$$

$$\mathcal{A}_+ = \{\sum_{i=0}^{\infty} c_i\lambda^i \mid m \in \mathbb{Z},\ c_i \in \mathbb{C}\}.$$

Notice that $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$.

Similar to the definitions of G and $\mathfrak{g}$, we define $G_{\mathcal{A}}$ to be the group of characters given by

$$G_{\mathcal{A}} = \{\phi : H \to \mathcal{A} \,|\, \phi \text{ is a linear map}, \phi(1) = 1, \phi(xy) = \phi(x)\phi(y) \text{ for any } x, y \in H\}.$$

and Lie algebra of infinitesimal characters

$$\mathfrak{g}_{\mathcal{A}} = \{Z : H \to \mathcal{A} \mid Z \text{ is a linear map, } \langle Z, hk\rangle = \langle Z, h\rangle\varepsilon(k) + \varepsilon(h)\langle Z, k\rangle\}.$$

Let $\pi : \mathcal{A} \to \mathcal{A}_-$ be the projection onto $\mathcal{A}_-$, let $P_- = -\pi$. Let $P_+ : \mathcal{A} \to \mathcal{A}_+$ be the projection onto $\mathcal{A}_+$. Notice that $P_+ = \mathrm{id} + P_-$.

Definition 2.4.3. A map $\pi : \mathcal{A} \to \mathcal{A}$ with $\pi^2 = \pi$ is said to satisfy the *Rota-Baxter equation* if

$$\pi(x)\pi(y) + \pi(xy) = \pi(\pi(x)y + x\pi(y)),$$

for any $x, y \in \mathcal{A}$.

Example 5. If $\mathcal{A}$ is the algebra of Laurent series from Example 4, then the projection $\pi : \mathcal{A} \to \mathcal{A}_-$ satisfies the Rota-Baxter equation.

Theorem 2.4.4 ([19]). *Let $\mathcal{H}$ be a connected graded Hopf algebra and let $\mathcal{A}$ be an algebra with a splitting $\mathcal{A} = \mathcal{A}_- \oplus \mathcal{A}_+$, $1 \in \mathcal{A}_+$. Then any $\varphi \in G_{\mathcal{A}}$ admits a unique Birkhoff decomposition:*

$$\varphi = \varphi_-^{-1} \star \varphi_+,$$

where $\varphi_-(1) = 1_{\mathcal{A}}$, $\varphi_-(\mathrm{Ker}\ \varepsilon) \subset \mathcal{A}_-$ and $\varphi_+(\mathcal{H}) \subset \mathcal{A}_+$.

Moreover φ_- and φ_+ are given by

$$\varphi_-(x) = -\pi(\bar{R}(x)),$$

$$\varphi_+(x) = \bar{R}(x) - \pi(\bar{R}(x)),$$

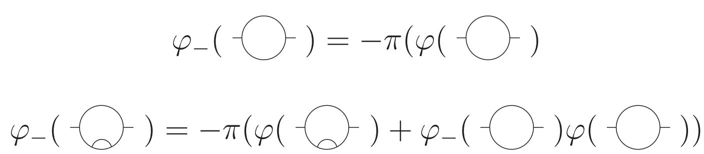
where $\bar{R}$ is the Bogoliubov-Parasiuk-Hepp preparation map given by

$$\bar{R}(x) = \varphi(x) + \sum_{(x)} \varphi_-(x')\varphi(x'')$$

If the algebra $\mathcal{A}$ is commutative and π satisfies the Rota-Baxter equation, e.g. Example 4 of Laurent series, then both φ_- and φ_+ are characters.

The Hopf algebra of rooted trees and the Hopf algebra of 1PI Feynman graphs are the most important examples of connected graded Hopf algebras. By Theorem 2.4.4, one gets the Connes-Kreimer Birkhoff decomposition in [7]. In fact the proof of Theorem 2.4.4 in [19] follows the ideas from [7].

Example 6.



2.5 The β-function

Following [8, 11, 19], we shall introduce the β-function of a character φ. In a later section, we shall find relations between β-functions and our Lax pair equations. Everywhere in this section, $\mathcal{A}$ will denote the algebra of Laurent series (given in Example 4).

Let $\mathcal{H} = \bigoplus_n \mathcal{H}_n$ be a connected graded Hopf algebra. Let Y be a biderivation on $\mathcal{H}$ given on homogeneous elements by

$$Y : \mathcal{H}_n \to \mathcal{H}_n, \qquad Y(x) = nx \quad \text{for } x \in \mathcal{H}_n.$$

Notice that $\varphi \mapsto \varphi \circ Y$ is a derivation of $G_{\mathcal{A}}$.

Let $\{\theta_t\}_{t\in\mathbb{C}}$ be the one-parameter group of $\mathcal{H}$ given by

$$\theta_t(x) = e^{nt}x, \text{ for } x \in \mathcal{H}_n.$$

Then $\varphi \mapsto \varphi \circ \theta_t$ is an automorphism of $G_{\mathcal{A}}$. Now, we define a different action of $\mathbb{C}$ on $G_{\mathcal{A}}$. For $t \in \mathbb{C}$ and $\varphi \in G_{\mathcal{A}}$ we define $\varphi^t(x)$ on an homogeneous element x by

$$\varphi^t(x)(\lambda) = e^{t\lambda|x|}\varphi(x)(\lambda),$$

for any $\lambda \in \mathbb{C}$, where $|x|$ is the degree of $x \in \mathcal{H}$. Let

$$G^{\Phi}_{\mathcal{A}} = \{\varphi \in G_{\mathcal{A}} \mid \frac{d}{dt}(\varphi^t)_- = 0\},$$

be the group of characters with the negative part of Birkhoff decomposition independent of t. The dimensional regularized Feynman rule character φ is in $G^{\Phi}_{\mathcal{A}}$. Referring to [8, 11], the physical meaning is that the counter term φ_- does not depend on the mass parameter μ, i.e. $\frac{\partial \varphi_-}{\partial \mu} = 0$.

Proposition 2.5.1. *Let $\varphi \in G^{\Phi}_{\mathcal{A}}$ and let $h_t = \varphi^{-1} \star \varphi^t$. Then the following limit*

$$F_t(x) = \lim_{\lambda\to 0} h_t(x)(\lambda)$$

exists and it is a one-parameter subgroup in $G_{\mathcal{A}} \cap G$ of scalar valued characters of $\mathcal{H}$.

Notice that $h_t(x) \in \mathcal{A}_+$ as $h_t = \varphi_+^{-1} \star \varphi_- \star (\varphi^t)_-^{-1} \star (\varphi^t)_+ = \varphi_+^{-1} \star (\varphi^t)_+$.

Definition 2.5.2. For any $\varphi \in G_{\mathcal{A}}^{\Phi}$, the beta-function of φ is defined to be

$$\beta(\varphi) = \frac{d}{dt}\Big|_{t=0} F_t(x)$$

for any $x \in \mathcal{H}$.

Using the Connes-Kreimer scattering formula, we show that φ_- can be given in terms of its residue.

Definition 2.5.3. We define $\tilde{R} : G_{\mathcal{A}} \to \mathfrak{g}_{\mathcal{A}}$ by

$$\tilde{R}(\varphi) = \varphi^{-1} \star (\varphi \circ Y).$$

Notice that $\tilde{R}$ is well defined. Indeed

$$\begin{aligned}
\tilde{R}(xy) &= \varphi^{-1}(x'y')\varphi(x''y'')(|x''| + |y''|) \\
&= \varphi^{-1}(x')\varphi(x'')|x''|\varepsilon(y) + \varphi^{-1}(y')\varphi(y'')|y''|\varepsilon(x) = \tilde{R}(x)\varepsilon(y) + \tilde{R}(y)\varepsilon(x).
\end{aligned}$$

Let $\tilde{\mathfrak{g}}_{\mathcal{A}}$ be the semidirect product

$$\tilde{\mathfrak{g}}_{\mathcal{A}} = \mathfrak{g}_{\mathcal{A}} \rtimes \mathbb{C}.Z_0,$$

where the action of $\mathbb{C}.Z_0$ on $\mathfrak{g}_{\mathcal{A}}$ is given by $Z_0(X) = X \circ Y$.

Let $\tilde{G}_{\mathcal{A}}$ be the semidirect product

$$\tilde{G}_{\mathcal{A}} = G_{\mathcal{A}} \rtimes \mathbb{C},$$

with the action of $\mathbb{C}$ on $G_{\mathcal{A}}$ given by $\varphi.t = \varphi \circ \theta_t$. $G_{\mathcal{A}}$ has Lie algebra $\mathfrak{g}_{\mathcal{A}}$.

Notice that $\tilde{R}$ is bijective and its inverse is given by the following theorem.

Theorem 2.5.4 ([19]). *Let $X \in \mathfrak{g}_{\mathcal{A}}$. Then*

$$\exp(-tZ_0)\exp(t(Z_0+X)) \in G_{\mathcal{A}} \quad \text{for any } t \in \mathbb{R}, \tag{2.5.1}$$

where $\exp : \tilde{\mathfrak{g}}_{\mathcal{A}} \to \tilde{G}_{\mathcal{A}}$ *is the exponential of $\tilde{\mathfrak{g}}$. The following limit exists and we have*

$$\tilde{R}^{-1}(X) = \lim_{t\to\infty} \exp(-tZ_0)\exp(t(Z_0+X)) \tag{2.5.2}$$

Let $G^{\Phi}_{\mathcal{A}_-} = \{\varphi \in G^{\Phi}_{\mathcal{A}} \mid \varphi(\mathrm{Ker}\ \varepsilon) \subset \mathcal{A}_-\}$.

By [11], we have various formulas for the beta-function:

$$\beta(\varphi) = \mathrm{Res}\ \tilde{R}(\varphi) = \mathrm{Res}\ \tilde{R}(\varphi_-^{-1}) = \mathrm{Res}(\varphi_-^{-1} \circ Y).$$

On the other hand, for any $\psi \in G^{\Phi}_{\mathcal{A}_-}$ we have (see [19])

$$\tilde{R}(\psi) = \frac{1}{\lambda}\mathrm{Res}(\psi \circ Y).$$

Then $\varphi_-^{-1}(\lambda) = \tilde{R}^{-1}(\frac{1}{\lambda}\mathrm{Res}(\varphi_-^{-1} \circ Y)) = \tilde{R}^{-1}(\frac{\beta}{\lambda})$. Therefore by Theorem 2.5.4 we get the Connes-Kreimer scattering formula:

Theorem 2.5.5. *If $\mathcal{H}$ is a connected graded Hopf algebra and $\varphi \in G^{\Phi}_{\mathcal{A}}$, then*

$$\varphi_-(\lambda) = \lim_{t\to\infty} \exp(-t(Z_0 + \frac{\beta}{\lambda}))\exp(tZ_0)$$

Thus the beta function encodes the "divergent" piece piece of φ_- of the character φ. In particular if $\beta = 0$ then φ_- is trivial. φ_- is determined by its residue, namely we have the following result.

Corollary 2.5.6. *If* $\varphi \in G_{\mathcal{A}}^{\Phi}$*, then*

$$\varphi_-(\lambda) = \lim_{t\to\infty} \exp\left(-t(Z_0 - \frac{\text{Res }(\varphi_- \circ Y)}{\lambda})\right) \exp(tZ_0)$$

2.6 The exponential map of $\mathfrak{g}_{\mathcal{A}}$

Let $\mathcal{H}$ be a connected graded Hopf algebra. Let

$$\bar{\mathfrak{g}}_{\mathcal{A}} = \{Z : \mathcal{H} \to \mathbb{C} \mid Z \text{ is linear, } Z(1) = 0\}$$

and

$$\bar{G}_{\mathcal{A}} = \{\varphi : \mathcal{H} \to \mathbb{C} \mid \varphi \text{ is linear, } \varphi(1) = 1\}$$

Let $\exp : \bar{\mathfrak{g}}_{\mathcal{A}} \to \bar{G}_{\mathcal{A}}$ be the exponential map of $\bar{\mathfrak{g}}_{\mathcal{A}}$.

Theorem 2.6.1 ([19])**.** *The exponential* $\exp : \bar{\mathfrak{g}}_{\mathcal{A}} \to \bar{G}_{\mathcal{A}}$ *is bijective,*

$$\exp(Z) = \sum_{n=0}^{\infty} \frac{1}{n!} Z^n,$$

and the inverse $\log : \bar{G}_{\mathcal{A}} \to \bar{\mathfrak{g}}_{\mathcal{A}}$ *is given by*

$$\log(1+Z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} Z^n.$$

Recall that Manchon's proof in [19, p. 32] uses the fact that the Hopf algebra is filtrated which implies that both the exponential and logarithm series evaluated on a given element are in fact finite sums.

This implies the following result which will be used in the main Theorem 4.4.3.

Corollary 2.6.2 ([19, p. 35])**.** *The exponential* $\exp : \mathfrak{g}_{\mathcal{A}} \to G_{\mathcal{A}}$ *is bijective and it is the restriction of the exponential on* $\bar{\mathfrak{g}}_{\mathcal{A}}$*.*

Chapter 3

Poisson-Lie structures and Lie bialgebra structures

In this chapter we review background material which will be needed for the results in the next chapter. We present the concepts of Poisson, Poisson-Lie and Lie bialgebra structures and discuss the relations among them. A Lie bialgebra is an infinitesimal analogue of a Poisson-Lie structure.

We also discuss integrable systems and an example the generalized open Toda lattice of a semisimple Lie algebra. We sketch how affine loop algebras of semisimple Lie algebras can be treated analogously to the semisimple case. The references for this chapter are the books by Adler, van Moerbeke & Vanhaecke [1], Babelon, Bernard & Talon [2], Chari & Pressley [4] and Suris [23] and the survey paper by Reyman and Semenov-Tian-Shansky [20]. In section §3.4 we briefly discuss the spectral curve.

3.1 Poisson structures

Definition 3.1.1. Let M be a smooth manifold. A *Poisson bracket* (or *Poisson structure*) on M is a bilinear operation $\{\cdot,\cdot\}$ on the set $C^\infty(M)$ of smooth functions on M which satisfies the following properties:

1. Skew-symmetry:
$$\{\varphi_1, \varphi_2\} = -\{\varphi_2, \varphi_1\}$$
for $\varphi_1, \varphi_2 \in C^\infty(M)$;

2. Jacobi identity

$$\{\varphi_1, \{\varphi_2, \varphi_3\}\} + \{\varphi_2, \{\varphi_3, \varphi_1\}\} + \{\varphi_3, \{\varphi_1, \varphi_2\}\} = 0$$

for $\varphi_1, \varphi_2, \varphi_3 \in C^\infty(M)$;

3. Leibniz rule:

$$\{\varphi_1, \varphi_2\varphi_3\} = \{\varphi_1, \varphi_2\}\varphi_3 + \{\varphi_1, \varphi_3\}\varphi_2$$

for $\varphi_1, \varphi_2, \varphi_3 \in C^\infty(M)$.

$(M, \{\cdot, \cdot\})$ is called a *Poisson manifold.*

Definition 3.1.2. A smooth map $F : (M, \{\cdot, \cdot\}_M) \to (N, \{\cdot, \cdot\}_N)$ between two Poisson manifolds is called a *Poisson map* if

$$\{\varphi_1, \varphi_2\}_N \circ F = \{\varphi_1 \circ F, \varphi_2 \circ F\}_M$$

for $\varphi_1, \varphi_2 \in C^\infty(N)$.

In local coordinates $(x_1, x_2, \cdots, x_n)$, the Poisson bracket can be written as

$$\{\varphi_1, \varphi_2\}(x) = \sum_{i,j=1}^{n} c_{ij}(x) \frac{\partial \varphi_1}{\partial x_i} \frac{\partial \varphi_2}{\partial x_j},$$

where $c_{ij}(x) = \{x_i, x_j\}$.

Let W be the skew-symmetric 2-tensor given by

$$W_x = \sum_{i,j=1}^{n} c_{ij}(x) \frac{\partial}{\partial x_i} \otimes \frac{\partial}{\partial x_j};$$

W is independent of local coordinates. W is called the *Poisson bivector.* Notice that $\{\varphi_1, \varphi_2\} = \langle d\varphi_1 \otimes d\varphi_2, W\rangle$.

Example 7. Let (M, ω) be a symplectic manifold. Then the bracket given by

$$\{\varphi_1, \varphi_2\} = X_{\varphi_1}(\varphi_2),$$

for $\varphi_1, \varphi_2 \in C^\infty(M)$, is a Poisson structure. Here X_{φ_1} is the vector field given by $i_{X_{\varphi_1}}\omega = d\varphi_1$.

Example 8. Let $(M_1, \{\cdot,\cdot\}_{M_1})$, $(M_2, \{\cdot,\cdot\}_{M_2})$ be two Poisson manifolds. The *product Poisson structure* $\{\cdot,\cdot\}_{M_1\times M_2}$ is given by

$$\{\varphi_1, \varphi_2\}_{M_1\times M_2}(x_1, x_2) = \{\varphi_1(\cdot, x_2), \varphi_2(\cdot, x_2)\}_{M_1}(x_1) + \{\varphi_1(x_1, \cdot), \varphi_2(x_1, \cdot)\}_{M_2}(x_2)$$

Definition 3.1.3. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold and let $f : M \to \mathbb{R}$ be a smooth function on M. X_f is called the *Hamiltonian* vector field of f if

$$X_f(g) = \{f, g\}$$

for all $g \in C^\infty(M)$. f is called the *Hamiltonian function* of X_f. The flow $\varphi_t : M \to M$ of X_f is called the *Hamiltonian flow* of f.

Note that any function on a symplectic manifold is Hamiltonian.

In local coordinates $(x_1, \cdots, x_n)$, the Hamiltonian vector field X_f of f is given by

$$X_f(x) = \sum_{i,j}^{n} c_{ij}(x)\frac{\partial f}{\partial x_i}\frac{\partial}{\partial x_j}.$$

Let $B : T^*M \to TM$ be the map given by

$$B(df) = X_f.$$

Notice this map is well defined.

If the Poisson structure $\{\cdot,\cdot\}$ is non-degenerate, then B is bijective and $\omega = (B^{-1} \otimes B^{-1})(W)$ is a closed two-form on M. The Jacobi identity for $\{\cdot,\cdot\}$ translates into $d\omega = 0$. Conversely a symplectic structure on M gives a non-degenerate Poisson structure on M (see Example 7). We call a non-degenerate Poisson structure *symplectic*.

Definition 3.1.4. Let $(M, \{\cdot,\cdot\})$ be a symplectic structure on a manifold M of (real) dimension $2n$. Let H be a Hamiltonian function (i.e. a function on M). The Hamiltonian system

$$\dot{F} = \{H, F\}$$

is called *completely (Liouville) integrable* if it has n independent conserved quantities $F_i : M \to \mathbb{R}$ (i.e. $\{H, F_i\} = 0$) that are in involution (i.e. $\{F_i, F_j\} = 0$). Here independent means that $dF_1, \ldots, dF_n$ are linearly independent 1-forms everywhere except possibly on a set of measure zero.

In the next section we give a more general definition and state the Arnorld-Liouville Theorem.

3.2 The Kirillov bracket on $\mathfrak{g}^*$ and Lax pair equation

In this section we introduce a natural Poisson bracket on the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$. We also show that the equations of motion on a Lie algebra with an ad-invariant non-degenerate bilinear form can be put in Lax pair form.

For $F \in C^\infty(\mathfrak{g}^*)$ and $L \in \mathfrak{g}^*$ we define $\nabla F(L) \in \mathfrak{g}$ as follows:

$$\langle \nabla F(L), X \rangle = \frac{d(F(L + \varepsilon X))}{d\varepsilon}\Big|_{\varepsilon=0}$$

for any $X \in \mathfrak{g}^*$, where $\langle \cdot,\cdot \rangle$ is the natural pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$. Notice that

$dF(L) = \nabla F(L)$ via the natural identification $\mathfrak{g}^{**} = \mathfrak{g}$.

Definition 3.2.1. For any two functions $F, G \in C^\infty(\mathfrak{g}^*)$ and $L \in \mathfrak{g}^*$ we define

$$\{F, G\}(L) = \langle L, [\nabla F(L), \nabla G(L)] \rangle.$$

Then $\{\cdot, \cdot\}$ is a Poisson bracket on $\mathfrak{g}^*$, called the *Kirillov (Lie-Poisson) bracket.*

The equations of motion $\dot{F} = \{H, F\}$ of a Hamiltonian function H with respect to the Kirillov bracket can be written as

$$\dot{F} = \mathrm{ad}^*(\nabla H(L))(L),$$

which we will write as $\dot{F} = \mathrm{ad}^* \nabla H(L) \cdot L$. Here ad^* is the *coadjoint representation* given by $\mathrm{ad}^*_X(Y^*)(Z) = -Y^*(\mathrm{ad}_X(Z))$ for $X, Z \in \mathfrak{g}$ and $Y \in \mathfrak{g}^*$.

Definition 3.2.2. A *Casimir function* on $\mathfrak{g}^*$ is a function $C : \mathfrak{g}^* \to \mathbb{R}$ satisfying the following identity

$$\mathrm{ad}^* C(L) \cdot L = 0.$$

The equations of motion are trivial for a Casimir function.

Proposition 3.2.3. *An Ad^*-invariant function is a Casimir function.*

If $\mathfrak{g}$ is endowed with an ad-invariant non-degenerate symmetric bilinear form, then we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$ and ad^* with $-\mathrm{ad}$, so the Hamiltonian equation becomes a Lax pair equation

$$\dot{L} = [L, \nabla H(L)].$$

3.3 Integrable Systems: Liouville Integrability

Definition 3.3.1. Let $(M, \{\cdot, \cdot\})$ be a Poisson manifold of rank $2r$ and let $F = (F_1, \ldots, F_s)$ be in involution (i.e. $\{F_i, F_j\} = 0$) and independent (i.e. $dF_1, \ldots, dF_n$ are linearly independent on a dense set), with $s = \dim M - r$. We say that F is *completely integrable* and that $(M, \{\cdot, \cdot\}, F)$ is an *integrable system* or *a completely integrable system*. The vector fields X_{F_i} are called *integrable vector fields* and the map F is called the *momentum map*. r is the *degree of freedom* of the integrable system and $2r$ is its *rank*.

Let X_{F_i} be the Hamiltonian vector field corresponding to F_i. Let $\mathcal{D}$ the distribution generated by $\{X_{F_1}, \ldots X_{F_s}\}$. The maximal integral manifold F'_m of $\mathcal{D}$ through m is called the *invariant manifold* of F through m.

Now we state the Arnorld-Liouville Theorem for real integrable systems. The Arnorld-Liouville Theorem is considered a good motivation for introducing the previous definition.

Theorem 3.3.2 (Arnorld-Liouville)**.** *Let $(M, \{\cdot, \cdot\}, F)$ be a real integrable system of rank $2r$, where $F = (F_1, \ldots, F_s)$. Let $m \in M$ such that $dF_1, \ldots, dF_s$ are linearly independent at m and F'_m be the invariant manifold of F that passes through m.*

1) If F'_m is compact then there exists a diffeomorphism from F'_m from F_m to the torus $T^r = (\mathbb{R}/\mathbb{Z})^r$, under which the vector fields $X_{F_1}, \ldots X_{F_s}$ are mapped to linear vector fields

2) If F'_m is not compact but the flow of each X_{F_i} is complete on F'_m then there exists a diffeomorphism from F'_m to $\mathbb{R}^{r-q} \times T^q$, $(0 \leq q < r)$, under which the vector fields $X_{F_1}, \ldots X_{F_s}$ are mapped to linear vector fields.

We will specially be interested in non-degenerate Poisson structures.

Now we present the generalized open Toda lattice associated to an arbitrary semisimple Lie algebra following [20].

Example 9. Let $\mathfrak{g}$ be a semi-simple Lie algebra, $\mathfrak{a}$ its Cartan subalgebra, Δ its root system, $P \subset \Delta$ the set of simple roots. Let g_α the corresponding root space of $\alpha \in \Delta$. Let $\mathfrak{g}$ be the root decomposition of $\mathfrak{g} = \mathfrak{a} + \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$. If $\beta = \sum_{\alpha \in P} k_\alpha \alpha$ we denote by $d(\beta) = \sum k_\alpha$.

Let $\mathfrak{g}_i = \bigoplus_{d(\alpha)=i} \mathfrak{g}_\alpha$ if $i \neq 0$ and $\mathfrak{g}_0 = \mathfrak{a}$. Let $\mathfrak{g}_+ = \bigoplus_{i \geq 0}$ and $\mathfrak{g}_- = \bigoplus_{i<0}$.

We denote by $(\cdot, \cdot)$ the Killing form of $\mathfrak{g}$. Set

$$H(X) = \frac{1}{2}(X, X)$$

and $f = \sum_{\alpha \in P} e_{-\alpha}$, where e_α is a root vector. Let $\mathcal{O}_f$ be the $\mathfrak{g}_+$-orbit of f in $\mathfrak{a} + \mathfrak{g}_{-1}$. Then

$$\mathcal{O}_f = \{p + \sum_{\alpha \in P} c_{-\alpha} e_\alpha,\ p \in \mathfrak{a}\},$$

where e_α is a root vector.

The Lie-Poisson bracket is given by $\{p_\beta, c_\alpha\} = (\alpha, \beta) c_\alpha$, $\{c_\alpha, c_\beta\} = \{p_\alpha, p_\beta\} = 0$.

A parametrization of $\mathcal{O}_f$ is given by

$$\xi = \sum_{i=1}^{l} p_i h_i + \sum_{\alpha \in P} \exp(\sum_{i=1}^{l} q_i (\alpha, g_i)).$$

Here $\{h_i\}$ is a basis of $\mathfrak{a}$, $\{g_i\}$ is its dual basis with respect to the Killing form i.e. $(h_i, g_i) = \delta_{i,j}$. Let $q_i = (q, h_i)$ and $p_i = (p, g_i)$. Let $\mathcal{O} = \mathcal{O}_f + e$, where $e = \sum_{\alpha \in P} e_\alpha$.

Computing the previous Hamiltonian for an element $L \in \mathcal{O}$ we get the Toda lattice Hamiltonian

$$H(L) = \frac{1}{2}(p, p) + \sum_{\alpha \in P} (e_{-\alpha}, e_\alpha) e^{(\alpha, q)}.$$

When $\{h_i\}$ is a orthogonal basis in $\mathfrak{a}$, the Toda lattice Hamiltonian is

$$H(L) = \frac{1}{2}\sum p_i^2 + \sum_{\alpha\in P} \exp(\sum_i q_i(\alpha, e_i)).$$

The Lax pair equation of the Toda lattice is:

$$\frac{dL}{dt} = [L, M_{\pm}],$$

where

$$L = p + \sum_{\alpha\in P} e^{(\alpha,q)} e_{-\alpha} + e,$$

$$M_+ = p + e, \quad M_- = M_+ - L.$$

Notice that the generalized Toda lattice Hamiltonian is Liouville (completely) integrable.

The ordinary nonperiodic Toda lattice is system of n interacting particles on a line with exponential interactions, the Hamiltonian is given by:

$$H = \frac{1}{2}\sum_{j=1}^{n} p_j^2 + \sum_{j=1}^{n-1} g_j^2 \exp[(2(q_j - q_{j+2})].$$

The equation of motions when we set $g_j = 1$ are:

$$\dot{q}_j = p_j, \quad j = 1, \ldots, n,$$

$$\dot{p}_1 = -2\exp[2(q_1 - q_2)], \quad \dot{p}_n = 2\exp[2(q_{n-1} - q_n)],$$

$$\dot{p}_j = -2\exp[2(q_j - q_{j-1})] + 2\exp[2(q_{j-1} - q_j)], \; j = 2, \ldots, n-1.$$

Affine Lie algebras

The periodic Toda lattice of a semisimple Lie algebra can written as a Lax pair equation on an affine Lie algebras. The theory of semisimple Lie algebras extends to the affine Lie algebras.

Let $\mathfrak{g}$ be a semisimple Lie algebra. Let $\mathfrak{a}$ be a Cartan Lie subalgebra. Let $\mathcal{L}(\mathfrak{g})$ be the loop algebra of $\mathfrak{g}$, i.e.

$$\mathcal{L}(\mathfrak{g}) = \{\sum_{i=M}^{N} L_i\lambda^i \mid M, N \in \mathbb{Z},\ L_i \in \mathfrak{g}\}$$

is the algebra of Laurent polynomials in λ with coefficients in $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{a} + \bigoplus_{\alpha\in\Delta} \mathfrak{g}_\alpha$ be the root decomposition of $\mathfrak{g}$. We have

$$\mathcal{L}(\mathfrak{g}) = \bigoplus_{i\in\mathbb{Z}} \mathfrak{a}\lambda^i + \bigoplus_{\alpha\in\Delta,\ i\in\mathbb{Z}} \mathfrak{g}_\alpha\lambda^i.$$

To keep track of the power of λ we need to add an extra element d to the loop algebra. This element is the analogue of the grading operator added to the infinitesimal Lie algebra $\mathfrak{g}_{\mathcal{A}}$ introduced in Chapter 2.

Set

$$[d, L_i\lambda^i] = iL_i\lambda^i.$$

Let $\hat{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) + \mathbb{C}d$, and let $\hat{\mathfrak{a}} = \mathfrak{a} + \mathbb{C}d$ be the extend Cartan subalgebra. Then

$$\hat{\mathfrak{g}} = \hat{\mathfrak{a}} + \bigoplus_{a\in\hat{\Delta}} \mathfrak{g}_\alpha,$$

where $\hat{\Delta}$ given below is called the affine root system:

$$\hat{\Delta} = \{(\alpha, i) \mid \alpha \in \Delta \cup \{0\}, i \in \mathbb{Z}, (\alpha, i) \neq (0, 0)\}.$$

Let

$$\Delta_+ = \{(\alpha, i \mid \text{either } i > 0 \text{ or } \alpha \in \Delta_+, i = 0\}.$$

The affine root systems are given in terms of (affine) Dynkin diagrams and these affine Dynkin diagrams are classified for all simple $\mathfrak{g}$. The periodic Toda lattice is a good example of an affine Lie algebra and the construction is analogous to the open Toda lattice discussed earlier.

3.4 Spectral curve

The presentation in this section follows [20] and [1]. Let $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Let $L(\lambda) = \sum x_i \lambda^i \in L\mathfrak{g}$. The *spectral curve* associated to $L(\lambda)$ is the algebraic curve

$$\Gamma_0 = \{(\lambda, \nu) \in \mathbb{C} \setminus \{0\} \times \mathbb{C} \mid \det(L(\lambda) - \nu \mathrm{Id}) = 0\}.$$

Assume $L(\lambda)$ has a simple spectrum for generic λ. For each nonsingular, non-branching point $p \in \Gamma_0$ we have a one-dimensional eigenspace $E(p) \subset \mathbb{C}^n$ associated to the eigenvalue $\nu(p)$. The disjoint union of $E(p)$ over such p give a holomorphic line bundle E over $\Gamma_0 \setminus \{\text{nonsingular points, non-branching points}\}$. Let Γ be a nonsingular compact model of Γ_0. One can extend the line bundle E to a line bundle over Γ which will be denoted also by E.

The evolution given by an equation of motion for a Casimir element on $\mathfrak{g}$ induces a flow of the line bundle E. Considered as a flow on the Jacobian of Γ this flow is linear. Thus spectral curve theory is an algebraic geometry analogue of the Arnord-Liouville Theorem and in good cases the spectral curve method "solves" the integrable system.

Now we discuss Lax pairs with parameter and a simple and efficient way of constructing some constants of motion.

Definition 3.4.1. Let M be a finite-dimensional affine subspace of the loop algebra $L(\mathfrak{g})$. *A Lax pair with parameter* λ on M is given by a differential equation on M of the form

$$\frac{d}{dt}X(\lambda) = [X(\lambda), Y(\lambda)]$$

where the coefficients Y_i of Y are polynomial function of the coefficients X_i of X.

Proposition 3.4.2. *Let* $\frac{d}{dt}X(\lambda) = [X(\lambda), Y(\lambda)]$ *be a Lax pair with parameter. Then the coefficients of the characteristic polynomial (spectral curve) are constants of motion of the Lax pair equation. Moreover the curve*

$$\Gamma_X = \{(\lambda, \nu) \mid \det(X(\lambda) - \nu \mathrm{Id}) = 0\}$$

is preserved by the flow.

3.5 Poisson-Lie structures

Definition 3.5.1. Let (G, μ) be a Lie group with multiplication map $\mu : G \times G \to G$, and let $\{\cdot, \cdot\}_G$ be a Poisson structure on G. Then $(G, \mu, \{\cdot, \cdot\})$ is called a *Poisson-Lie group* if $\mu : (G \times G, \{\cdot, \cdot\}_{G\times G}) \to (G, \{\cdot, \cdot\}_G)$ is a Poisson map.

Definition 3.5.2. Let $(G, \mu_G, \{\cdot, \cdot\}_G)$ and $(H, \mu_H, \{\cdot, \cdot\}_H)$ be two Poisson-Lie groups. A smooth map $F : (G, \mu_G, \{\cdot, \cdot\}_G) \to (H, \mu_H, \{\cdot, \cdot\}_H)$ is called a homomorphism of Poisson-Lie groups if it is a morphism of Lie groups and a Poisson map.

Let $L_g : G \to G$ be left multiplication $L_g(g') = gg'$ and $R_{g'} : G \to G$ be right multiplication $R_{g'}(g) = gg'$. We denote by $d(L_g)_{g'} : T_{g'}G \to T_{gg'}G$ the differential of L_g at g'.

Proposition 3.5.3. *Let* (G, μ) *be a Lie group.* $(G, \mu, \{\ ,\ \})$ *is a Poisson-Lie group* *if and only if its Poisson bivector W satisfies the following relation:*

$$W_{gg'} = (d(L_g)_{g'} \otimes d(L_g)_{g'})(W_{g'}) + (d(R_{g'})_g \otimes d(R_{g'})_g)(W_g). \tag{3.5.1}$$

Remark 3.5.4. Let G be a Poisson-Lie group with Lie algebra $\mathfrak{g}$. Then the Poisson structure induces a Lie algebra structure on $\mathfrak{g}^*$:

$$[U_1, U_2]_{\mathfrak{g}^*} = (d\{\varphi_1, \varphi_2\})_e$$

for any $U_1, U_2 \in \mathfrak{g}^*$ and with $(d\varphi_1)_e = U_1$ and $(d\varphi_2)_e = U_2$. Here e is the unit element of G.

Remark 3.5.5. Let $W^R : G \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ be given by

$$W^R(g) = (d(R_{g^{-1}})_{g^{-1}} \otimes d(R_{g^{-1}})_{g^{-1}})(W_g).$$

Let $\gamma : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ be the differential of W^R at $g = e$. Then

$$[U_1, U_2]_{\mathfrak{g}^*} = \gamma^*(U_1 \otimes U_2)$$

Applying $(d(R_{(gg')^{-1}})_{(gg')^{-1}} \otimes d(R_{(gg')^{-1}})_{(gg')^{-1}})$ to relation (3.5.1) we get

$$W^R(gg') = (\mathrm{Ad}_g \otimes \mathrm{Ad}_g)(W^R(g')) + W^R(g) \tag{3.5.2}$$

Then taking its derivative at e we get (see [4, p. 25]):

$$\gamma[x, y] = (\mathrm{ad}_x \otimes 1 + 1 \otimes \mathrm{ad}_x)\gamma(y) - (\mathrm{ad}_y \otimes 1 + 1 \otimes \mathrm{ad}_y)\gamma(x) \tag{3.5.3}$$

for any $x, y \in \mathfrak{g}$.

These two remarks motivate introducing the concept of a Lie bialgebra structure below.

3.6 Lie bialgebra structures

We recall the definition of a Lie bialgebra structure.

Definition 3.6.1. A Lie bialgebra is a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ with a linear map $\gamma : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ such that

a) ${}^t\gamma : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ defines a Lie bracket on $\mathfrak{g}^*$,

b) γ is a 1-cocycle of $\mathfrak{g}$, i.e.

$$\mathrm{ad}_x^{(2)}(\gamma(y)) - \mathrm{ad}_y^{(2)}(\gamma(x)) - \gamma([x,y]) = 0,$$

where $\mathrm{ad}_x^{(2)} : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ is given by

$$\mathrm{ad}_x^{(2)}(y \otimes z) = \mathrm{ad}_x(y) \otimes z + y \otimes \mathrm{ad}_x(z) = [x,y] \otimes z + y \otimes [x,z].$$

Definition 3.6.2. Let $(\mathfrak{g}, \gamma_{\mathfrak{g}})$ and $(\mathfrak{h}, \gamma_{\mathfrak{h}})$ be two Lie bialgebras. A homomorphism of Lie algebras $F : \mathfrak{g} \to \mathfrak{h}$ is called a homomorphism of Lie bialgebras if

$$(F \otimes F) \circ \gamma_{\mathfrak{g}} = \gamma_{\mathfrak{h}} \circ F.$$

Remark 3.6.3. A Lie bialgebra $(\mathfrak{g}, [\cdot,\cdot], \gamma)$ induces an Lie algebra structure on the *double Lie algebra* $\mathfrak{g} \oplus \mathfrak{g}^*$ by

$$[X, Y]_{\mathfrak{g}\oplus\mathfrak{g}^*} = [X, Y],$$

$$[X^*, Y^*]_{\mathfrak{g}\oplus\mathfrak{g}^*} = {}^t\gamma(X \otimes Y),$$

$$[X, Y^*] = \mathrm{ad}^*_X(Y^*),$$

for X, $Y \in \mathfrak{g}$ and X^*, $Y^* \in \mathfrak{g}^*$.

The following theorem gives the relation between Poisson-Lie groups and Lie bialgebras. Moreover, we have an equivalence of categories between the category of connected simply-connected Poisson-Lie groups and the category of Lie bialgebras structures.

Theorem 3.6.4 ([4])**.** *Let G be a Lie group with Lie algebra $\mathfrak{g}$.*

i) If G is a Poisson-Lie group, then $\mathfrak{g}$ has a natural Lie bialgebra structure, called the tangent Lie bialgebra of G. A homomorphism of Poisson-Lie groups induces a homomorphism of Lie bialgebras between their corresponding tangent Lie bialgebras.

ii) If G is connected and simply-connected then every Lie bialgebra structure on $\mathfrak{g}$ is the tangent Lie bialgebra of a unique Poisson structure on G which makes G a Poisson-Lie group. A homomorphism of Lie bialgebras induces a homomorphism of Poisson-Lie groups between their corresponding connected simply-connected Lie groups.

The proof of i) follows from the construction in Remarks 3.5.4 and 3.5.5.

Conversely, the connectness and simply-connectness of G imply that the 1-cocycle condition (3.5.3) at the Lie algebra level can be lifted to a 1-cocycle condition (3.5.2) at the Lie group level.

Chapter 4

Main results on Lax pair equations

In this chapter we combine the material from Chapters 2 and 3 to relate the Connes-Kreimer factorization to Lax pair equations. The main result, Theorem 4.4.3, gives a Lax pair equation whose solution is provided by this factorization.

If the Lie algebra of infinitesimal characters were semisimple, this process would be straightforward. There is a well known method to associate a Lax pair equation to a Casimir element on the dual $\mathfrak{g}^*$ of a semisimple Lie algebra $\mathfrak{g}$ [20]. The semisimplicity is used to produce an Ad-invariant, symmetric, non-degenerate bilinear form on $\mathfrak{g}$, allowing an identification of $\mathfrak{g}$ with $\mathfrak{g}^*$.

The Lie algebra of infinitesimal characters is not semisimple. For a general Lie algebra $\mathfrak{g}$, there may be no Ad-invariant, symmetric, non-degenerate bilinear form. To produce a Lax pair, we need to extend $\mathfrak{g}$ to a larger Lie algebra with such a bilinear form. We do this by constructing a Lie bialgebra structure on $\mathfrak{g}$ and extending $\mathfrak{g}$ to $(\mathfrak{g} \oplus \mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}^*})$, where $[\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}^*}$ is the Lie bracket induced by the Lie bialgebra.

Once we have a Lax pair equation for infinitesimal characters, it is natural to exponentiate this flow to a flow of characters and to ask how the beta function changes along the flow. This is discussed in Section 4.5.

4.1 The double Lie algebra and its associated Lie Group

Since it is difficult to construct explicitly the Lie group associated to the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$, we will choose the trivial Lie bialgebra given by the cocycle $\gamma = 0$ and denote by $\delta = \mathfrak{g} \oplus \mathfrak{g}^*$ the associated Lie algebra. Let $\{Y_i, i = 1, \dots, l\}$ be a basis of $\mathfrak{g}$, with dual basis $\{Y_i^*\}$. The Lie bracket $[\cdot, \cdot]_\delta$ on δ is given by

$$[Y_i, Y_j]_\delta = [Y_i, Y_j],\ [Y_i^*, Y_j^*]_\delta = 0,\ [Y_i, Y_j^*]_\delta = -\sum_k c_{ik}^j Y_k^*,$$

where the c_{ik}^j are the structure constants: $[Y_i, Y_j] = \sum_k c_{ij}^k Y_k$.

The main point of this construction is that the natural pairing $\langle \cdot, \cdot \rangle : \delta \otimes \delta \to \mathbb{C}$ given by $\langle Y_i, Y_j^* \rangle = \delta_{ij}$ is an Ad-invariant symmetric non-degenerate bilinear form on δ.

In the case of the trivial Lie bialgebra structure $\gamma = 0$, a Lie group naturally corresponding to the double Lie algebra δ is given by the following proposition.

Proposition 4.1.1. *Let $\theta : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the coadjoint representation $\theta(g, X) = \mathrm{Ad}_G^*(g)(X)$. Then the Lie algebra of the semi-direct product $\tilde{G} = G \ltimes_\theta \mathfrak{g}^*$ is the double Lie algebra δ.*

Proof. The Lie group law on the semi-direct product $\tilde{G}$ is given by

$$(g, h) \cdot (g', h') = (gg', h + \theta_g(h')).$$

Let $\tilde{\mathfrak{g}}$ be the Lie algebra of $\tilde{G}$. Then the bracket on $\tilde{\mathfrak{g}}$ is given by

$$[X, Y^*]_{\tilde{\mathfrak{g}}} = d\theta(X, Y^*), \quad [X, Y]_{\tilde{\mathfrak{g}}} = [X, Y], \quad [X^*, Y^*]_{\tilde{\mathfrak{g}}} = 0,$$

for left-invariant vector fields X, Y of G and $X^*, Y^* \in \mathfrak{g}^*$. We have

$$d\theta(X, Y^*) = d\mathrm{Ad}^*_G(X)(Y^*) = [X, Y^*]_\delta$$

since $d\mathrm{Ad}^*_G = \mathrm{ad}^*_{\mathfrak{g}}$. □

4.2 The loop algebra of a Lie algebra

Following [1], we consider the loop algebra

$$L\delta = \{L(\lambda) = \sum_{j=M}^{N} \lambda^j L_j \mid M, N \in \mathbb{Z}, L_j \in \delta\}.$$

The natural Lie bracket on $L\delta$ is given by

$$\left[\sum \lambda^i L_i, \sum \lambda^j L'_j\right] = \sum_k \lambda^k \sum_{i+j=k} [L_i, L'_j].$$

Set

$$\begin{aligned} L\delta_+ &= \{L(\lambda) = \sum_{j=0}^{N} \lambda^j L_j \mid N \in \mathbb{Z}^+ \cup \{0\}, L_j \in \delta\} \\ L\delta_- &= \{L(\lambda) = \sum_{j=-M}^{-1} \lambda^j L_j \mid M \in \mathbb{Z}^+, L_j \in \delta\}. \end{aligned}$$

Let $P_+ : L\delta \to L\delta_+$ and $P_- : L\delta \to L\delta_-$ be the natural projections and set $R = P_+ - P_-$.

The natural pairing $\langle \cdot, \cdot \rangle$ on δ yields to an Ad-invariant, symmetric, non-degenerate pairing on $L\delta$ by setting

$$\left\langle \sum_{i=M}^{N} \lambda^i L_i, \sum_{j=M'}^{N'} \lambda^j L'_j \right\rangle = \sum_{i+j=-1} \langle L_i, L'_j \rangle.$$

We denote by $L\delta^*$ the loop algebra of δ^* and by $L\delta_- = P_-(L(\delta^*))$. For our choice of basis $\{Y_i\}$ of $\mathfrak{g}$, the pairing induces an isomorphism

$$I : L\delta^* \to L\delta \tag{4.2.1}$$

with

$$I\left(\sum L_i^j Y_j \lambda^i\right) = \sum L_i^j Y_j^* \lambda^{-1-i}.$$

The identification I induces the following identifications:

$$L\delta_+ = L(\delta^*)_- \text{ and } L\delta_- = L(\delta^*)_+.$$

The following lemma gives a procedure to obtain Casimir functions on the loop algebra.

Lemma 4.2.1. *[20, Lemma 4.1] Let φ be an* Ad*-invariant polynomial on δ. Then*

$$\varphi_{m,n}[L(\lambda)] = \operatorname{Res}_{\lambda=0}(\lambda^{-n}\varphi(\lambda^m L(\lambda)))$$

is an Ad*-invariant polynomial on $L\delta$ for $m, n \in \mathbb{Z}$.*

As a double Lie algebra, δ has an Ad-invariant polynomial, the quadratic polynomial

$$\psi(Y) = \langle Y, Y\rangle$$

associated to the natural pairing. Let $Y_{l+i} = Y_i^*$ for $i \in \{1, \ldots, l\}$, so elements of $L\delta$ can be written $L(\lambda) = \sum\limits_{j=1}^{2l} \sum\limits_{i=-M}^{N} L_i^j Y_j \lambda^i$. Then the Ad-invariant polynomials

$$\psi_{m,n}(L(\lambda)) = \operatorname{Res}_{\lambda=0}(\lambda^{-n}\psi(\lambda^m L(\lambda))), \tag{4.2.2}$$

defined as in Lemma 4.2.1 are given by

$$\psi_{m,n}(L(\lambda)) = 2\sum_{j=1}^{l}\sum_{i+k-n+2m=-1} L_i^j L_k^{j+l}. \tag{4.2.3}$$

Note that powers of ψ are also Ad-invariant polynomials on δ, so

$$\psi_{m,n}^k(L(\lambda)) = \mathrm{Res}_{\lambda=0}(\lambda^{-n}\psi^k(\lambda^m L(\lambda))) \tag{4.2.4}$$

are Ad-invariant polynomials on $L\delta$.

It would be interesting to classify all Ad-invariant polynomials on $L\delta$ in general.

4.3 The Lax pair equation

From [20, Theorem 2.1], if we have endomorphisms $P_\pm$ and $R = P_+ - P_-$ on a Lie algebra $\mathfrak{h}$ such that

$$[X, Y]_R = [P_+X, P_+Y] - [P_-X, P_-Y]$$

is a Lie bracket on $\mathfrak{h}$ then the equations of motion induced by a Casimir function φ on the dual of a Lie algebra $\mathfrak{h}$ are given by

$$\frac{dL}{dt} = -\mathrm{ad}_{\mathfrak{h}}^* M \cdot L, \tag{4.3.1}$$

where $L \in \mathfrak{h}^*, M = \frac{1}{2}R(d\varphi(L)) \in \mathfrak{h}$.

Now we take $\mathfrak{h} = (L\delta)^* = L(\delta^*)$. Here $(L\delta)^*$ denotes the graded dual. Let $P_\pm$ be the projections of $L\delta^*$ onto $L\delta_\pm^*$. After identifying $L\delta^* = L\delta$ and $\mathrm{ad}^* = -\mathrm{ad}$ via the map I in (4.2.1), the equations of motion (4.3.1) can be written in Lax pair form

$$\frac{dL}{dt} = [M, L], \tag{4.3.2}$$

where $M = \frac{1}{2}R(I(d\varphi(L(\lambda)))) \in L\delta$, and φ is a Casimir function on $L\delta^* = L\delta$.

Finding a solution for (4.3.2) reduces to the Riemann-Hilbert (or Birkhoff) factorization problem. The following theorem is a corollary of [20, Theorem 2.2].

Theorem 4.3.1. *Let φ be a Casimir function on $L\delta$ and set $X = I(d\varphi(L(\lambda))) \in L\delta$, for $L(\lambda) = L(0)(\lambda) \in L\delta$. Let $g_\pm(t)$ be smooth curves in $L\tilde{G}$ which solve the factorization problem*

$$\exp(-tX) = g_-(t)^{-1} g_+(t),$$

with $g_\pm(0) = e$, and with $g_+(t) = g_+(t)(\lambda)$ holomorphic in $\lambda \in \mathbb{C}$ and $g_-(t) - e$ a polynomial in $1/\lambda$ with no constant term. Here e is the identity element of $L\tilde{G}$. Let $M = \frac{1}{2}R(I(d\varphi(L(\lambda)))) \in L\delta$. Then the integral curve $L(t)$ of the Lax pair equation

$$\frac{dL}{dt} = [M, L]$$

is given by

$$L(t) = \mathrm{Ad}^*_{L\tilde{G}} g_\pm(t) \cdot L(0).$$

Notice that at $t = 0$ we have $g_+(0) = g_-(0) = e$.

This Lax pair equation projects to a Lax pair equation on the loop algebra of the original Lie algebra $\mathfrak{g}$. Let π_1 be either the projection of $\tilde{G}$ onto G or its differential from δ onto $\mathfrak{g}$. This extends to a projection of $L\delta$ onto $L\mathfrak{g}$. The projection of (4.3.2) onto $L\mathfrak{g}$ is

$$\frac{d(\pi_1(L(t))}{dt} = [\pi_1(L), \pi_1(M)], \tag{4.3.3}$$

since $\pi_1 = d\pi_1$ commutes with the bracket. Thus the equations of motion (4.3.2) induce a Lax pair equation on $L\mathfrak{g}$, although this is not the equations of motion for a Casimir on $L\mathfrak{g}$.

Theorem 4.3.2. *The Lax pair equation of Theorem 4.3.1 projects to a Lax pair equation on $L\mathfrak{g}$.*

When $\psi_{m,n}$ is the Casimir function on $L\delta$ given by (4.2.2), X can be written nicely in terms of $L(\lambda)$.

Proposition 4.3.3. *Let $X = I(d\psi_{m,n}(L(\lambda)))$. Then*

$$X = 2\lambda^{-n+2m} L(\lambda). \tag{4.3.4}$$

Proof. Write $L(\lambda) = \sum_{i,j} L_i^j \lambda^i Y_j$. By (4.2.3), we have

$$\frac{\partial \psi_{m,n}}{\partial L_p^t} = \begin{cases} 2L_{n-1-2m-p}^{t+l}, & \text{if } \ t \leq l, \\ 2L_{n-1-2m-p}^{t-l}, & \text{if } \ t > l. \end{cases} \tag{4.3.5}$$

Therefore

$$\begin{aligned} X &= I(d\psi_{m,n}(L(\lambda))) = \sum_{p,t} \frac{\partial \psi_{m,n}}{\partial L_p^t} \lambda^{-1-p} Y_t^* \\ &= 2\lambda^{-n+2m} \sum_p (\sum_{t=1}^{l} L_{n-1-2m-p}^{t+l} Y_{t+l} \lambda^{n-1-2m-p} + \sum_{t=l+1}^{2l} L_{n-1-2m-p}^{t-l} Y_{t-l} \lambda^{n-1-2m-p}) \\ &= 2\lambda^{-n+2m} L(\lambda). \end{aligned}$$

□

4.4 The main theorem for Hopf algebras

In this section we prove the main result Theorem 4.4.3 relating our Lax pair equation with the Connes-Kreimer decomposition.

Let $(H, 1 = \emptyset, \mu, \Delta, \varepsilon, S)$ be a connected graded Hopf algebra, in particular we can take H to be the Hopf algebra Feynman graphs or the Hopf algebra of rooted trees introduced in Chapter 2. In these cases, for later computations in Chapter 5, we also consider a Hopf subalgebra H_1 generated by a finite number of Feynman graphs $A_0 = \emptyset,\ A_1,\ A_2, \ldots, A_l$.

Let G be the Lie group of characters of H, and let G_1 be the Lie group of characters of H_1. The Lie algebra of infinitesimal characters $\mathfrak{g}, \mathfrak{g}_1$ of H, H_1 are precisely the Lie algebras of G, G_1, respectively.

For any $T \in \{A_1, \ldots, A_l\}$, let Z_T be the infinitesimal character given by $Z_T(T') = \delta_{T,T'}$. The Lie algebra $\mathfrak{g}_1$ is generated by $Z_{A_1}, \ldots,\ Z_{A_l}$. Set $Y_i = Z_{A_i}$ for $i \in \{1, \ldots, l\}$. Let G_0 be the semi-direct product $G_1 \rtimes \mathbb{C}$ given by

$$(g, t) \cdot (g', t') = (g \cdot \theta_t(g'), t + t'),$$

where $\theta_t(g)(\Gamma) = e^{t\#(\Gamma)}g(\Gamma)$ for $\Gamma \in H_1$, and $\#(\Gamma)$ is the number of independent loops of Γ. Set $Z_0 = \frac{\partial}{\partial t}\Big|_{t=0}$, so $[Z_0, Z_{A_i}] = \#(A_i)Z_{A_i}$. The Lie algebra $\mathfrak{g}_0$ of G_0 is generated by $Z_0, Z_{A_1}, \ldots,\ Z_{A_l}$.

In the next lemma, $\tilde{G}$ refers either to $G_0 \ltimes_\theta \mathfrak{g}_0^*$ as in Prop. 4.1.1 or to $G \ltimes_\theta \mathfrak{g}^*$.

Lemma 4.4.1. *Let (g, α) be an element in $L\tilde{G}$. If $(g, \alpha) = (g_-, \alpha_-)^{-1}(g_+, \alpha_+)$ then $g = g_-^{-1}g_+$ and $\alpha = \mathrm{Ad}^*(g_-^{-1})(-\alpha_- + \alpha_+)$.*

Proof. We recall that $(g_1, \alpha_1)(g_2, \alpha_2) = (g_1g_2, \alpha_1 + \mathrm{Ad}^*(g_1)(\alpha_2))$. Notice that

$(g_-, \alpha_-)^{-1} = (g_-^{-1}, -\mathrm{Ad}^*(g_-^{-1})(\alpha_-))$, so

$$(g_-, \alpha_-)^{-1}(g_+, \alpha_+) = (g_-^{-1}g_+, -\mathrm{Ad}^*(g_-^{-1})(\alpha_-) + \mathrm{Ad}^*(g_-^{-1})(\alpha_+)).$$

$\square$

We prove the existence of a Birkhoff decomposition for any element $(g, \alpha) \in L\tilde{G}$.

Theorem 4.4.2. *Every $(g, \alpha) \in L\tilde{G}$ has a Birkhoff decomposition*

$$(g, \alpha) = (g_-, \alpha_-)^{-1}(g_+, \alpha_+)$$

with (g_+, α_+) a polynomial in λ and $(g_-, \alpha_-) - (e, 0)$ a polynomial in λ^{-1} without constant term.

Proof. Let $g = g_-^{-1}g_+$ be the Birkhoff decomposition of g in LG given in [9]. Let $\alpha_+ = P_+(\mathrm{Ad}^*(g_-)(\alpha))$ and $\alpha_- = -P_-(\mathrm{Ad}^*(g_-)(\alpha))$. Then, by Lemma 4.4.1, $(g, \alpha) = (g_-, \alpha_-)^{-1}(g_+, \alpha_+)$. $\square$

In [7], Connes and Kreimer give a Birkhoff decomposition for the character group of the Feynman graph Hopf algebra, and in particular for the normalized loop character $\bar{\varphi}(\lambda, q)$ of dimensional regularization.

Here

$$\bar{\varphi}(\lambda, q) = \frac{\varphi(\lambda, q)}{q^2},$$

where $\varphi(\lambda, q)$ is the usual character given by dimensional regularization and Feynman rules. We consider the algebra of Laurent series with coefficients in δ:

$$\Omega\delta = \{L(\lambda) = \sum_{j=-M}^{\infty} \lambda^j L_j \mid L_j \in \delta, M \in \mathbb{Z}_+\}.$$

The natural Lie bracket on $\Omega\delta$ is

$$\left[\sum \lambda^i L_i, \sum \lambda^j L'_j\right] = \sum_k \lambda^k \sum_{i+j=k} [L_i, L'_j].$$

Set

$$\begin{aligned}\Omega\delta_+ &= \{L(\lambda) = \sum_{j=0}^{\infty} \lambda^j L_j \mid L_j \in \delta\} \\ \Omega\delta_- &= \{L(\lambda) = \sum_{j=-M}^{-1} \lambda^j L_j \mid L_j \in \delta, M \in \mathbb{Z}_+\}.\end{aligned}$$

Recall that π_1 denotes either the projection of the double Lie group $\tilde{G}$ to its first factor G, its differential, or its extension to the loop group and loop algebra. We denote the image of an element by adding a tilde, e.g. $\pi_1(L(\lambda)) = \tilde{L}(\lambda)$.

Theorem 4.4.3. *Let H be a connected graded Hopf algebra, e.g. the Hopf algebra of 1PI Feynman graphs (or the Hopf algebra of rooted trees). Let $\psi_{m,n}$ be the Casimir function on $\Omega\delta$ given by*

$$\psi_{m,n}(L(\lambda)) = \mathrm{Res}_{\lambda=0}(\lambda^m \langle \lambda^n L(\lambda), \lambda^n L(\lambda)\rangle).$$

For $L_0(\lambda) \in \Omega\mathfrak{g}$, set $X = I(d\psi_{m,n}(L_0(\lambda)))$. Then the solution of

$$\frac{dL}{dt} = [M, L], \quad M = \frac{1}{2} R(I(d\psi_{m,n}(L(\lambda)))) \tag{4.4.1}$$

with initial condition $L(0) = L_0$ is given by

$$L(t) = \mathrm{Ad}^* g_\pm(t) \cdot L_0, \tag{4.4.2}$$

where $\exp(-tX)$ has the Connes-Kreimer Birkhoff factorization

$\exp(-tX) = g_-(t)^{-1}g_+(t)$. *The same results hold for any finitely generated Hopf subalgebra* H_1 *of* H.

Proof. By Proposition 4.3.3 we have

$$X = I(d\psi_{m,n}(L_0(\lambda))) = 2\lambda^{-n+2m}L_0(\lambda).$$

Since $2\lambda^{-n+2m}L_0(\lambda) \in \Omega\mathfrak{g} = \mathfrak{g}_{\mathcal{A}}$, we get $\exp(-tX) \in G_{\mathcal{A}}$ and therefore by Proposition 2.4.4 there exists a unique Connes-Kreimer Birkhoff decomposition of $\exp(-tX) \in G_{\mathcal{A}}$. The theorem then follows from Theorem 4.3.1 and (4.3.3) applied to the natural pairing on δ and the uniqueness of the Birkhoff factorization. □

Remark 4.4.4. a) In this theorem, the initial infinitesimal character is arbitrary. We can find the Birkhoff factorization of the Feynman rule character (or any fixed character) $\bar{\varphi}$ itself within this framework by adjusting the initial condition. Namely, set

$$L_0(\lambda) = \frac{1}{2}\lambda^{n-2m}\exp^{-1}(\bar{\varphi}(\lambda)).$$

Since the exponential is bijective $\exp^{-1}(\bar{\varphi}(\lambda))$ is well defined and $L_0 \in \Omega\mathfrak{g}$. Then $\exp(X) = \bar{\varphi}$ by Proposition 4.3.3, so the solution of (4.4.1) involves the Birkhoff factorization of $\bar{\varphi}(\lambda)$:

$$\bar{\varphi} = g_-(-1)^{-1}g_+(-1).$$

b) As a special case, if $-n+2m = 0$, then

$$L_0 = \frac{1}{2}\exp^{-1}(\bar{\varphi}).$$

This gives a Lax pair flow of (half) of $\exp^{-1}(\bar{\varphi})$ with solution determined by the Birkhoff factorization of $\exp(-t\exp^{-1}(\bar{\varphi}))$. In particular, at time $t = -1$, the Birkhoff factorization of $\bar{\varphi}$ solves the flow: $L_{t=-1} = \mathrm{Ad}^* g_{\pm}(-1)\exp^{-1}(\bar{\varphi})$.

c) In the special case $-n+2m=0$ to get the flow of $\exp^{-1}\bar{\varphi}$, we can set

$$L_0 = \exp^{-1}\bar{\varphi}.$$

Then $\exp(-tX) = \exp(-2t\exp^{-1}(\bar{\varphi}))$. In particular at $t=-\frac{1}{2}$, the Birkhoff factorization of $\bar{\varphi}$ solves the flow: $L_{t=-\frac{1}{2}} = \mathrm{Ad}^* g_\pm(-\frac{1}{2})\exp^{-1}(\bar{\varphi})$.

d) We can also replace G by $G_0 = G \rtimes \mathbb{C}$ in Theorem 4.4.3.

Remark 4.4.5. It would interesting to know whether there exists a bigger connected graded Hopf algebra $\tilde{\mathcal{H}}$ having the original Hopf algebra $\mathcal{H}$ as a Hopf subalgebra and whose infinitesimal Lie algebra is the double δ, associated to a connected graded Hopf algebra, in particular for the Kreimer Hopf algebra of 1PI Feynman graphs or the Hopf algebra of rooted trees. This would provide a Lax pair equation, which comes from an equation of motion, on the infinitesimal Lie algebra of $\tilde{\mathcal{H}}$. The most natural candidate, the Drinfeld double $\mathcal{D}(\mathcal{H})$ of $\mathcal{H}$, does not work since the dimension of the Lie algebra associated to $\mathcal{D}(\mathcal{H})$ is larger than the dimension of δ.

4.5 The β-function

In this section we get relations between our Lax pair equations and the β-function. In fact, we consider two flows for β-function. First, we extend the (scalar) beta function to a meromorphic function of the character. Under the condition that the minus part of the meromorphic beta function is independent of loop scaling, the meromorphic beta function is an infinitesimal character (Lemma 4.5.1), and we can use it as an initial condition for a Lax pair flow. For certain Casimir elements, we show that the meromorphic beta function is a fixed point of the flow (Theorem 4.5.3).

It is more natural to consider a second flow for the beta function. Namely, given the Lax pair flow $\phi(s)$ of infinitesimal characters, we have a corresponding curve of characters $\psi(s) = \exp(\phi(s))$. We would like to understand the corresponding beta functions $\beta_{\psi(s)}$. In Theorem 4.5.5, we give a differential equation for $\beta_{\psi(s)}$.

We recall from Section 2.5 the definition of $\beta(\varphi)$-function of a character $\varphi \in G_{\mathcal{A}}^{\Phi}$, where $\mathcal{A}$ is the algebra of Laurent series:

$$\beta(\varphi)(x) = \frac{d}{dt}\Big|_{t=0} \lim_{\lambda \to 0} (\varphi^{-1} \star \varphi^{t})(x)(\lambda),$$

where $\varphi^t(x) = e^{t\lambda|x|}\varphi(x)$. To relate the β-function to Lax pair equations, we need an element in the loop algebra. For $\varphi \in G_{\mathcal{A}}^{\Phi}$, set

$$\tilde{\beta}_{\varphi}(x)(\lambda) = \frac{dh_t}{dt}\Big|_{t=0} = \frac{d}{dt}\Big|_{t=0} (\varphi^{-1} \star \varphi^{t})(x)(\lambda).$$

The following lemma establishes that $\tilde{\beta}$ is an infinitesimal character. Later we shall consider the flow associated to our Lax pair.

Lemma 4.5.1. *If $\varphi \in G_{\mathcal{A}}^{\Phi}$, then i) $\tilde{\beta}_{\varphi}$ is an infinitesimal character in $\mathfrak{g}_{\mathcal{A}}$.*
ii) $\tilde{\beta}_{\varphi}$ is holomorphic (i.e. $\tilde{\beta}_{\varphi}(x) \in \mathcal{A}_{+}$).

Proof. i) For two homogeneous elements $x, y \in \mathcal{H}$, we have:

$$\varphi^t(xy) = e^{t|xy|\lambda}\varphi(xy) = e^{t|x|\lambda}\varphi(x)e^{t|y|\lambda}\varphi(y) = \varphi^t(x)\varphi^t(y).$$

Therefore $\varphi \star \varphi^t \in G_{\mathcal{A}}$. Since $\varphi^{-1} \star \varphi^0 = e$ we get

$$\frac{d}{dt}\Big|_{t=0} \varphi^{-1} \star \varphi^{t} \in \mathfrak{g}_{\mathcal{A}}.$$

ii) Since $\frac{d}{dt}(\varphi^t)_{-} = 0$, we get

$$\tilde{\beta}_{\varphi} = (\varphi_{+})^{-1} \star \varphi_{-} \star ((\varphi^t)_{-})^{-1} \star (\varphi^t)_{+} = (\varphi_{+})^{-1} \star (\varphi^t)_{+}$$

Then

$$\tilde{\beta}_\varphi(x) = (\varphi_+)^{-1}(x')(\varphi^t)_+(x'') = (\varphi_+)(S(x'))(\varphi^t)_+(x'')$$

Therefore $\tilde{\beta}_\varphi(x) \in \mathcal{A}_+$. $\square$

Now we can apply Theorem 4.4.3 for $L_0 = \tilde{\beta}_\varphi$.

Proposition 4.5.2. *If $\varphi \in G^\Phi_\mathcal{A}$ then the Lax pair equation in Theorem 4.4.3 for $L_0 = \tilde{\beta}_\varphi$ is*

$$\frac{d}{ds}\tilde{\beta}_\varphi(s) = [M, \tilde{\beta}_\varphi(s)],$$

where $M = \frac{1}{2}R(\lambda^{n-2m}\tilde{\beta}_\varphi(s))$ and the solution is given by

$$\tilde{\beta}_\varphi(s) = \mathrm{Ad}^*(g_+(s))\tilde{\beta}_\varphi(0)$$

where $g_\pm(s)$ are given by the Birkhoff decomposition

$$\exp(s\lambda^{n-2m}\tilde{\beta}_\varphi) = g_-^{-1}(s) \star g_+(s).$$

The next theorem shows that the β-function is a fixed point of the Lax pair flow for certain Casimirs.

Theorem 4.5.3. *$\tilde{\beta}_\varphi$ and therefore $\beta_\varphi = \tilde{\beta}_\varphi\Big|_{\lambda=0}$ are constant under the Lax flow if $n - 2m \geq 0$.*

Proof. If $n - 2m \geq 0$ then

$$M = R(\frac{1}{2}\lambda^{n-2m}\tilde{\beta}_\varphi) = \frac{1}{2}\lambda^{n-2m}\tilde{\beta}_\varphi,$$

since $\tilde{\beta}_\varphi$ is holomorphic by Lemma 4.5.1. So the Lax pair equation becomes

$$\frac{d}{ds}\tilde{\beta}_\varphi = [\frac{1}{2}\lambda^{n-2m}\tilde{\beta}_\varphi, \tilde{\beta}_\varphi] = \frac{1}{2}\lambda^{n-2m}[\tilde{\beta}_\varphi, \tilde{\beta}_\varphi] = 0.$$

□

Now, we consider the more interesting case of the flow $\beta_{\psi(s)}$ for the beta function of exponentiated infinitesimal characters, as defined in the beginning of this section.

We first establish some simple and useful properties of φ^t that which be used to characterize the second flow involving the β-function.

Lemma 4.5.4. *Let $\varphi \in G_{\mathcal{A}}$. Then*

1) $(\varphi \star \psi)^t = \varphi^t \star \psi^t$,

2) $(\varphi^{-1})^t = (\varphi^t)^{-1}$,

Proof. We have

$$\begin{aligned}
(\varphi \star \psi)^t(x) &= e^{t|x|\lambda}(\varphi \star \psi)(x) = \sum_{(x)} e^{t|x|\lambda}\varphi(x')\psi(x'') \\
&= \sum_{(x)} e^{t(|x'|+|x''|)\lambda}\varphi(x')\psi(x'') = e^{t|x'|\lambda}\varphi(x')\ e^{t|x''|\lambda}\psi(x'') = \varphi^t(x')\psi^t(x'') \\
&= (\varphi^t \star \psi^t)(x).
\end{aligned}$$

Therefore

$$\varphi^t \star (\varphi^{-1})^t = (\varphi \star \varphi^{-1})^t = \varepsilon^t = \varepsilon = \varphi^t \star (\varphi^t)^{-1},$$

so $(\varphi^{-1})^t = (\varphi^t)^{-1}$. □

In the next theorem we study the beta-function $\beta_{\varphi(s)}$ of the flow of characters $\varphi(s) = \exp(\psi(s))$, with $\psi(0) = \log \varphi$.

Theorem 4.5.5. *Let* $\varphi \in G^{\Phi}_{\mathcal{A}}$. *Let* $\psi = \log \varphi$ *and*

$$\dot{\psi} = [M, \psi]$$

be the Lax pair from Theorem 4.4.3 and $\psi(0) = \psi$. *Let* $\varphi(s) = \exp(\psi(s))$. *For*

$$\tilde{\beta}_{\varphi(s)} = \frac{d}{dt}\Big|_{t=0} \varphi(s)^{-1} \star (\varphi(s))^t,$$

we have

$$\frac{d}{ds}\tilde{\beta}_{\varphi(s)} = [\tilde{\beta}_{\varphi(s)}, \varphi^{-1} \star d\exp[M, \log\varphi]] + \lambda(\varphi^{-1} \star d\exp[M, \log\varphi]) \circ Y.$$

Moreover if we assume that $\varphi(s) \in G^{\Phi}_{\mathcal{A}}$ *for every* s *(so the* β*-function of* $\varphi(s)$ *is well defined for each* s*), then*

$$\frac{d}{ds}\beta_{\varphi(s)} = [\tilde{\beta}_{\varphi(s)}, \varphi^{-1}(s) \star d\exp[M, \log\varphi]]_+\Big|_{\lambda=0} + \mathrm{Res}\ \left((\varphi^{-1}(s) \star d\exp[M, \log\varphi]) \circ Y\right).$$

Proof. We have

$$\begin{aligned}
\frac{d}{ds}\tilde{\beta}_{\varphi(s)}(x) &= \frac{d}{ds}\frac{d}{dt}\Big|_{t=0}(\varphi^{-1}(s) \star \varphi^t(s))(x) \\
&= \frac{d}{dt}\Big|_{t=0}(-\varphi^{-1}(s)\frac{d}{ds}\exp\psi(s)\varphi^{-1}(s)\varphi^t(s) + \varphi^{-1}(s)(\frac{d}{ds}\exp\psi(s))^t)(x) \\
&= \frac{d}{dt}\Big|_{t=0}(-\varphi^{-1}(s)d\exp\dot{\psi}(s)\varphi^{-1}\varphi^t(s) + \varphi^{-1}(s)(d\exp\dot{\psi}(s))^t)(x) \\
&= (-\varphi^{-1}(s)d\exp[M, \psi]\tilde{\beta}_{\varphi(s)} + \frac{d}{dt}\Big|_{t=0}\varphi^{-1}(s)(d\exp[M, \psi(s)])^t)(x).
\end{aligned} \tag{4.5.1}$$

The second term in (4.5.1) is

$$\begin{aligned}
&\tfrac{d}{dt}\Big|_{t=0}\left(\varphi^{-1}(s)(d\exp[M,\psi(s)])^t\right)(x)\\
= \quad & \frac{d}{dt}\Big|_{t=0}\left(\varphi^{-1}(s)\varphi(s)^t(\varphi(s)^t)^{-1}(d\exp[M,\psi(s)])^t\right)(x)\\
= \quad & \frac{d}{dt}\Big|_{t=0}\left(\varphi^{-1}(s)\varphi(s)^t\right)\ \left((\varphi(s)^t)^{-1}(d\exp[M,\psi(s)])^t\right)\Big|_{t=0}(x)\\
& +\left(\varphi^{-1}(s)\varphi(s)^t\right)\Big|_{t=0}\ \frac{d}{dt}\Big|_{t=0}\left((\varphi(s)^{-1})^t(d\exp[M,\psi(s)])^t\right)(x)\\
= \quad & \left(\tilde{\beta}_{\varphi(s)}\star(\varphi(s)^{-1}d\exp[M,\psi(s)])\right)(x)+\frac{d}{dt}\Big|_{t=0}\left((\varphi(s)^{-1}(d\exp[M,\psi(s)]))^t\right)(x)\\
= \quad & \left(\tilde{\beta}_{\varphi(s)}\star(\varphi(s)^{-1}d\exp[M,\psi(s)])\right)(x)+|x|\lambda\left(\varphi(s)^{-1}(d\exp[M,\psi(s)])\right)(x)\\
= \quad & \left(\tilde{\beta}_{\varphi(s)}\star(\varphi(s)^{-1}d\exp[M,\psi(s)])\right)(x)+\left(\lambda(\varphi(s)^{-1}(d\exp[M,\psi(s)]))\circ Y\right)(x)
\end{aligned} \tag{4.5.2}$$

Substituting (4.5.2) back into (4.5.1) gives

$$\frac{d}{ds}\tilde{\beta}_{\varphi(s)}(x)=[\tilde{\beta}_\varphi,\varphi^{-1}\star d\exp[M,\log\varphi]]+\lambda(\varphi^{-1}\star d\exp[M,\log\varphi])\circ Y.$$

If $\varphi(s)\in G^{\Phi}_{\mathcal{A}}$ for every s, then $\beta_{\varphi(s)}$ is well defined for every s. So at $\lambda=0$ we get

$$\frac{d}{ds}\beta_{\varphi(s)}=[\tilde{\beta}_\varphi,\varphi^{-1}\star d\exp[M,\log\varphi]]_+(\lambda=0)+\text{Res}\ \left((\varphi^{-1}\star d\exp[M,\log\varphi])\circ Y\right).$$

□

Thus the meromorphic β-function satisfies a Lax pair-type equation with an additional term.

Remark 4.5.6. A simple example that satisfies the condition $\varphi(s)\in G^{\Phi}_{\mathcal{A}}$ from Theorem 4.5.5 is given by a holomorphic φ (i.e $\varphi(x)\in\mathcal{A}_+$) with $m-2n=0$. Indeed $(\varphi^t)_-$ is the identity as φ^t is holomorphic, so $(\varphi^t)_-$ does not depend on t. Using the Taylor series of the exponential one can see that $\exp(-s\log(\varphi))$ has only holomorphic part so $g_-(s) = e$. Therefore the solutions of the Lax pair equation $\psi(s)$ are constant, so $\varphi(s) = \varphi(0) \in G^{\Phi}_{\mathcal{A}}$.

It is not clear that the condition $\varphi(s) \in G^{\Phi}_{\mathcal{A}}$ (for any s or for any s sufficiently close to zero) in Theorem 4.5.5 is satisfied in general or even in the particular case of the flow of the Feynman rule character $\bar{\varphi}$. Notice that this condition is necessary for the existence of the β-function.

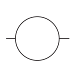
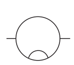
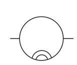
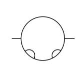
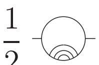
Chapter 5

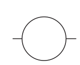
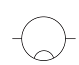
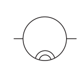
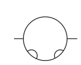
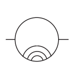
A worked example

In Sections 5.1-5.8 we discuss a specific example of the main theorem. In §5.9 we discuss how a general finite dimensional example should work. The reader who wants to skip the details of the worked example can just refer to §5.9. In §5.10 we discuss the relevance of Lie algebra cohomology computations for a five dimensional Lie algebra associated to specific Feynman diagrams. Then in Section 5.11 we approach the spectral curve. Most of the computations are supported by Mathematica file presented in appendices.

5.1 A finite dimensional group of characters

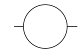
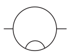
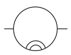
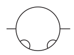
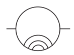
Let

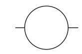
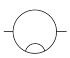
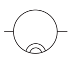
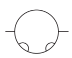
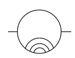
$A_1 = \frac{1}{2}$  , $A_2 = \frac{1}{2}$  , $A_3 = \frac{1}{2}$  , $A_4 = \frac{1}{4}$  , $A_5 = \frac{1}{2}$  .

Remark 5.1.1. Here we choose to divide the graphs  ,  ,  ,  ,  by their symmetry factors. The symmetry factor $\mathrm{sym}(\Gamma)$ is the rank of the automorphism group of the graph. The advantage of dividing by their symmetry factor would show up when we consider the Green function of an amplitude a; we have (see [16])

$$G^a = 1 \pm \sum_{\Gamma \in \mathcal{M}_a} \alpha^{|\Gamma|} \frac{\phi(\Gamma)}{\mathrm{sym}(\Gamma)}$$

where $\mathcal{M}_a$ is the set of all 1PI graphs containing the amplitude a. The Green functions are not considered in this thesis.

For the computations given in this chapter we can work as well as with  ,  ,  ,  ,  . For these graphs one can adjust some coefficients in definition of the map F from Proposition 5.1.2 below, and then we can identify the group of characters of the Hopf algebra generated by these graphs and $\emptyset$ with the group $(\mathbb{C}^5, \oplus)$ given below.

Let H be the Hopf algebra of 1PI Feynman graphs. Let H_1 be the Hopf subalgebra generated by $A_0 = \emptyset$, A_1, A_2, A_3, A_4, A_5. Notice that the graphs  ,  ,  ,  ,  do not have overlapping divergences. In Section 5.9 we discuss how the computations from this chapter can be extended to a Hopf subalgebras generated by a finite set of Feynman graphs.

Let G_1 be the Lie groups of characters of H_1. The Lie algebra of infinitesimal characters $\mathfrak{g}_1$ of H_1 is the Lie algebra of G_1. In particular, the Lie algebra $\mathfrak{g}_1$ of G_1 is generated by Z_{A_1}, Z_{A_2}, Z_{A_3}, Z_{A_4}, Z_{A_5}.

We identify G_1 with $\mathbb{C}^5$ using the normal coordinates defined in [5]. The group law induced on $\mathbb{C}^5$ is given by the following lemma.

Proposition 5.1.2. *Let $\oplus : \mathbb{C}^5 \times \mathbb{C}^5 \to \mathbb{C}^5$ be the group law on $\mathbb{C}^5$ given by*

$$(x_1, x_2, x_3, x_4, x_5) \oplus (y_1, y_2, y_3, y_4, y_5)$$

$$= (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4 + x_1 y_2 - x_2 y_1, x_5 + y_5).$$

Define $F : G_1 \to (\mathbb{C}^5, \oplus)$ by

$$\begin{aligned} F(\varphi) &= (\varphi(A_1), \varphi(A_2) - \varphi(A_1)^2, \varphi(A_3) - 2\varphi(A_1)\varphi(A_2) + \frac{4}{3}\varphi(A_1)^3, \\ &\quad \varphi(A_4) - \varphi(A_1)\varphi(A_2) + \frac{1}{3}\varphi(A_1)^3, \\ &\quad \varphi(A_5) - 2\varphi(A_1)\varphi(A_3) - \varphi(A_2)^2 + 4\varphi(A_1)^2\varphi(A_2) - 2\varphi(A_1)^4). \end{aligned}$$

Then F is a group morphism.

Proof. Let $(F_1, F_2, F_3, F_4, F_5) = F$. Closely following [5], we set

$$C_1 = A_1, \quad C_2 = A_2 - A_1 \cdot A_1, \quad C_3 = A_3 - 2A_1 \cdot A_2 + \frac{4}{3}A_1 \cdot A_1 \cdot A_1$$

$$C_4 = A_4 - A_1 \cdot A_2 + \frac{1}{3}A_1 \cdot A_1 \cdot A_1, \quad C_5 = A_5 - 2A_1 \cdot A_3 - A_2 \cdot A_2 + 4A_1 \cdot A_1 \cdot A_2 - 2A_1 \cdot A_1 \cdot A_1 \cdot A_1.$$

Notice that $F_i = \varphi(C_i)$ for any $i \in \{1, \cdots, 5\}$. C_1, C_2, C_3, C_5 are primitive, so $F_k(\phi_1 \star \phi_2) = F_k(\phi_1) + F_k(\phi_2)$ for $k \in \{1, 2, 3, 5\}$. We have

$$\Delta(C_4) = C_4 \otimes 1 + 1 \otimes C_4 + C_1 \otimes C_2 - C_2 \otimes C_1,$$

which implies

$$F_4(\phi_1 \star \phi_2) = \langle \phi_1 \otimes \phi_2, \Delta(C_4) \rangle = F_4(\phi_1) + F_4(\phi_2) + F_1(\phi_1)F_2(\phi_2) - F_2(\phi_1)F_1(\phi_2).$$

□

Identifying G_1 with $(\mathbb{C}^5, \oplus)$, we can identify G_0 with $(\mathbb{C}^6, \oplus)$, where

$$(x_1, x_2, x_3, x_4, x_5, t) \oplus (y_1, y_2, y_3, y_4, y_5, t') =$$

$$(x_1 + e^t y_1, x_2 + e^{2t} y_2, x_3 + e^{3t} y_3, x_4 + e^{3t} y_4 + e^{2t} x_1 y_2 - e^t x_2 y_1, x_5 + e^{4t} y_5, t + t').$$

The following lemma gives a basis of the left invariant vector fields on G_0 and the structure constants of $\mathfrak{g}_0$.

Lemma 5.1.3. *Let*

$$Y_1 = e^t \left(\frac{\partial}{\partial y_1} - y_2 \frac{\partial}{\partial y_4} \right), \quad Y_2 = e^{2t} \left(\frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial y_4} \right),$$

$$Y_3 = e^{3t}\frac{\partial}{\partial y_3}, \qquad Y_4 = e^{3t}\frac{\partial}{\partial y_4}, \qquad Y_5 = e^{4t}\frac{\partial}{\partial y_5}, \qquad Z_0 = \frac{\partial}{\partial t}.$$

where $(y_1, y_2, y_3, y_4, y_5, t)$ are the standard coordinates on $\mathbb{C}^5$. Then $\{Y_1,\ Y_2,\ Y_3,\ Y_4,\ Y_5,\ Z_0\}$ is a basis of the left invariant vector fields on G_0. We have $[Y_i, Y_j] = 0$ for any $(i,j) \neq (1,2), (2,1)$, and $[Y_1, Y_2] = -[Y_2, Y_1] = 2Y_4$, $[Z_0, Y_1] = Y_1$, $[Z_0, Y_2] = 2Y_2$, $[Z_0, Y_3] = 3Y_3$, $[Z_0, Y_4] = 3Y_4$ and $[Z_0, Y_5] = 4Y_5$.

Proof. This follows from the easily computed formulas

$$L_x\left(\frac{\partial}{\partial y_1}\right) = e^t\left(\frac{\partial}{\partial y_1} - x_2\frac{\partial}{\partial y_4}\right), \quad L_x\left(\frac{\partial}{\partial y_2}\right) = e^{2t}\left(\frac{\partial}{\partial y_2} + x_1\frac{\partial}{\partial y_4}\right),$$

$$L_x\left(\frac{\partial}{\partial y_3}\right) = e^{3t}\frac{\partial}{\partial y_3}, \quad L_x\left(\frac{\partial}{\partial y_4}\right) = e^{3t}\frac{\partial}{\partial y_4}, \quad L_x\left(\frac{\partial}{\partial y_5}\right) = e^{4t}\frac{\partial}{\partial y_5},$$

$$L_x\left(\frac{\partial}{\partial t'}\right) = \frac{\partial}{\partial t'}.$$

Here L_g is left multiplication i.e. $L_g(g') = gg'$ for $g, g' \in G_0$. □

5.2 The exponential map and the adjoint and coadjoint representations

Lemma 5.2.1. *The exponential* exp : $\mathfrak{g}_0 \to G_0$ *is given by*

$$\exp(a_1Y_1 + a_2Y_2 + a_3Y_3 + a_4Y_4 + a_5Y_5 + a_6Z_0) =$$

$$= \begin{cases} (a_1, a_2, a_3, a_4, a_5, 0), & \text{if } a_6 = 0, \\ \Big(\frac{a_1(e^{a_6}-1)}{a_6}, \frac{a_2(e^{2a_6}-1)}{2a_6}, \frac{a_2(e^{3a_6}-1)}{3a_6}, & \\ \frac{a_4(e^{3a_6}-1)}{3a_6} + \frac{a_1a_2}{2a_6^2}(\frac{e^{3a_6}-1}{3} - e^{2a_6t} + e^{a_6t}), \frac{a_5(e^{4a_6}-1)}{4a_6}, a_6\Big) & \text{if } a_6 \neq 0, \end{cases}$$

and exp *is bijective.*

Proof. Let $Y \in \mathfrak{g}_0$ and let $\gamma(t)$ be the 1-parameter subgroup of G_0 generated by Y.

Set

$$\begin{aligned} Y &= a_1Y_1 + a_2Y_2 + a_3Y_3 + a_4Y_4 + a_5Y_5 + a_6Z_0, \\ \gamma(t) &= (g_1(t), g_2(t), g_3(t), g_4(t), g_5(t), g_6(t)). \end{aligned}$$

To find $\gamma(t) = \exp(tY)$, we solve the differential equation

$$L_{\gamma(t)^{-1}}\dot{\gamma}(t) = a_1Y_1 + a_2Y_2 + a_3Y_3 + a_4Y_4 + a_5Y_5 + a_6Z_0$$

with the initial condition $\gamma(0) = 0$. First notice that

$$L_{\gamma(t)^{-1}}\dot{\gamma}(t) = (\dot{g}_1e^{-g_6}, \dot{g}_2e^{-2g_6}, \dot{g}_3e^{-3g_6}, e^{-3g_6}(\dot{g}_1g_2 - \dot{g}_2g_1 + \dot{g}_4), \dot{g}_5e^{-4g_6}, \dot{g}_6).$$

Then $a_k = \dot{g}_ke^{-kg_6}$ for $k \in \{1,2,3\}$, $a_5 = \dot{g}_5e^{-4g_6}$, $a_6 = \dot{g}_6$ and

$$a_4 = (\dot{g}_4 + \dot{g}_1g_2 - \dot{g}_2g_1)e^{-3g_6},$$

with the initial conditions

$$g_1(0) = g_2(0) = g_3(0) = g_4(0) = g_5(0) = g_6(0) = 0.$$

Therefore $g_6 = a_6t$, $g_k(t) = a_k(e^{ka_6t} - 1)/(ka_6)$ for $k \in \{1,2,3\}$,

$$g_5(t) = a_5(e^{4a_6t} - 1)/(4a_6)$$

and

$$g_4(t) = \frac{a_4(e^{3a_6t} - 1)}{3a_6} - \int_0^t \frac{a_1e^{a_6x}a_2(e^{2a_6x} - 1)}{2a_6}\,dx + \int_0^t \frac{a_2e^{2a_6x}a_1(e^{a_6x} - 1)}{a_6}\,dx$$

$$= \quad \frac{a_4(e^{3a_6t}-1)}{3a_6} + \frac{a_1a_2}{2a_6^2}\left(\frac{e^{3a_6t}-1}{3} - e^{2a_6t} + e^{a_6t}\right).$$

If $a_6 \neq 0$, then

$$\exp\left(\sum_{k=1}^{5} a_kY_k + a_6Z_0\right) = \left(\frac{a_1(e^{a_6}-1)}{a_6}, \frac{a_2(e^{2a_6}-1)}{2a_6}, \frac{a_2(e^{3a_6}-1)}{3a_6},\right.$$

$$\left.\frac{a_4(e^{3a_6}-1)}{3a_6} + \frac{a_1a_2}{2a_6^2}(\frac{e^{3a_6}-1}{3} - e^{2a_6t} + e^{a_6t}), \frac{a_5(e^{4a_6}-1)}{4a_6}, a_6\right).$$

If $a_6 = 0$ then

$$\exp\left(\sum_{k=1}^{5} a_kY_k\right) = (a_1, a_2, a_3, a_4, a_5, 0).$$

□

The adjoint and coadjoint representations of G_0 are given by the following lemmas.

Lemma 5.2.2. *i) The adjoint representation* $\mathrm{Ad}_{G_0} : G_0 \to \mathrm{GL}(\mathfrak{g}_0)$ *is given by*

$$\mathrm{Ad}_{G_0}(g_1,\ldots,g_5,g_6) = \begin{pmatrix} e^{g_6} & 0 & 0 & 0 & 0 & -g_1 \\ 0 & e^{2g_6} & 0 & 0 & 0 & -2g_2 \\ 0 & 0 & e^{3g_6} & 0 & 0 & -3g_3 \\ -2g_2e^{g_6} & 2g_1e^{2g_6} & 0 & e^{3g_6} & 0 & -g_1g_2 - 3g_4 \\ 0 & 0 & 0 & 0 & e^{4g_6} & -4g_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

ii) The coadjoint representation $\mathrm{Ad}^*_{G_0} : G_0 \to \mathrm{GL}(\mathfrak{g}_0^*)$ *is given by*

$$\mathrm{Ad}^*_{G_0}(g_1,\ldots,g_5,g_6) =$$

$$\begin{pmatrix} e^{-g_6} & 0 & 0 & 2g_2e^{-3g_6} & 0 & 0 \\ 0 & e^{-2g_6} & 0 & -2g_1e^{-3g_6} & 0 & 0 \\ 0 & 0 & e^{-3g_6} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-3g_6} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-4g_6} & 0 \\ e^{-g_6}g_1 & 2e^{-2g_6}g_2 & 3e^{-3_6}g_3 & -e^{-3_6}g_1g_2 + 3e^{-3g_6}g_4 & 4e^{-4g_6}g_5 & 1 \end{pmatrix}$$

Proof. To show i), we straightforwardly compute $\mathrm{Ad}_{G_0}(g) = dC_g$ where $C_g(h) = ghg^{-1}$. In Appendix B we present a commented Mathematica file for the computation of Ad_{G_0}. Note that

$$g^{-1} = (x_1, x_2, x_3, x_4, x_5, t)^{-1} = (-e^{-t}x_1, -e^{-2t}x_2, -e^{-3t}x_3, -e^{-3t}x_4, -e^{-4t}x_5, -t).$$

ii) then follows from i) and $\mathrm{Ad}^*(g) = (\mathrm{Ad}(g^{-1}))^t$. □

Lemma 5.2.3. *1) For the basis $\{Y_1, \dots, Y_6\}$ of Lemma 5.1.3,* $\mathrm{ad} : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_0)$ *is given by*

$$\mathrm{ad}\left(\sum_{i=1}^{6} c_iY_i\right) = \begin{pmatrix} c_6 & 0 & 0 & 0 & 0 & -c_1 \\ 0 & 2c_6 & 0 & 0 & 0 & -2c_2 \\ 0 & 0 & 3c_6 & 0 & 0 & -3c_3 \\ -2c_2 & 2c_1 & 0 & 3c_6 & 0 & -3c_4 \\ 0 & 0 & 0 & 0 & 4c_6 & -4c_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

2) $\mathrm{ad}^* : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g}_0^*)$ *is given by*

$$\mathrm{ad}^*\left(\sum_{i=1}^{6} c_i Y_i\right) = \begin{pmatrix} -c_6 & 0 & 0 & 2c_2 & 0 & 0 \\ 0 & -2c_6 & 0 & -2c_1 & 0 & 0 \\ 0 & 0 & -3c_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3c_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4c_6 & 0 \\ c_1 & 2c_2 & 3c_3 & 3c_4 & 4c_5 & 0 \end{pmatrix}$$

Since $d\mathrm{Ad} = \mathrm{ad}$, the proof follows by differentiating the formulas in Lemma 5.2.2.

5.3 The double Lie algebra and its associated Lie group

The conditions a) and b) in Definition 3.6.1 of a Lie bialgebra can be written in a basis as a system of quadratic equations. Solving this system explicitly, in our case via Mathematica, gives the following proposition.

Proposition 5.3.1. *There are* 43 *families of Lie bialgebra structures* γ *on* $\mathfrak{g}_0$*.*

Remark 5.3.2. In fact, the system of quadratic equations involves 90 variables. Referring Appendix A for a sample, using Mathematica we get 1 solution with 82 linear relations (and so 8 degrees of freedom), 7 solutions with 83 linear relations, 16 solutions with 84 linear relations, 13 solutions with 85 linear relations, 5 solutions with 86 linear relations, and 1 solution with 87 linear relations.

Remark 5.3.3. Since the Lie group G_0 is connected and simply-connected, each family of Lie bialgebras has a corresponding family of Poisson-Lie groups. It is possible that some Lie bialgebras of our set of families are isomorphic as Lie bialgebras.

As it is difficult to construct explicitly a Lie group corresponding to the Lie algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_0^*$ for an arbitrary choice of γ, as in §4.1 we take the simplest choice $\gamma = 0$ and let $\tilde{G} = G_0 \ltimes \mathfrak{g}_0^*$ be the corresponding Lie group of $\delta = \mathfrak{g} \oplus \mathfrak{g}^*$.

Remark 5.3.4. The group law on $\tilde{G}$ is given by

$$((g_1, g_2, g_3, g_4, g_5, g_6), (h_1, h_2, h_3, h_4, h_5, h_6))\cdot$$

$$((g_1', g_2', g_3', g_4', g_5', g_6'), (h_1', h_2', h_3', h_4', h_5', h_6')) =$$

$$(g_1 + e^{g_6} g_1', g_2 + e^{2g_6} g_2', g_3 + e^{3g_6} g_3', g_4 + e^{3g_6} g_4' + g^{2g_6} g_1 g_2' - g^{g_6} g_2 g_1', g_5 + e^{4g_6} g_5', g_6 + g_6',$$

$$h_1 + e^{g_6} h_1' - 2g_2 e^{g_6} h_4', h_2 + e^{2g_6} h_2' + 2g_2 e^{2g_6} h_4', h_3 + e^{3g_6} h_3', h_4 + e^{3g_6} h_4', h_5 + e^{4g_6} h_5',$$

$$h_6 - g_1 h_1' - 2g_2 h_2' - 3g_3 h_3' - 3g_4 h_4' - 4g_5 h_5' + h_6').$$

5.4 The adjoint representations ad_δ and $\mathrm{Ad}_{\tilde{G}}$

Let $\mathrm{ad}_\delta : \delta \to \mathfrak{gl}(\delta)$ be the adjoint representation of δ. Computing ad_δ explicitly, for example via Mathematica, we get

$$\mathrm{ad}_\delta \left(\sum_{i=1}^{12} x_i Y_i \right) =$$

$$\begin{pmatrix} x_6 & 0 & 0 & 0 & 0 & -x_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2x_6 & 0 & 0 & 0 & -2x_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3x_6 & 0 & 0 & -3x_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2x_2 & 2x_1 & 0 & 3x_6 & 0 & -3x_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4x_6 & -4x_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2x_{10} & 0 & 0 & 0 & x_7 & -x_6 & 0 & 0 & 2x_2 & 0 & 0 \\ 2x_{10} & 0 & 0 & 0 & 0 & 2x_8 & 0 & -2x_6 & 0 & -2x_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3x_9 & 0 & 0 & -3x_6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3x_{10} & 0 & 0 & 0 & -3x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4x_{11} & 0 & 0 & 0 & 0 & -4x_6 & 0 \\ -x_7 & -2x_8 & -3x_9 & -3x_{10} & -4x_{11} & 0 & x_1 & 2x_2 & 3x_3 & 3x_4 & 4x_5 & 0 \end{pmatrix}$$

where $Y_{6+t} = Y_t$ for $t \in \{1, \ldots, 6\}$.

Corollary 5.4.1. $\mathrm{Ker}(\mathrm{ad}_\delta) = \mathrm{Span}\{Y_{12}\}$.

The adjoint and coadjoint representations of $\tilde{G}$ are given in the following proposition.

Proposition 5.4.2.

$$\mathrm{Ad}_{\tilde{G}}(g_1, g_2, \ldots, g_{12}) = \begin{pmatrix} \mathrm{Ad}_{G_0}(g_1, g_2, \ldots, g_6) & 0 \\ H(g_1, g_2, \ldots, g_{12}) & \mathrm{Ad}^*_{G_0}(g_1, g_2, \ldots, g_6) \end{pmatrix}$$

$$\mathrm{Ad}^*_{\tilde{G}}(g_1, g_2, \ldots, g_{12}) = \begin{pmatrix} \mathrm{Ad}^*_{G_0}(g_1, g_2, \ldots, g_6) & H(g_1, g_2, \ldots, g_{12}) \\ 0 & \mathrm{Ad}_{G_0}(g_1, g_2, \ldots, g_6) \end{pmatrix}$$

where $H(g_1, g_2, \ldots, g_{12})$ *is a* 6×6 *matrix given by*

$$\begin{pmatrix} 0 & -2e^{2g_6}g_{10} & 0 & 0 & 0 & 4g_{10}g_2 + g_7 \\ 2e^{g_6}g_{10} & 0 & 0 & 0 & 0 & -2g_1g_{10} + 2g_8 \\ 0 & 0 & 0 & 0 & 0 & 3g_9 \\ 0 & 0 & 0 & 0 & 0 & 3g_{10} \\ 0 & 0 & 0 & 0 & 0 & 4g_{11} \\ e^{g_6}(6g_{10}g_2 - g_7) & -2e^{2g_6}(3g_1g_{10} - g_8) & -3e^{3g_6}g_9 & -3e^{3g_6}g_{10} & -4e^{4g_6}g_{11} & z \end{pmatrix}$$

and

$$z = 3g_1g_{10}g_2 + 9g_{10}g_4 + 16g_{11}g_5 + g_1g_7 + 4g_2g_8 + 9g_3g_9.$$

The computations are given in Appendix B.

5.5 Some $\mathrm{Ad}_{\tilde{G}}$-invariant polynomials on δ and Lδ

We note that $\mathrm{Tr}(\mathrm{ad}(a)^k), k \in \mathbb{Z}^+$, are Ad-invariant polynomials on δ. By Lemma 4.2.1, these induce Ad-invariant polynomials on $L\delta$, i.e. Casimir functions (constants of motions) on $L\delta$. Explicit computations in Chapter 9 give the following lemma.

Lemma 5.5.1. *Let* $\varphi : \delta \to \mathbb{C}$ *be the map given by* $\varphi(a) = \mathrm{Tr}(\mathrm{ad}(a)^k)$. *For odd positive integers* k, $\varphi(a) = 0$. *For even positive integers* k, $\varphi(a) = C(a_6)^k$ *for some constant* $C = C_k$, *where* $a = \sum_{i=1}^{12} a_i Y_i$.

This gives the following corollary, which can also be checked directly.

Corollary 5.5.2. *For any even positive integer* k *and any constant* C

$$\varphi(a) = C(a_6)^k$$

is an $\mathrm{Ad}_{\tilde{G}}$*-invariant polynomial on* δ.

Proof. Let $\pi_6 : \delta \to \mathbb{C}$ be the projection given by

$$\pi_6 \left(\sum_{j=1}^{12} a_i Y_i \right) = a_6. \tag{5.5.1}$$

Since

$$\pi_6(\mathrm{Ad}_{\tilde{G}}(\sum_{j=1}^{12} a_i Y_i)) = a_6,$$

we see that $\varphi(a) = C(a_6)^k$ is an Ad-invariant polynomial on δ. □

Example 1.

In the notation of (4.2.2), for integers M and N we get Casimir elements on $L\delta$

$$\varphi_{m,n} \left(\sum_{i=M}^{N} \sum_{j=1}^{12} L_i^j Y_j \lambda^i \right) = \mathrm{Res}_{\lambda=0}(\lambda^{-n} C(\lambda^m L_6^i(\lambda^i))^{2k})$$

$$= C \sum_S \begin{pmatrix} & & k & \\ i_M & i_{M+1} & \cdots & i_N \end{pmatrix} (L_M^6)^{i_M} \cdots (L_N^6)^{i_N}$$

for all nonnegative integers m, n, where

$$S = \{(i_M, \ldots, i_N) \mid i_M \geq 0, \ldots, i_N \geq 0,\ i_M + i_{M+1} + \ldots + i_N = k,$$

$$M i_M + (M+1) i_{M+1} + \ldots + N i_N = -1 + n - km\}.$$

Example 2.

The natural pairing $\langle \cdot, \cdot \rangle$ on δ induces an Ad-invariant polynomial

$$\psi(Y) = \langle Y, Y \rangle = 2 \sum_{i=1}^{6} a_i a_{i+6}$$

for $Y = \sum_{i=1}^{12} a_i Y_i$. Then $\psi_{m,n} : L\delta \to \mathbb{C}$ given by (4.2.2) becomes

$$\psi_{m,n}(L(\lambda)) = 2\sum_{j=1}^{6} \sum_{i+k-n+2m=-1} L_i^j L_k^{j+6}.$$

Example 3.

(4.2.4) gives other Ad-invariant polynomials on $L\delta$. An explicit computation gives

$$\psi_{m,n}^k(L(\lambda)) = \sum_S \begin{pmatrix} k \\ \{i_{a,b}\}_{a,b\in\{-M,\ldots,N\}} \end{pmatrix} \prod_{a,b\in\{-M,\ldots,N\}} (2\sum_{j=1}^{6} L_a^j L_b^{j+6})^{i_{a,b}},$$

where

$$S = \{\{i_{a,b}\}_{a,b\in\{-M,\ldots,N\}} \mid \sum_{a,b\in\{-M,\ldots,N\}} i_{a,b} = k \text{ and } \sum_{a,b\in\{-M,\ldots,N\}} i_{a,b}(a+b) - n = -1\}.$$

5.6 The Lax pair equation in local coordinates

We can write the Lax pair equation

$$\frac{dL}{dt} = [M, L], \quad M = \frac{1}{2}R(I(d\kappa(L(\lambda)))) \tag{5.6.1}$$

in local coordinates when the Casimir function κ is $\varphi_{m,n}$ or $\psi_{m,n}$ given above.

Note that

$$M = \frac{1}{2}R(I(d\kappa(L(\lambda)))) = \frac{1}{2}\frac{\partial(\kappa(L(\lambda)))}{\partial L_p^t} Y_t \lambda^{-1-p} r(-1-p), \tag{5.6.2}$$

where $r(s) = 1$ if $s \geq 0$ and $r(s) = -1$ if $s < 0$.

The case $\kappa = \varphi_{m,n}$

Lemma 5.6.1. *In local coordinates the Lax pair equation (5.6.1) becomes*

$$\frac{dL^j_{i+p}}{dt} = 0.$$

for $1 \leq j \leq 12$. Thus $L(t) = L(0)$ for any t.

Proof.

$$\frac{\partial \varphi_{mn}(\sum_{i=M}^{N}\sum_{j=1}^{12} L^j_i Y_j \lambda^i)}{\partial L^6_p} =$$

$$C \sum_S \begin{pmatrix} & & k & \\ i_M & i_{M+1} & \cdots & i_N \end{pmatrix} (L^6_M)^{i_M} \cdots ((i_p)(L^6_p)^{i_p - 1}) \cdot (L^6_N)^{i_N},$$

where S is some set of multi-indices. Then

$$M = R(I(\frac{\partial \varphi_{mn}(\sum_{i=M}^{N}\sum_{j=1}^{12} L^j_i Y_j \lambda^i)}{\partial L^6_p}) Y_6 \lambda^p) = r(-1-p) f(L(\lambda)) Y_{12}.$$

Since $\mathrm{ad}(Y_{12}) = 0$, we have $[L, M] = 0$, and so $\frac{dL(t)}{dt} = 0$. □

Thus the Lax pair equation is trivial in this case.

The case $\kappa = \psi_{m,n}$.

By Proposition 4.3.3, $M = R(X) = R(\lambda^{-n+2m} L(\lambda))$.

Let q^k_{ij} be the structure constants of δ in the usual basis.

Theorem 5.6.2. *In local coordinates the Lax pair equation (5.6.1) becomes*

$$\frac{dL^k_{i+p-n+2m}}{dt} = r(-n+2m+p) \sum_{j=1}^{12}\sum_{t=1}^{12} L^j_i L^t_p q^k_{tj}$$

for $j \in \{1, \ldots, 12\}$ *and all* i, p*, where* $r(s) = 1$ *if* $s \geq 0$ *and* $r(s) = -1$ *if* $s < 0$.

Note that the Lax pair equation $\pi_1(\dot{L}(t)) = [\pi_1(L), \pi_1(M)]$ on $L\mathfrak{g}_0$ given by (4.3.3) has the local coordinate form

$$\frac{dL^k_{i+p-n+2m}}{dt} = r(-n+2m+p)\sum_{j=1}^{6}\sum_{t=1}^{6} L^j_i L^t_p c^k_{tj}$$

for $j \in \{1, \ldots, 6\}$ and all i, p.

5.7 The Birkhoff factorization of $\exp(-tX)$

We compute the factorization of $\exp(-tX)$ for the interesting case of

$$X = I(d\psi_{m,n}(L(\lambda))),$$

for $\pi_6(L(\lambda)) = 0$, where π_6 is the extension to $L\delta$ of π_6 in (5.5.1). Then $X = \lambda^{-n+2m}L(\lambda) = \sum_{j,i}(L^j_i\lambda^{i-n+2m})Y_j$, and

$$\exp(-tX) = \exp(\sum_{j=1}^{12}(-\sum_i L^j_i\lambda^{i-n+2m}t)Y_j).$$

Let $z_j = -\sum_i L^j_i\lambda^{i-n+2m}$ for $j \in \{1, \ldots, 12\}$.

Our assumption is that $z_6 = 0$, as the exponential of $L(\lambda)$ has a simpler form in this case; in fact this is the only case needed for our main theorem below. The exponential of $L\delta$ on $z_6 = 0$ is given by (see Appendix B)

$$\exp(tz_1, tz_2, tz_3, tz_4, tz_5, 0, tz_7, tz_8, tz_9, tz_{10}, tz_{11}, tz_{12}) =$$

$$\big(tz_1, tz_2, tz_3, tz_4, tz_5, 0, t^2z_{10}z_2 + tz_7, -t^2z_1z_{10} + tz_8, tz_9, tz_{10}, tz_{11},$$

$$tz_{12} - \frac{1}{3}t^3 z_1 z_{10} z_2 + \frac{3}{2}t^2 z_{10} z_4 + 2t^2 z_{11} z_5 + \frac{1}{2}t^2 z_1 z_7 + t^2 z_2 z_8 + \frac{3}{2}t^2 z_3 z_9 \Big).$$

By Lemma 4.4.1, the Birkhoff decomposition $(g, \alpha) = (g_-, \alpha_-)^{-1}(g_+, \alpha)_+$, with $g \in G_0$ and $\alpha \in \mathfrak{g}_0^*$ is given by $g = g_-^{-1} g_+$, and $-\alpha_- + \alpha_+ = \mathrm{Ad}^*(g_-)\alpha$.

Let g_{j-} and g_{j+}, $j \in \{1, \dots, 12\}$, be the components of g_- and g_+ respectively. Therefore, for $j \in \{1, 2, 3, 4, 5, 9, 10, 11\}$, we have

$$g_{j+} = -\sum_{i \geq n-2m} L_i^j \lambda^{i-n+2m} t$$

and

$$g_{j-} = -\sum_{i < n-2m} L_i^j \lambda^{i-n+2m} t. \tag{5.7.1}$$

Then

$$g_{i+} = P_+(tz_i), \quad g_{i-} = -P_-(tz_i) \tag{5.7.2}$$

for $i \in \{1, 2, 3, 5\}$. We also get

$$g_{4+} = P_+(tz_4 + t^2(P_-(z_1)P_+(z_2) - P_-(z_1)P_+(z_2))), \tag{5.7.3}$$

$$g_{4-} = P_-(tz_4 + t^2(P_-(z_1)P_+(z_2) - P_-(z_1)P_+(z_2))), \tag{5.7.4}$$

$$g_{6-} = 0, \qquad g_{6+} = 0, \tag{5.7.5}$$

$$g_{7-} = -P_-(2tg_{2-}z_{10} + t^2 z_{10} z_2 + tz_7), \tag{5.7.6}$$

$$g_{8-} = -P_-(-2tg_{1-}z_{10} - t^2 z_1 z_{10} + tz_8), \tag{5.7.7}$$

$$\begin{aligned} g_{12-} \quad = \quad & -P_-(-tg_{1-}g_{2-}z_{10} + 3tg_{4-}z_{10} - 2t^2 g_{2-} z_1 z_{10} + 4tg_{5-}z_{11} \\ & +tz_{12} + t^2 g_{1-} z_{10} z_2 - \frac{1}{3}t^3 z_1 z_{10} z_2 + \frac{3}{2}t^2 z_{10} z_4 + 2t^2 z_{11} z_5 + tg_{1-} z_7 \end{aligned} \tag{5.7.8}$$

$$+\frac{1}{2}t^2 z_1 z_7 + 2tg_{2-} z_8 + t^2 z_2 z_8 + 3tg_{3-} z_9 + \frac{3}{2}t^2 z_3 z_9),$$

$$g_{k+} = z_k + g_{k-},$$

for $k \in \{7, 8, 12\}$. Then $g_- = (g_{1-}, \ldots, g_{12-})$ and $g_+ = (g_{1+}, \ldots, g_{12+})$ satisfy $g_-^{-1} g_+ = \exp(-tX)$.

We now assemble the final formulas needed to compute the solution to the Lax pair equation given by Theorem 4.3.1. Let $\pi : \delta \to \mathfrak{g}$ and $\theta : \delta \to \mathfrak{g}^*$ be the projections onto $\mathfrak{g}$ respectively $\mathfrak{g}^*$. Then

$$\pi(\mathrm{Ad}^*(g_{1-}, \ldots, g_{12-})L(\lambda)) = \mathrm{Ad}^*_{G_0}(g_{1-}, \ldots, g_{6-})\pi(L(\lambda)) + H(g_{1-}, \ldots, g_{12-})\theta(L(\lambda)) \tag{5.7.9}$$

and

$$\theta(\mathrm{Ad}^*(g_{1-}, \ldots, g_{12-})L(\lambda)) = \mathrm{Ad}_{G_0}(g_{1-}, \ldots, g_{6-})\theta(L(\lambda)), \tag{5.7.10}$$

where H is given by Proposition 5.4.2. It is fortunate that $\mathrm{Ad}^*_{\tilde{G}}$ does not depend on g_{12-}, which by (5.7.8) is difficult to compute explicitly.

5.8 The Feynman rules characters and the main result

For any graph γ, the Feynman rules integral $\varphi(\lambda, q)(\gamma)$ can be written in term of Γ-functions. Some explicit formulas are as follows.

$$\begin{aligned}
\varphi(\lambda, q)(A_1) &= \frac{1}{2}\pi^3 (q^2)^{1-\lambda} B_1(\lambda) \\
\varphi(\lambda, q)(A_2) &= \frac{1}{2}\pi^6 (q^2)^{1-2\lambda} B_1(\lambda) B_2(\lambda) \\
\varphi(\lambda, q)(A_3) &= \frac{1}{2}\pi^9 (q^2)^{1-3\lambda} B_1(\lambda) B_2(\lambda) B_3(\lambda) \\
\varphi(\lambda, q)(A_4) &= \frac{1}{4}\pi^9 (q^2)^{1-3\lambda} B_1(\lambda)^2 B_2(\lambda)
\end{aligned}$$

$$\varphi(\lambda,q)(A_5) = \frac{1}{2}\pi^{12}(q^2)^{1-4\lambda}B_1(\lambda)B_2(\lambda)B_3(\lambda)B_4(\lambda)$$

where $B_j(\lambda) = \frac{-1}{j\lambda(1-j\lambda)(2-j\lambda)(3-j\lambda)}$, $j \in \{1,2,3,4\}$.

Theorem 5.8.1. *Let H_1 be the Hopf subalgebra generated by A_1, A_2, A_3, A_4, A_5, $\emptyset$, and choose a character $\bar{\varphi}(\lambda) \in \Omega\tilde{G}$ with $\pi_6(\bar{\varphi}) = 0$. Pick $L_0(\lambda) \in \Omega\delta$ and set $X = I(d\psi_{m,n}(L_0(\lambda)))$, where $\psi_{m,n}$ is the Casimir function on $\Omega\delta$ given by*

$$\psi_{m,n}(L(\lambda)) = \mathrm{Res}_{\lambda=0}(\lambda^m\langle\lambda^n L(\lambda), \lambda^n L(\lambda)\rangle).$$

Then the solution of

$$\frac{dL}{dt} = [M,L], \quad M = \frac{1}{2}R(I(d\psi_{m,n}(L(\lambda)))) \tag{5.8.1}$$

with the initial condition $L(0) = L_0$ is given by

$$L(t) = \mathrm{Ad}^* g_{\pm}(t)\cdot L_0, \tag{5.8.2}$$

with $\exp(-tX) = g_-(t)^{-1}g_+(t)$ where g_- and g_+ are given by (5.7.1), (5.7.5), (5.7.6), (5.7.7), (5.7.8), and where the $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$ components of Ad are given by (5.7.9) and (5.7.10).

In the particular case where $\bar{\varphi}$ is the normalized Feynman rule characters $\bar{\varphi}(\lambda,q) = \frac{\varphi(\lambda,q)}{q^2}$ and $L_0(\lambda) = \frac{1}{2}\lambda^{n-2m}\exp^{-1}(\bar{\varphi}(\lambda))$, $g_-(t)$ and $g_+(t)$ are given by (5.7.2), (5.7.3) and (5.7.4). If in addition $n-2m=0$, then the solution $L(t)$ of (5.8.1) is the flow of half of the logarithm of Feynman rules.

This follows from Theorem 4.4.3.

5.9 Generalizations to arbitrary finitely generated Hopf subalgebras

In this section we see how the previous computations can carried over to Hopf subalgebra generated by a finite number of Feynman diagrams.

Let H_2 be the Hopf algebra generated by a finite number of Feynman graphs:

$$\emptyset, B_1, B_2, B_3, \ldots, B_l.$$

We shall denote this list by $\mathcal{L}$. Recall that $L(B_i)$ is the loop number of B_i. We assume the following:

1) $L(B_i) \leq L(B_j)$ for any $1 \leq i < j \leq n$,

2) If $L(B_i) = L(B_j)$ and B_i is a ladder graph then $i < j$,

3) Any Feynman subgraph of a graph $B_i \in \mathcal{L}$ is contained in the list $\mathcal{L}$.

Notice that condition 3) is essential. Let G_2 be the group of characters of H_2.

5.9.1 Normal coordinates and the character group

To follow the construction in Section 5.1 we need to construct normal coordinates (see [5]).

To associate to any B_i, $i \in \{1, \ldots, l\}$ an element B_i', we start by associating to any $B_i \in \mathcal{L}$ its decorated rooted tree $T_i = T(B_i)$ (if B_i has overlapping divergences one needs to associate its sum of decorated rooted trees instead (see [6])). To the decorated rooted tree T_i (or the sum of trees if overlappings occur) one can associate a polynomial of rooted tree T_i' following the normal coordinates construction in [5]. Finally to T_i' we can associate $B_i' \in H_2$.

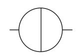
Notice that

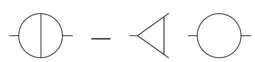
$$B'_i = B_i + \text{a polynomial of } B_{i_k}$$

with the loop number of each B_{i_k} less than the loop number of B_i. The B'_i-graph corresponding to a ladder graph B_i is primitive. In particular, the group of characters associated to a set of ladder graphs is abelian, so its Lie algebra has trivial bracket.

Notice that vertex graphs are excluded from our discussion.

Now, we give an example of computation of a B_i for an overlapping divergences. We do not adjust the computation by the symmetry factor.

Example 10. If $B =$  $\in \mathcal{L}$ then the corresponding B'-graph mentioned above is given by

$P =$  .

Let $F : G_2 \longrightarrow (\mathbb{C}^l, \oplus)$ be given by

$$F(\varphi) = \{\varphi(B'_i)\}_{i\in\{1,\ldots,l\}}.$$

Let $\oplus : \mathbb{C}^l \times \mathbb{C}^l \longrightarrow \mathbb{C}^l$ be the group law induced by the group law on G_2. Then we can identify G_2 with $(\mathbb{C}^l, \oplus)$.

We define the semidirect product $G_0 = \mathbb{C}^l \rtimes \mathbb{C}$ given by the following action of $\mathbb{C}$ on $\mathbb{C}^l$:

$$t.\{x_i\}_i = \{e^{L(B_i)t} x_i\}_i,$$

where $L(B_i)$ is the loop number of B_i. This corresponds in local coordinates to the action θ_i defined in Chapter 2.

We then compute the differential of the left translation and we get the left invariant vector fields on our Lie group. To get the infinitesimal Lie algebra structure, we compute the bracket between the left invariants vector fields on the semi-direct group $(\mathbb{C}^{l+1}, \oplus)$. We shall discuss at the end of this section an alternative way to compute the Lie bracket of the infinitesimal Lie algebra.

5.9.2 The exponential map

The exponential of the Lie algebra $\mathfrak{g}_0$ can be obtained by solving a system of ordinary differential equations since $\gamma(t) = \exp(tY)$ is equivalent to $Y = (L_{\gamma(t)^{-1}})_* \dot{\gamma}$, where L_* denotes the differential of the left translation of G_2.

The ODE $Y = (L_{\gamma(t)^{-1}})_* \dot{\gamma}$ is easily programmed and solved in Mathematica as long as the dimensions of the dimension of the Hopf algebra is not to large. To speed up Mathematica computations, one might want to split the system of differential equations into partially decoupled subsystems, like in our worked example (see Appendix B).

5.9.3 The adjoint and coadjoint representations on G_2

The computations for the coadjoint representation Ad^* are needed to define the law group on the Lie group associated to the double Lie algebra.

The adjoint action Ad on the group G_2 is obtained by differentiating $C_g(h) = ghg^{-1}$. The coadjoint action Ad^* is the given by $\mathrm{Ad}^*(g) = (\mathrm{Ad}(g^{-1}))^t$. Differentiating Ad and Ad^*, one gets ad and ad^*, respectively. All these computation can be carried out by Mathematica. Notice that the matrix block in Ad obtained by removing the last row and the last column (i.e the column corresponding to the grading element Z_0) is lower triangular and the similar block corresponding to Ad^* is upper triangular.

5.9.4 The double Lie algebra and its associated group

The computation to find all families of Lie bialgebra structures on $\mathfrak{g}_0$ can carried out in a fashion similar to our worked example. However, as in the thesis it is easiest to choose the zero bialgebra structure (i.e. the abelian Lie algebra on $\mathfrak{g}^*$). The Lie algebra structure of the double Lie algebra δ is then determined. The coadjoint representation Ad^* of G_0 yields the group law on the associated Lie group of the double Lie algebra.

5.9.5 The adjoint representation ad and Ad on δ

The structure constants of δ determines ad. Notice that Z_0^* is still in the kernel of ad_δ. The adjoint representation Ad can be obtained using Mathematica as before by taking the differential of C_g given by $C_g(h) = ghg^{-1}$.

5.9.6 Ad-invariant polynomials, Lax pair equations, Birkhoff decomposition

Closely following the proof of Corollary 5.5.2, one can directly show that $\varphi(a) = C(a_{l+1})^k$ is an Ad_δ-invariant polynomial on δ. The rest of the computation in Sections 5.5-8 can be easily adjusted to our case.

In summary, the techniques of this chapter extend to any finitely generated Hopf algebra with conditions 1), 2), 3) up to the limits of machine computations

It is unclear whether results for the spectral curve (and Lie algebra cohomology) change when we consider other Hopf algebra. In other several cases considered we have computed the spectral curve and have always obtained a reducible spectral curve.

5.9.7 A different approach to find the Lie group law

A different approach to compute the group law on G_0 is given by Baker-Campbell-Hausdorff formula (see [10]). The bracket on the infinitesimal Lie algebra is given by the insertion of (decorated) rooted trees. Using this combinatorial approach, the infinitesimal Lie algebra structure is much easier to find, but finding the group law from the Baker-Campbell-Hausdorff formula seems more difficult than the approach in §5.9.2.

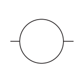
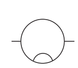
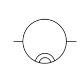
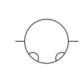
A different approach to find the normal coordinates in 5.9.1 would be a combinatorial method that produces the independent generators B_i' of H_2 with $S(B_i') = -B_i'$.

5.10 Lie algebra cohomology

In this section we compute the Lie algebra cohomology of the infinitesimal Lie algebra of a Hopf algebra generated by a finite number of Feynman graphs. We also discuss the possibility of producing a non-degenerate symmetric bilinear form on a deformation of a semidirect product of the infinitesimal Lie algebra.

The underlying idea is that if we can deform the Lie bracket to one which has a nondegenerate symmetric bilinear form, we can apply our Lax pair technique on the deformed algebra without passing to the double Lie algebra.

From this point of view the motivation for calculating Lie algebra cohomology is that the existence of non-trivial infinitesimal deformation and the obstruction to integrating infinitesimal deformations are controlled by $H^2(\mathfrak{g}, \mathfrak{g})$ and $H^3(\mathfrak{g}, \mathfrak{g})$ respectively. All computations rely on the Mathematica file given in Appendix E.

In this section we consider the Hopf algebra H_3 generated by the empty set and and the following Feynman graphs:  ,  ,  ,  .

Let $\mathfrak{g}_3$ be the infinitesimal Lie algebra of H_3. Notice that $\mathfrak{g}_3$ is a Lie subalgebra of $\mathfrak{g}_1$ introduced in Section 5.1. We denote by Y_1, Y_2, Y_3 and Y_4 the generators of $\mathfrak{g}_2$ introduced in Section 5.1. Let $\mathfrak{g} = \mathfrak{g}_2 \rtimes \mathbb{C}.Z_0$ given by $[Z_0, Y_i] = iY_i$ for $i \in \{1, 2, 3\}$ and $[Z_0, Y_4] = 3Y_4$.

The reason to restrict ourself to a lower dimensional Lie algebra is just for the economy of computations: Mathematica already takes an unreasonable amount of time and computer resources in the five-dimensional case.

We begin by recalling the theory of deformations of Lie algebra following the ideas from [12]. Then we state the results obtained in Appendix E.

Let $\mathfrak{g}$ be a Lie algebra. Let $\mathfrak{g}[[t]]$ be the algebra of power series with coefficients in $\mathfrak{g}$. Let $f : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be the Lie bracket of $\mathfrak{g}$.

Let $f_t : \mathfrak{g}[[t]] \times \mathfrak{g}[[t]] \to \mathfrak{g}[[t]]$ be a bilinear map such that

$$f_t(a,b) = f(a,b) + tF_1(a,b) + t^2 F_2(a,b) + \cdots , \tag{5.10.1}$$

$(\mathfrak{g}[[t]], f_t)$ is called a *deformation of the Lie algebra* $(\mathfrak{g}, f)$ if $(\mathfrak{g}[[t]], f_t)$ is a Lie algebra. We denote by $F_0 = f$.

By the definition of the Lie algebra we get the following:

$$F_\mu(a,b) = -F_\mu(b,a), \tag{5.10.2}$$

$$\sum_{\lambda+\mu=\nu,\lambda,\mu\geq 0} F_\lambda(F_\mu(a,b),c) + F_\lambda(F_\mu(b,c),a) + F_\lambda(F_\mu(c,a),b) = 0. \tag{5.10.3}$$

Let

$$\mathcal{C}^n(\mathfrak{g},\mathfrak{g}) = \{F : V^{\otimes n} \to V \mid B \text{ is a n-linear map}\}.$$

We define the coderivation $\delta^n : \mathcal{C}^n(\mathfrak{g},\mathfrak{g}) \to \mathcal{C}^{n+1}(\mathfrak{g},\mathfrak{g})$ by

$$\begin{aligned}(\delta^n)F(x_0,x_1,\ldots,x_n) &= \sum_{i=0}^{n}(-1)^i[x_i, F(x_0,\ldots,\hat{x}_i,\ldots,x_n)] \\ &+ \sum_{0\leq i<j\leq n}(-1)^{i+j}F([x_i,x_j],x_0,\ldots,\hat{x}_i,\ldots,\hat{x}_j,\ldots,x_n)\end{aligned}$$

Using (5.10.2), can be rewrite (5.10.3) as

$$(\delta^2 F_\nu)(a,b,c) = \sum_{\lambda+\mu=\nu,\lambda,\mu>0} F_\lambda(F_\mu(a,b),c) + F_\lambda(F_\mu(b,c),a) + F_\lambda(F_\mu(c,a),b) \quad (5.10.4)$$

In particular we have $\delta^2 F_1 = 0$ and

$$(\delta^2 F_2)(a,b,c) = F_1(F_1(a,b),c) + F_1(F_1(b,c),a) + F_1(F_1(c,a),b)$$

For any integer $n \geq 0$, we denote by

$$Z^n(\mathfrak{g},\mathfrak{g}) = \text{Ker } \delta^n, \qquad B^n(\mathfrak{g},\mathfrak{g}) = \text{Im } \delta^{n-1}, \qquad H^n(\mathfrak{g},\mathfrak{g}) = Z^n(\mathfrak{g},\mathfrak{g})/B^n(\mathfrak{g},\mathfrak{g}),$$

with the convention $\delta^n = 0$ for $n = -1$.

The right hand side of (5.10.4) is always an element in $Z^3(\mathfrak{g},\mathfrak{g})$ and is called the *obstruction* of F_ν. We recall the following definition.

Definition 5.10.1. A deformation $(\mathfrak{g}[[t]], f_t)$ of a Lie algebra $(\mathfrak{g}, f)$ is called *trivial* if there exist the linear maps $\varphi_i : \mathfrak{g}[[t]] \to \mathfrak{g}[[t]]$, $i \geq 1$ such that $\phi_t : \mathfrak{g}[[t]] \to \mathfrak{g}[[t]]$ given by

$$\phi_t(a) = a + t\varphi_1(a) + t^2\varphi_2(a) + \cdots$$

is an linear isomorphism satisfying

$$f_t(a,b) = \psi_t^{-1}([\psi_t(a), \psi_t(b)]).$$

Proposition 5.10.2. *If $H^2(\mathfrak{g},\mathfrak{g}) = 0$ then any deformation of $\mathfrak{g}$ is trivial.*

Proposition 5.10.3. *If $H^3(\mathfrak{g},\mathfrak{g}) = 0$ then the obstructions can be resolved, i.e. the relation* (5.10.4) *can solved for F_ν.*

The significance of the previous proposition is that we can recursively choose

$F_1,\ldots,F_\nu$. Namely, since Ker $\delta^3 = \text{Im } \delta^2$ and the right hand side of equation (5.10.4) is in Ker δ_3, we can pick a F_ν such that $\delta_2 F_\nu$ is equal to the right hand side of equation (5.10.4). Thus we can find a deformation f_t as in (5.10.1).

In Appendix E, for our choice of five-dimensional Lie algebra $\mathfrak{g}$, we get the following results:

Theorem 5.10.4. *$\dim H^2(\mathfrak{g}, \mathfrak{g}) = 3$.*

Theorem 5.10.5. *$H^3(\mathfrak{g}, \mathfrak{g}) = 0$.*

For notational convenience, let

$$F_\lambda \circ F_\mu = F_\lambda(F_\mu(a,b),c) + F_\lambda(F_\mu(b,c),a) + F_\lambda(F_\mu(c,a),b)$$

Since we have too many choices for F_ν, we take the simplest choose

$$F_3 = F_4 = \ldots = 0. \tag{5.10.5}$$

Then the equations (5.10.3) become

$$\delta_2 F_2 = F_1 \circ F_1,\ 0 = F_1 \circ F_2 + F_2 \circ F_1 \text{ and } 0 = F_2 \circ F_2.$$

This gives the quadratic deformations of our Lie algebra. In particular we can additionally choose $F_2 = 0$ to obtain an affine deformation of our Lie algebra. Then one can find the ad-invariant bilinear nondegenerate symmetric form on the deformed Lie algebra. Even for the five dimensional algebra H_3 of Mathematica takes an unreasonable time. This is the main reason we abandoned this approach.

Interesting questions and future directions are:

- How does the Lie algebra deformations of the five dimensional Lie algebra or of the double Lie algebra relate to deformations of the initial Hopf algebra?

- How does the Lie algebra cohomology relates to the Hopf algebra cohomology?

5.11 Spectral curve approach

Remark 5.11.1. To any Lax equation with a spectral parameter, one can associate a spectral curve and study its algebro-geometric properties (see [20]). In our case, we consider the adjoint representation ad : $\delta \to \mathfrak{g}l(\delta)$ and the induced adjoint representation of the loop algebra. The spectral curve is given by the characteristic equation of ad$(L\lambda)$:

$$\Gamma_0 = \{(\lambda, \nu) \in \mathbb{C} - \{0\} \times \mathbb{C} \mid \det(\mathrm{ad}(L(\lambda) - \nu \mathrm{I}d) = 0\}.$$

The theory of the spectral curve and its Jacobian usually assumes that the spectral curve is irreducible. Unfortunately, for all 43 families of Lie bialgebra structures on δ, on the associated twelve-dimensional Lie algebra ad δ all eigenvalues of the characteristic equation are zero, and the zero eigenspace is eight dimensional. The spectral curve itself is the union of degree one curves. Thus each irreducible component has a trivial Jacobian, and the spectral curve theory breaks down.

Our solutions in §5.7 for the Lax pair flow for our example are relatively simple, in that they are polynomials in z. For general Lax pair equations solvable by spectral curve techniques, the solutions involve transcendental functions such as Baker-Akhiezer functions, which are a type of theta function on the Jacobian of the spectral curve [20]. Since our spectral curve is the union of degree one curves, the associated Jacobians are just points, so there are no theta functions available.

We expect that the spectral curve is reducible for all examples of finitely generated Hopf algebras. Looking at Lemma 5.2.2 and Proposition 5.4.2, we see that the spectral curve is always reducible to a product of degree one curves. Therefore spectral curve techniques do not seem promising.

Chapter 6

Appendix A

In this chapter we solve explicitly the system given by conditions a) and b) in Definition 3.6.1. We prove Proposition 5.3.1 and Remark 5.3.2 from section §5.3, namely we establish the following result.

number of solutions	number of linear relations	degrees of freedom
1	82	8
7	83	7
16	84	6
13	85	5
5	86	4
1	87	3

Let $w[i, j, l]$ with $i, j, k\{1, \cdots, 6\}$, be the structure constants of the six-dimensional Lie algebra $\mathfrak{g}_0$.

$\boldsymbol{c = \textbf{Array}[w, \{6, 6, 6\}];}$

$\boldsymbol{\textbf{Do}[w[i, j, k] = 0, \{i, 6\}, \{j, 6\}, \{k, 6\}]; w[1, 2, 4] = 2;}$

$\boldsymbol{w[2, 1, 4] = -2;}$

$\boldsymbol{\textbf{Do}[w[6, i, i] = i, \{i, 3\}]; \textbf{Do}[w[i, 6, i] = -i, \{i, 3\}];}$

$\boldsymbol{w[6, 4, 4] = 3; w[4, 6, 4] = -3; w[6, 5, 5] = 4; w[5, 6, 5] = -4;}$

Let g be the Lie bialgebra structure, i.e. $g[i,j,k]$ constants of $\mathfrak{g}_0^*$. To deal with condition a) of Definition 3.6.1, i.e. Jacobi identity, we write it in coordinates. First, we define

$$\mathbf{jacobicond}[i,j,k,l]Y_l = [[Y_i,Y_j],Y_k]] + [[Y_j,Y_k],Y_i]] + [[Y_k,Y_i],Y_j]]$$

ga = Array[g, {6, 6, 6}];

Do[mmm = Sort[{i1, i2}];

g[i1, i2, i3] = Signature[{i1, i2}]g[mmm[[1]], mmm[[2]], i3],{i1, 1, 6}, {i2, 1, 6}, {i3, 1, 6}]

jacobicond = Array[jac, {6, 6, 6, 6}];

Do[jac[i, j, k, s] = Sum[g[i, j, l]g[l, k, s], {l, 1, 6}]+ Sum[g[j, k, l]g[l, i, s], {l, 1, 6}]+

Sum[g[k, i, l]g[l, j, s], {l, 1, 6}], {i, 1, 6}, {j, 1, 6},{k, 1, 6}, {s, 1, 6}]

For the second condition b) of Definition 3.6.1, we define **adcond[i,j,k,l]**.

adcond = Array[adc, {6, 6, 6, 6}];

Do[adc[i, j, m, q] = Sum[g[l, q, j]c[[i, l, m]] + g[m, l, j]c[[i, l, q]] − g[l, q, i]c[[j, l, m]]

−g[m, l, i]c[[j, l, q]] − g[m, q, l]c[[i, j, l]], {l, 1, 6}], {i, 1, 6}, {j, 1, 6}, {m, 1, 6}, {q, 1, 6}]

Conditions a) and b) are equivalent to the following system.

Solve[jacobicond == 0&&adcond == 0]

Solve :: svars: Equations may not give solutions for all solve variables. More...

$\{\{g[3,4,3] \to 0, g[2,5,5] \to 0, g[3,4,4] \to 0, g[2,5,2] \to 0, g[1,4,1] \to 0, g[4,5,5] \to$ $0,$

$g[1,2,1] \to 0, g[3,5,5] \to 0, g[2,4,2] \to 0, g[1,3,1] \to 0, g[3,5,3] \to 0, g[1,5,1] \to 0,$

$g[1,5,5] \to 0, g[1,2,2] \to 0, g[2,3,2] \to 0, g[2,4,4] \to 0, g[1,3,3] \to 0, g[4,6,6] \to 0,$

$g[2,3,3] \to 0, g[3,6,6] \to 0, g[4,5,4] \to 0, g[5,6,5] \to 0, g[1,4,4] \to 0, g[1,6,1] \to 0,$

$g[3,6,3] \to 0, g[5,6,6] \to 0, g[2,6,6] \to 0, g[2,6,2] \to 0, g[2,4,6] \to 0, g[4,6,4] \to 0,$

$g[1,6,6] \to 0, g[2,5,6] \to 0, g[1,3,6] \to 2g[3,4,2], g[3,4,1] \to 0, g[2,4,1] \to 0,$

$g[3,4,5] \to 0, g[2,3,6] \to 0, g[2,4,5] \to 0, g[2,4,3] \to 0, g[4,5,3] \to 0, g[1,4,3] \to 0,$

$g[4,5,1] \to 0, g[1,4,2] \to 0, g[2,5,1] \to 0, g[1,4,5] \to 0, g[2,3,1] \to 0,$

$g[4,5,2] \to -\frac{2}{5}g[1,5,6], g[2,3,5] \to 0, g[3,5,1] \to 0, g[3,5,2] \to 0, g[1,2,6] \to 0,$

$g[3,5,4] \to 0, g[1,3,5] \to 0, g[1,5,2] \to 0, g[1,3,2] \to 0, g[2,5,3] \to 0, g[3,6,5] \to 0,$

$g[4,6,3] \to 0, g[4,6,1] \to 0, g[2,6,1] \to 0, g[1,5,3] \to 0, g[2,3,4] \to 0, g[4,6,5] \to 0,$

$g[1,2,5] \to 0, g[2,6,5] \to 0, g[1,2,3] \to 0, g[3,6,1] \to 0, g[3,6,4] \to 0, g[2,5,4] \to 0,$

$g[4,6,2] \to 0, g[1,3,4] \to 0, g[5,6,3] \to 0, g[2,6,3] \to 0, g[5,6,1] \to 0,$

$g[1,6,2] \to 0, g[1,6,5] \to 0, g[1,2,4] \to 0, g[3,6,2] \to 0, g[2,6,4] \to 0,$

$g[1,6,3] \to 0, g[5,6,2] \to 0, g[5,6,4] \to 0, g[1,5,4] \to 0, g[1,6,4] \to 0\},$

$\{g[3,4,3] \to 0, g[2,5,5] \to 0, g[3,4,4] \to 0, g[2,5,2] \to 0, g[1,4,1] \to 0,$

$g[4,5,5] \to 0, g[1,2,1] \to 0, g[3,5,5] \to 0, g[2,4,2] \to 0, g[1,3,1] \to 0, g[3,5,3] \to 0,$

$g[1,5,1] \to 0, g[1,5,5] \to 0, g[1,2,2] \to 0, g[2,3,2] \to 0, g[2,4,4] \to 0, g[1,3,3] \to 0,$

$g[4,6,6] \to 0, g[2,3,3] \to 0, g[3,6,6] \to 0, g[4,5,4] \to 0, g[5,6,5] \to 0, g[1,4,4] \to 0,$

$g[1,6,1] \to 0, g[3,6,3] \to 0, g[5,6,6] \to 0, g[2,6,6] \to 0, g[2,6,2] \to 0,$

$g[4,6,4] \to 0, g[1,4,6] \to 0, g[1,6,6] \to 0, g[2,5,6] \to 3g[4,5,1], g[1,3,6] \to 0,$

$g[3,4,1] \to -\frac{2}{5}g[2,3,6], g[2,4,1] \to 0, g[3,4,2] \to 0, g[3,4,5] \to 0, g[2,4,5] \to 0,$

$g[2,4,3] \to 0, g[4,5,3] \to 0, g[1,4,3] \to 0, g[1,4,2] \to 0, g[2,5,1] \to 0, g[1,4,5] \to 0,$

$g[2,3,1] \to 0, g[4,5,2] \to 0, g[2,3,5] \to 0, g[1,5,6] \to 0, g[3,5,1] \to 0, g[3,5,2] \to 0,$

$g[1,2,6] \to 0, g[3,5,4] \to 0, g[1,3,5] \to 0, g[1,5,2] \to 0, g[1,3,2] \to 0, g[2,5,3] \to 0,$

$g[3,6,5] \to 0, g[4,6,3] \to 0, g[4,6,1] \to 0, g[2,6,1] \to 0, g[1,5,3] \to 0, g[2,3,4] \to 0,$

$g[4,6,5] \to 0, g[1,2,5] \to 0, g[2,6,5] \to 0, g[1,2,3] \to 0, g[3,6,1] \to 0, g[3,6,4] \to 0,$

$g[2,5,4] \to 0, g[4,6,2] \to 0, g[1,3,4] \to 0, g[5,6,3] \to 0, g[2,6,3] \to 0, g[5,6,1] \to 0,$

$g[1,6,2] \to 0, g[1,6,5] \to 0, g[1,2,4] \to 0, g[3,6,2] \to 0, g[2,6,4] \to 0,$

$g[1,6,3] \to 0, g[5,6,2] \to 0, g[5,6,4] \to 0, g[1,5,4] \to 0, g[1,6,4] \to 0\},$

$\cdots$ another 40 solutions here (the Mathematica output is shrunk) $\cdots$

$\{g[3,4,3] \to 0, g[2,5,5] \to 0, g[3,4,4] \to 0, g[2,5,2] \to \frac{1}{2}g[5,6,6], g[1,4,1] \to 0,$

$g[4,5,5] \to 0, g[1,2,1] \to 0, g[3,5,5] \to 0, g[2,4,2] \to 0, g[1,3,1] \to 0,$

$g[3,5,3] \to \frac{3}{4}g[5,6,6], g[1,5,1] \to \frac{1}{4}g[5,6,6], g[1,5,5] \to 0, g[1,2,2] \to 0,$

$g[2,3,2] \to 0, g[2,4,4] \to 0, g[1,3,3] \to 0, g[4,6,6] \to 0, g[3,4,6] \to 0, g[2,3,3] \to 0,$

$g[3,6,6] \to 0, g[4,5,4] \to \frac{3}{4}g[5,6,6], g[5,6,5] \to 0, g[1,4,4] \to 0, g[1,6,1] \to 0,$

$g[3,6,3] \to 0, g[2,6,6] \to 0, g[2,6,2] \to 0, g[2,4,6] \to 0, g[4,6,4] \to 0, g[1,4,6] \to 0,$

$g[1,6,6] \to 0, g[2,5,6] \to 0, g[1,3,6] \to 0, g[3,4,1] \to 0, g[2,4,1] \to 0, g[3,4,2] \to 0,$

$g[3,4,5] \to 0, g[2,3,6] \to 0, g[2,4,5] \to 0, g[2,4,3] \to 0, g[4,5,3] \to 0,$

$g[1,4,3] \to 0, g[4,5,1] \to 0, g[1,4,2] \to 0, g[2,5,1] \to 0, g[1,4,5] \to 0,$

$g[2,3,1] \to 0, g[4,5,2] \to 0, g[2,3,5] \to 0, g[1,5,6] \to 0, g[3,5,1] \to 0,$

$g[3,5,2] \to 0, g[1,2,6] \to 0, g[3,5,4] \to 0, g[1,3,5] \to 0, g[1,5,2] \to 0,$

$g[1,3,2] \to 0, g[2,5,3] \to 0, g[3,6,5] \to 0, g[4,6,3] \to 0, g[4,6,1] \to 0,$

$g[2,6,1] \to 0, g[1,5,3] \to 0, g[2,3,4] \to 0, g[4,6,5] \to 0, g[1,2,5] \to 0,$

$g[2,6,5] \to 0, g[1,2,3] \to 0, g[3,6,1] \to 0, g[3,6,4] \to 0, g[2,5,4] \to 0,$

$g[4,6,2] \to 0, g[1,3,4] \to 0, g[5,6,3] \to 0, g[2,6,3] \to 0, g[5,6,1] \to 0,$

$g[1,6,2] \to 0, g[1,6,5] \to 0, g[1,2,4] \to 0, g[3,6,2] \to 0, g[2,6,4] \to 0,$

$g[1,6,3] \to 0, g[5,6,2] \to 0, g[5,6,4] \to 0, g[1,5,4] \to 0, g[1,6,4] \to 0\}\}$

Chapter 7

Appendix B

In this chapter, we give the Mathematica file for the computations from sections 5.5 and 5.7. All Mathematica code is shown in **bold** fonts, all comments are in *italic*, while the Mathematica output is given in roman fonts.

First, we define the group law on G_0.

```
f[{{x1_, x2_, x3_, x4_, x5_, t_}, {y1_, y2_, y3_, y4_, y5_, y6_}}]:=
{x1 + Exp[t]y1, x2 + Exp[2t]y2, x3 + Exp[3t]y3,
x4 + Exp[3t]y4 + Exp[2t]x1y2 − Exp[t]x2y1, x5 + Exp[4t]y5, t + y6}
```

We find the inverse of an element.

```
Solve[f[{{x1, x2, x3, x4, x5, t}, {y1, y2, y3, y4, y5, y6}}] == 0, {y1, y2, y3, y4, y5, y6}]
```

$\{\{\mathrm{y4} \to -e^{-3t}\mathrm{x4}, \mathrm{y3} \to -e^{-3t}\mathrm{x3}, \mathrm{y5} \to -e^{-4t}\mathrm{x5}, \mathrm{y6} \to -t, \mathrm{y1} \to -e^{-t}\mathrm{x1},$
$\mathrm{y2} \to -e^{-2t}\mathrm{x2}\}\}$

```
inverseoff[{x1_, x2_, x3_, x4_, x5_, t_}]:={y1, y2, y3, y4, y5, y6}/.
{y4 → −e^(−3t)x4, y3 → −e^(−3t)x3, y5 → −e^(−4t)x5, y6 → −t, y1 → −e^(−t)x1,
y2 → −e^(−2t)x2}
inverseoff[{x1, x2, x3, x4, x5, t}]
```

$\{-e^{-t}\mathrm{x1}, -e^{-2t}\mathrm{x2}, -e^{-3t}\mathrm{x3}, -e^{-3t}\mathrm{x4}, -e^{-4t}\mathrm{x5}, -t\}$

Define the conjugation C_g *as* Conj, *i.e.* $C_g(h) = ghg^{-1}$.

```
Conj[{{x1_, x2_, x3_, x4_, x5_, t_}, {y1_, y2_, y3_, y4_, y5_, y6_}}]:=
f[{f[{{x1, x2, x3, x4, x5, t}, {y1, y2, y3, y4, y5, y6}}],
inverseoff[{x1, x2, x3, x4, x5, t}]}]
Conj[{{x1, x2, x3, x4, x5, t}, {y1, y2, y3, y4, y5, y6}}]
```

$\{\mathrm{x1} - e^{\mathrm{y6}}\mathrm{x1} + e^{t}\mathrm{y1}, \mathrm{x2} - e^{-2t+2(t+\mathrm{y6})}\mathrm{x2} + e^{2t}\mathrm{y2}, \mathrm{x3} - e^{-3t+3(t+\mathrm{y6})}\mathrm{x3} + e^{3t}\mathrm{y3},$

$\mathrm{x4} - e^{-3t+3(t+\mathrm{y6})}\mathrm{x4} - e^{t}\mathrm{x2y1} - e^{-2t+2(t+\mathrm{y6})}\mathrm{x2}\left(\mathrm{x1} + e^{t}\mathrm{y1}\right) + e^{2t}\mathrm{x1y2} +$

$e^{\mathrm{y6}}\mathrm{x1}\left(\mathrm{x2} + e^{2t}\mathrm{y2}\right) + e^{3t}\mathrm{y4}, \mathrm{x5} - e^{-4t+4(t+\mathrm{y6})}\mathrm{x5} + e^{4t}\mathrm{y5}, \mathrm{y6}\}$

Now we take the differential of C_g.

Jac = Array[d, {6, 6}];

Do[[i, 1] = Expand[D[Conj[{{x1, x2, x3, x4, x5, t}, {y1, y2, y3, y4, y5, y6}}][[i]], y1]], {i, 1, 6}]

Do[[i, 2] = Expand[D[Conj[{{x1, x2, x3, x4, x5, t}, {y1, y2, y3, y4, y5, y6}}][[i]], y2]], {i, 1, 6}]

Do[[i, 3] = Expand[D[Conj[{{x1, x2, x3, x4, x5, t}, {y1, y2, y3, y4, y5, y6}}][[i]], y3]], {i, 1, 6}]

Do[[i, 4] = Expand[D[Conj[{{x1, x2, x3, x4, x5, t}, {y1, y2, y3, y4, y5, y6}}][[i]], y4]], {i, 1, 6}]

Do[[i, 5] = Expand[D[Conj[{{x1, x2, x3, x4, x5, t}, {y1, y2, y3, y4, y5, y6}}][[i]], y5]], {i, 1, 6}]

Do[[i, 6] = Expand[D[Conj[{{x1, x2, x3, x4, x5, t}, {y1, y2, y3, y4, y5, y6}}][[i]], y6]], {i, 1, 6}]

Expand[Jac]

$\{\{e^{t}, 0, 0, 0, 0, -e^{\mathrm{y6}}\mathrm{x1}\}, \{0, e^{2t}, 0, 0, 0, -2e^{-2t+2(t+\mathrm{y6})}\mathrm{x2}\},$

$\{0, 0, e^{3t}, 0, 0, -3e^{-3t+3(t+\mathrm{y6})}\mathrm{x3}\}, \{-e^{t}\mathrm{x2} - e^{-t+2(t+\mathrm{y6})}\mathrm{x2}, e^{2t}\mathrm{x1} + e^{2t+\mathrm{y6}}\mathrm{x1}, 0, e^{3t},$

$0, e^{\mathrm{y6}}\mathrm{x1x2} - 2e^{-2t+2(t+\mathrm{y6})}\mathrm{x1x2} - 3e^{-3t+3(t+\mathrm{y6})}\mathrm{x4} - 2e^{-t+2(t+\mathrm{y6})}\mathrm{x2y1} + e^{2t+\mathrm{y6}}\mathrm{x1y2}\},$

$\{0, 0, 0, 0, e^{4t}, -4e^{-4t+4(t+\mathrm{y6})}\mathrm{x5}\}, \{0, 0, 0, 0, 0, 1\}\}$

The adjoint representation Ad *is given below.*

Adjoint = Array[adj, {6, 6}];

CoAdjoint = Array[coadj, {6, 6}];

Adjoint = Jac/.{y1 → 0, y2 → 0, y3 → 0, y4 → 0, y5 → 0, y6 → 0};

Adjoint

$\{\{e^t, 0, 0, 0, 0, -\text{x1}\}, \{0, e^{2t}, 0, 0, 0, -2\text{x2}\},$

$\{0, 0, e^{3t}, 0, 0, -3\text{x3}\}, \{-2e^t\text{x2}, 2e^{2t}\text{x1}, 0, e^{3t}, 0, -\text{x1x2} - 3\text{x4}\},$

$\{0, 0, 0, 0, e^{4t}, -4\text{x5}\}, \{0, 0, 0, 0, 0, 1\}\}$

Now we prepare for the computations of ad.

littleadoff[{x1_, x2_, x3_, x4_, x5_, t_}] =

$\{\{e^t, 0, 0, 0, 0, -\text{x1}\}, \{0, e^{2t}, 0, 0, 0, -2\text{x2}\}, \{0, 0, e^{3t}, 0, 0, -3\text{x3}\},$

$\{-2e^t\text{x2}, 2e^{2t}\text{x1}, 0, e^{3t}, 0, -\text{x1x2} - 3\text{x4}\}, \{0, 0, 0, 0, e^{4t}, -4\text{x5}\}, \{0, 0, 0, 0, 0, 1\}\};$

af1 = D[littleadoff[{x1, x2, x3, x4, x5, t}], x1]/.

$\{\text{x1} \to 0, \text{x2} \to 0, \text{x3} \to 0, \text{x4} \to 0, \text{x5} \to 0, t \to 0\};$

af2 = D[littleadoff[{x1, x2, x3, x4, x5, t}], x2]/.

$\{\text{x1} \to 0, \text{x2} \to 0, \text{x3} \to 0, \text{x4} \to 0, \text{x5} \to 0, t \to 0\};$

af3 = D[littleadoff[{x1, x2, x3, x4, x5, t}], x3]/.

$\{\text{x1} \to 0, \text{x2} \to 0, \text{x3} \to 0, \text{x4} \to 0, \text{x5} \to 0, t \to 0\};$

af4 = D[littleadoff[{x1, x2, x3, x4, x5, t}], x4]/.

$\{\text{x1} \to 0, \text{x2} \to 0, \text{x3} \to 0, \text{x4} \to 0, \text{x5} \to 0, t \to 0\};$

af5 = D[littleadoff[{x1, x2, x3, x4, x5, t}], x5]/.

$\{\text{x1} \to 0, \text{x2} \to 0, \text{x3} \to 0, \text{x4} \to 0, \text{x5} \to 0, t \to 0\};$

af6 = D[littleadoff[{x1, x2, x3, x4, x5, t}], t]/.

$\{\text{x1} \to 0, \text{x2} \to 0, \text{x3} \to 0, \text{x4} \to 0, \text{x5} \to 0, t \to 0\};$

littleadjointoff = x1af1 + x2af2 + x3af3 + x4af4 + x5af5 + x6af6;

The adjoint representation ad on the 6-dimensional Lie algebra is given below.

MatrixForm[littleadjointoff]

$$\begin{pmatrix} x6 & 0 & 0 & 0 & 0 & -x1 \\ 0 & 2x6 & 0 & 0 & 0 & -2x2 \\ 0 & 0 & 3x6 & 0 & 0 & -3x3 \\ -2x2 & 2x1 & 0 & 3x6 & 0 & -3x4 \\ 0 & 0 & 0 & 0 & 4x6 & -4x5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

CoAdjoint =

Transpose[Jac]/. {x1 → −e^{−t}x1, x2 → −e^{−2t}x2, x3 → −e^{−3t}x3, x4 → −e^{−3t}x4,

x5 → −e^{−4t}x5, t → −t, y1 → 0, y2 → 0, y3 → 0, y4 → 0, y5 → 0, y6 → 0};

MatrixForm[Adjoint]

$$\begin{pmatrix} e^t & 0 & 0 & 0 & 0 & -x1 \\ 0 & e^{2t} & 0 & 0 & 0 & -2x2 \\ 0 & 0 & e^{3t} & 0 & 0 & -3x3 \\ -2e^t x2 & 2e^{2t} x1 & 0 & e^{3t} & 0 & -x1x2 - 3x4 \\ 0 & 0 & 0 & 0 & e^{4t} & -4x5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

MatrixForm[CoAdjoint]

$$\begin{pmatrix} e^{-t} & 0 & 0 & 2e^{-3t}x2 & 0 & 0 \\ 0 & e^{-2t} & 0 & -2e^{-3t}x1 & 0 & 0 \\ 0 & 0 & e^{-3t} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-3t} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-4t} & 0 \\ e^{-t}x1 & 2e^{-2t}x2 & 3e^{-3t}x3 & -e^{-3t}x1x2 + 3e^{-3t}x4 & 4e^{-4t}x5 & 1 \end{pmatrix}$$

CoAdjoint

$\{\{e^{-t}, 0, 0, 2e^{-3t}x2, 0, 0\}, \{0, e^{-2t}, 0, -2e^{-3t}x1, 0, 0\},$

$\{0, 0, e^{-3t}, 0, 0, 0\}, \{0, 0, 0, e^{-3t}, 0, 0\}, \{0, 0, 0, 0, e^{-4t}, 0\},$

$\{e^{-t}\text{x1}, 2e^{-2t}\text{x2}, 3e^{-3t}\text{x3}, -e^{-3t}\text{x1x2} + 3e^{-3t}\text{x4}, 4e^{-4t}\text{x5}, 1\}\}$

Now, we define the law on $\tilde{G}$, then we compute the exponential.

g[{{x1_, x2_, x3_, x4_, x5_, t_, x7_, x8_, x9_, x10_, x11_, x12_},

{y1_, y2_, y3_, y4_, y5_, y6_, y7_, y8_, y9_, y10_, y11_, y12_}}]:=

Join[{x1 + Exp[t]y1, x2 + Exp[2t]y2, x3 + Exp[3t]y3,

x4 + Exp[3t]y4 + Exp[2t]x1y2 − Exp[t]x2y1, x5 + Exp[4t]y5, t + y6},

Expand[{x7, x8, x9, x10, x11, x12}+

$\{\{e^{-t}, 0, 0, 2e^{-3t}\text{x2}, 0, 0\}, \{0, e^{-2t}, 0, -2e^{-3t}\text{x1}, 0, 0\},$

$\{0, 0, e^{-3t}, 0, 0, 0\}, \{0, 0, 0, e^{-3t}, 0, 0\}, \{0, 0, 0, 0, e^{-4t}, 0\},$

$\{e^{-t}\text{x1}, 2e^{-2t}\text{x2}, 3e^{-3t}\text{x3}, -e^{-3t}\text{x1x2} + 3e^{-3t}\text{x4}, 4e^{-4t}\text{x5}, 1\}\}.$

{y7, y8, y9, y10, y11, y12}]]

g[{{x1, x2, x3, x4, x5, t, x7, x8, x9, x10, x11, x12},

{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}]

$\{\text{x1} + e^{t}\text{y1}, \text{x2} + e^{2t}\text{y2}, \text{x3} + e^{3t}\text{y3}, \text{x4} - e^{t}\text{x2y1} + e^{2t}\text{x1y2} + e^{3t}\text{y4},$

$\text{x5} + e^{4t}\text{y5}, t + \text{y6}, \text{x7} + 2e^{-3t}\text{x2y10} + e^{-t}\text{y7}, \text{x8} - 2e^{-3t}\text{x1y10} + e^{-2t}\text{y8},$

$\text{x9} + e^{-3t}\text{y9}, \text{x10} + e^{-3t}\text{y10}, \text{x11} + e^{-4t}\text{y11}, \text{x12} - e^{-3t}\text{x1x2y10} +$

$3e^{-3t}\text{x4y10} + 4e^{-4t}\text{x5y11} + \text{y12} + e^{-t}\text{x1y7} + 2e^{-2t}\text{x2y8} + 3e^{-3t}\text{x3y9}\}$

Solve[g[{{x1, x2, x3, x4, x5, t, x7, x8, x9, x10, x11, x12},

{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}] == 0,

{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}]

$\{\{\text{y12} \to -\text{x12} + \text{x1x10x2} + 3\text{x10x4} + 4\text{x11x5} + \text{x1x7} + 2\text{x2x8} + 3\text{x3x9},$

$\text{y3} \to -e^{-3t}\text{x3}, \text{y4} \to -e^{-3t}\text{x4}, \text{y5} \to -e^{-4t}\text{x5}, \text{y6} \to -t, \text{y1} \to -e^{-t}\text{x1},$

$\text{y2} \to -e^{-2t}\text{x2}, \text{y7} \to e^{t}(2\text{x10x2} - \text{x7}), \text{y8} \to -e^{2t}(2\text{x1x10} + \text{x8}),$

$\text{y9} \to -e^{3t}\text{x9}, \text{y11} \to -e^{4t}\text{x11}, \text{y10} \to -e^{3t}\text{x10}\}\}$

inverseofg[{x1_, x2_, x3_, x4_, x5_, t_, x7_, x8_, x9_, x10_, x11_, x12_}]:=

{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}/.

$\{y12 \to -x12 + x1x10x2 + 3x10x4 + 4x11x5 + x1x7 + 2x2x8 + 3x3x9,$

$y3 \to -e^{-3t}x3, y4 \to -e^{-3t}x4, y5 \to -e^{-4t}x5, y6 \to -t, y1 \to -e^{-t}x1,$

$y2 \to -e^{-2t}x2, y7 \to e^{t}(2x10x2 - x7), y8 \to -e^{2t}(2x1x10 + x8), y9 \to -e^{3t}x9,$

$y11 \to -e^{4t}x11, y10 \to -e^{3t}x10\}$

inverseofg[{x1, x2, x3, x4, x5, *t*, x7, x8, x9, x10, x11, x12}]

$\{-e^{-t}x1, -e^{-2t}x2, -e^{-3t}x3, -e^{-3t}x4, -e^{-4t}x5, -t,$

$e^{t}(2x10x2 - x7), -e^{2t}(2x1x10 + x8), -e^{3t}x9, -e^{3t}x10, -e^{4t}x11,$

$- x12 + x1x10x2 + 3x10x4 + 4x11x5 + x1x7 + 2x2x8 + 3x3x9\}$

We define the conjugate $C_g(h) = ghg^{-1}$ *as* Conjofg.

Conjofg[{{x1_, x2_, x3_, x4_, x5_, t_, x7_, x8_, x9_, x10_, x11_, x12_},

{y1_, y2_, y3_, y4_, y5_, y6_, y7_, y8_, y9_, y10_, y11_, y12_}}]:=

***g*[{*g*[{{x1, x2, x3, x4, x5, *t*, x7, x8, x9, x10, x11, x12},**

{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}],

inverseofg[{x1, x2, x3, x4, x5, *t*, x7, x8, x9, x10, x11, x12}]}]

Jacofg = Array[dofg, {12, 12}];

leftstar = Array[lstar, {12, 12}];

Do[lstar[*i*, 1] = *D*[*g*[{{x1, x2, x3, x4, x5, *t*, x7, x8, x9, x10, x11, x12},

{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}][[*i*]], y1], {*i*, 1, 12}]

Do[lstar[*i*, 2] = *D*[*g*[{{x1, x2, x3, x4, x5, *t*, x7, x8, x9, x10, x11, x12},

{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}][[*i*]], y2], {*i*, 1, 12}]

Do[lstar[*i*, 3] = *D*[*g*[{{x1, x2, x3, x4, x5, *t*, x7, x8, x9, x10, x11, x12},

{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}][[*i*]], y3], {*i*, 1, 12}]

Do[lstar[*i*, 4] = *D*[*g*[{{x1, x2, x3, x4, x5, *t*, x7, x8, x9, x10, x11, x12},

{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}][[*i*]], y4], {*i*, 1, 12}]

Do[lstar[*i*, 5] = *D*[*g*[{{x1, x2, x3, x4, x5, *t*, x7, x8, x9, x10, x11, x12},

{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}][[*i*]], y5], {*i*, 1, 12}]

Do[lstar[i, 6] = D[g[{{x1, x2, x3, x4, x5, t, x7, x8, x9, x10, x11, x12},
{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}][[i]], y6], {i, 1, 12}]
Do[lstar[i, 7] = D[g[{{x1, x2, x3, x4, x5, t, x7, x8, x9, x10, x11, x12},
{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}][[i]], y7], {i, 1, 12}]
Do[lstar[i, 8] = D[g[{{x1, x2, x3, x4, x5, t, x7, x8, x9, x10, x11, x12},
{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}][[i]], y8], {i, 1, 12}]
Do[lstar[i, 9] = D[g[{{x1, x2, x3, x4, x5, t, x7, x8, x9, x10, x11, x12},
{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}][[i]], y9], {i, 1, 12}]
Do[lstar[i, 10] = D[g[{{x1, x2, x3, x4, x5, t, x7, x8, x9, x10, x11, x12},
{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}][[i]], y10], {i, 1, 12}]
Do[lstar[i, 11] = D[g[{{x1, x2, x3, x4, x5, t, x7, x8, x9, x10, x11, x12},
{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}][[i]], y11], {i, 1, 12}]
Do[lstar[i, 12] = D[g[{{x1, x2, x3, x4, x5, t, x7, x8, x9, x10, x11, x12},
{y1, y2, y3, y4, y5, y6, y7, y8, y9, y10, y11, y12}}][[i]], y12], {i, 1, 12}]
leftstar

$\{\{e^t, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$
$\{0, e^{2t}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, e^{3t}, 0, 0, 0, 0, 0, 0, 0, 0, 0\},$
$\{-e^t\mathrm{x2}, e^{2t}\mathrm{x1}, 0, e^{3t}, 0, 0, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, e^{4t}, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, e^{-t}, 0, 0, 2e^{-3t}\mathrm{x2}, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0, e^{-2t}, 0, -2e^{-3t}\mathrm{x1}, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0, 0, e^{-3t}, 0, 0, 0\},$
$\{0, 0, 0, 0, 0, 0, 0, 0, 0, e^{-3t}, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, e^{-4t}, 0\},$
$\{0, 0, 0, 0, 0, 0, e^{-t}\mathrm{x1}, 2e^{-2t}\mathrm{x2}, 3e^{-3t}\mathrm{x3}, -e^{-3t}\mathrm{x1x2} + 3e^{-3t}\mathrm{x4}, 4e^{-4t}\mathrm{x5}, 1\}\}$

$\boldsymbol{\{\{e^t, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, e^{2t}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\},}$

$\{0,0,e^{3t},0,0,0,0,0,0,0,0,0\}$,

$\{-e^{t}\text{x2},e^{2t}\text{x1},0,e^{3t},0,0,0,0,0,0,0,0\}$,

$\{0,0,0,0,e^{4t},0,0,0,0,0,0,0\}$, $\{0,0,0,0,0,1,0,0,0,0,0,0\}$,

$\{0,0,0,0,0,0,e^{-t},0,0,2e^{-3t}\text{x2},0,0\}$,

$\{0,0,0,0,0,0,0,e^{-2t},0,-2e^{-3t}\text{x1},0,0\}$,

$\{0,0,0,0,0,0,0,0,e^{-3t},0,0,0\}$, $\{0,0,0,0,0,0,0,0,0,e^{-3t},0,0\}$,

$\{0,0,0,0,0,0,0,0,0,0,e^{-4t},0\}$,

$\{0,0,0,0,0,0,e^{-t}\text{x1},2e^{-2t}\text{x2},3e^{-3t}\text{x3},-e^{-3t}\text{x1x2}+3e^{-3t}\text{x4},4e^{-4t}\text{x5},1\}\}$ /.

$\{\text{x1}\to -e^{-t}\text{x1},\text{x2}\to -e^{-2t}\text{x2},\text{x3}\to -e^{-3t}\text{x3},\text{x4}\to -e^{-3t}\text{x4},\text{x5}\to -e^{-4t}\text{x5},$

$t\to -t,\text{x7}\to e^{t}(2\text{x10x2}-\text{x7}),\text{x8}\to -e^{2t}(2\text{x1x10}+\text{x8}),\text{x9}\to -e^{3t}\text{x9},$

$\text{x10}\to -e^{3t}\text{x10},\text{x11}\to -e^{4t}\text{x11},$

$\text{x12}\to -\text{x12}+\text{x1x10x2}+3\text{x10x4}+4\text{x11x5}+\text{x1x7}+2\text{x2x8}+3\text{x3x9}\}$

$\{\{e^{-t},0,0,0,0,0,0,0,0,0,0,0\}$,

$\{0,e^{-2t},0,0,0,0,0,0,0,0,0,0\}$, $\{0,0,e^{-3t},0,0,0,0,0,0,0,0,0\}$,

$\{e^{-3t}\text{x2},-e^{-3t}\text{x1},0,e^{-3t},0,0,0,0,0,0,0,0\}$,

$\{0,0,0,0,e^{-4t},0,0,0,0,0,0,0\}$, $\{0,0,0,0,0,1,0,0,0,0,0,0\}$,

$\{0,0,0,0,0,0,e^{t},0,0,-2e^{t}\text{x2},0,0\}$,

$\{0,0,0,0,0,0,0,e^{2t},0,2e^{2t}\text{x1},0,0\}$, $\{0,0,0,0,0,0,0,0,e^{3t},0,0,0\}$,

$\{0,0,0,0,0,0,0,0,0,e^{3t},0,0\}$, $\{0,0,0,0,0,0,0,0,0,0,e^{4t},0\}$,

$\{0,0,0,0,0,0,-\text{x1},-2\text{x2},-3\text{x3},-\text{x1x2}-3\text{x4},-4\text{x5},1\}\}$

$\text{Leftl}=\{\{e^{-t},0,0,0,0,0,0,0,0,0,0,0\}$,

$\{0,e^{-2t},0,0,0,0,0,0,0,0,0,0\}$, $\{0,0,e^{-3t},0,0,0,0,0,0,0,0,0\}$,

$\{e^{-3t}\text{x2},-e^{-3t}\text{x1},0,e^{-3t},0,0,0,0,0,0,0,0\}$,

$\{0,0,0,0,e^{-4t},0,0,0,0,0,0,0\}$, $\{0,0,0,0,0,1,0,0,0,0,0,0\}$,

$\{0,0,0,0,0,0,e^{t},0,0,-2e^{t}\text{x2},0,0\}$,

$\{0,0,0,0,0,0,0,e^{2t},0,2e^{2t}\text{x1},0,0\}$, $\{0,0,0,0,0,0,0,0,e^{3t},0,0,0\}$,

$\{0,0,0,0,0,0,0,0,0,e^{3t},0,0\},\{0,0,0,0,0,0,0,0,0,0,e^{4t},0\},$

$\{0,0,0,0,0,0,-\mathrm{x1},-2\mathrm{x2},-3\mathrm{x3},-\mathrm{x1x2}-3\mathrm{x4},-4\mathrm{x5},1\}\}$

$\{\{e^{-t},0,0,0,0,0,0,0,0,0,0,0\},$

$\{0,e^{-2t},0,0,0,0,0,0,0,0,0,0\},\{0,0,e^{-3t},0,0,0,0,0,0,0,0,0\},$

$\{e^{-3t}\mathrm{x2},-e^{-3t}\mathrm{x1},0,e^{-3t},0,0,0,0,0,0,0,0\},$

$\{0,0,0,0,e^{-4t},0,0,0,0,0,0,0\},\{0,0,0,0,0,1,0,0,0,0,0,0\},$

$\{0,0,0,0,0,0,e^{t},0,0,-2e^{t}\mathrm{x2},0,0\},$

$\{0,0,0,0,0,0,0,e^{2t},0,2e^{2t}\mathrm{x1},0,0\},\{0,0,0,0,0,0,0,0,e^{3t},0,0,0\},$

$\{0,0,0,0,0,0,0,0,0,e^{3t},0,0\},\{0,0,0,0,0,0,0,0,0,0,e^{4t},0\},$

$\{0,0,0,0,0,0,-\mathrm{x1},-2\mathrm{x2},-3\mathrm{x3},-\mathrm{x1x2}-3\mathrm{x4},-4\mathrm{x5},1\}\}$

$\{\{e^{-t},0,0,0,0,0,0,0,0,0,0,0\},\{0,e^{-2t},0,0,0,0,0,0,0,0,0,0\},$

$\{0,0,e^{-3t},0,0,0,0,0,0,0,0,0\},$

$\{e^{-3t}\mathrm{x2},-e^{-3t}\mathrm{x1},0,e^{-3t},0,0,0,0,0,0,0,0\},$

$\{0,0,0,0,e^{-4t},0,0,0,0,0,0,0\},\{0,0,0,0,0,1,0,0,0,0,0,0\},$

$\{0,0,0,0,0,0,e^{t},0,0,-2e^{t}\mathrm{x2},0,0\},$

$\{0,0,0,0,0,0,0,e^{2t},0,2e^{2t}\mathrm{x1},0,0\},\{0,0,0,0,0,0,0,0,e^{3t},0,0,0\},$

$\{0,0,0,0,0,0,0,0,0,e^{3t},0,0\},\{0,0,0,0,0,0,0,0,0,0,e^{4t},0\},$

$\{0,0,0,0,0,0,-\mathrm{x1},-2\mathrm{x2},-3\mathrm{x3},-\mathrm{x1x2}-3\mathrm{x4},-4\mathrm{x5},1\}\}.$

$\{\mathrm{y1},\mathrm{y2},\mathrm{y3},\mathrm{y4},\mathrm{y5},\mathrm{y6},\mathrm{y7},\mathrm{y8},\mathrm{y9},\mathrm{y10},\mathrm{y11},\mathrm{y12}\}$

$\{e^{-t}\mathrm{y1},e^{-2t}\mathrm{y2},e^{-3t}\mathrm{y3},e^{-3t}\mathrm{x2y1}-e^{-3t}\mathrm{x1y2}+e^{-3t}\mathrm{y4},$

$e^{-4t}\mathrm{y5},\mathrm{y6},-2e^{t}\mathrm{x2y10}+e^{t}\mathrm{y7},2e^{2t}\mathrm{x1y10}+e^{2t}\mathrm{y8},e^{3t}\mathrm{y9},e^{3t}\mathrm{y10},$

$e^{4t}\mathrm{y11},(-\mathrm{x1x2}-3\mathrm{x4})\mathrm{y10}-4\mathrm{x5y11}+\mathrm{y12}-\mathrm{x1y7}-2\mathrm{x2y8}-3\mathrm{x3y9}\}$

Fl[{{x1_, x2_, x3_, x4_, x5_, t_, x7_, x8_, x9_, x10_, x11_, x12_},

{y1_, y2_, y3_, y4_, y5_, y6_, y7_, y8_, y9_, y10_, y11_, y12_}}]:=

$\{e^{-t}\mathrm{y1},e^{-2t}\mathrm{y2},e^{-3t}\mathrm{y3},e^{-3t}\mathrm{x2y1}-e^{-3t}\mathrm{x1y2}+e^{-3t}\mathrm{y4},e^{-4t}\mathrm{y5},\mathrm{y6},$

$-2e^{t}\mathrm{x2y10}+e^{t}\mathrm{y7},2e^{2t}\mathrm{x1y10}+e^{2t}\mathrm{y8},e^{3t}\mathrm{y9},e^{3t}\mathrm{y10},e^{4t}\mathrm{y11},$

$(-\text{x1x2} - 3\text{x4})\text{y10} - 4\text{x5y11} + \text{y12} - \text{x1y7} - 2\text{x2y8} - 3\text{x3y9}\}$

b1[t_]:=g1[t];

b2[t_]:=g2[t];

b3[t_]:=g3[t];

b4[t_]:=g4[t];

b5[t_]:=g5[t];

b6[t_]:=g6[t];

b7[t_]:=g7[t];

b8[t_]:=g8[t];

b9[t_]:=g9[t];

b10[t_]:=g10[t];

b11[t_]:=g11[t];

b12[t_]:=g12[t];

Evaluate[Fl[{{b1[t], b2[t], b3[t], b4[t], b5[t], b6[t], b7[t], b8[t], b9[t], b10[t], b11[t], b12[t]}, {b1′[t], b2′[t], b3′[t], b4′[t], b5′[t], b6′[t], b7′[t], b8′[t], b9′[t], b10′[t], b11′[t], b12′[t]}}]]

$\{e^{-\text{g6}[t]}\text{g1}'[t], e^{-2\text{g6}[t]}\text{g2}'[t], e^{-3\text{g6}[t]}\text{g3}'[t],$
$e^{-3\text{g6}[t]}\text{g2}[t]\text{g1}'[t] - e^{-3\text{g6}[t]}\text{g1}[t]\text{g2}'[t] + e^{-3\text{g6}[t]}\text{g4}'[t], e^{-4\text{g6}[t]}\text{g5}'[t], \text{g6}'[t],$
$-2e^{\text{g6}[t]}\text{g2}[t]\text{g10}'[t] + e^{\text{g6}[t]}\text{g7}'[t], 2e^{2\text{g6}[t]}\text{g1}[t]\text{g10}'[t] + e^{2\text{g6}[t]}\text{g8}'[t],$
$e^{3\text{g6}[t]}\text{g9}'[t], e^{3\text{g6}[t]}\text{g10}'[t], e^{4\text{g6}[t]}\text{g11}'[t], (-\text{g1}[t]\text{g2}[t] - 3\text{g4}[t])\text{g10}'[t] -$
$4\text{g5}[t]\text{g11}'[t] + \text{g12}'[t] - \text{g1}[t]\text{g7}'[t] - 2\text{g2}[t]\text{g8}'[t] - 3\text{g3}[t]\text{g9}'[t]\}$

g6[t_]:=tz6

To obtain the exponential, we first solve a system of differential equation which is a subsystem of the system of ODE that gives the exponential.

DSolve [{$e^{-t\text{z6}}$g1′[t] == z1, $e^{-2t\text{z6}}$g2′[t] == z2, $e^{-3t\text{z6}}$g3′[t] == z3, $e^{-4t\text{z6}}$g5′[t] == z5,

$-2e^{tz6}g2[t]g10'[t]+e^{tz6}g7'[t]==z7, 2e^{2tz6}g1[t]g10'[t]+e^{2tz6}g8'[t]==z8,$

$e^{3tz6}g9'[t]==z9, e^{3tz6}g10'[t]==z10, e^{4tz6}g11'[t]==z11\}$,

$\{g1[t], g2[t], g3[t], g5[t], g7[t], g8[t], g9[t], g10[t], g11[t]\}, t]$

$\Big\{\Big\{g8[t]\to -\frac{e^{-2tz6}(-2z1z10+z6z8)}{2z6^2}+\frac{2e^{-3tz6}z10C[1]}{3z6}+C[2],$

$g1[t]\to \frac{e^{tz6}z1}{z6}+C[1], g2[t]\to \frac{e^{2tz6}z2}{2z6}+C[3],$

$g7[t]\to -\frac{e^{-tz6}(z10z2+z6z7)}{z6^2}-\frac{2e^{-3tz6}z10C[3]}{3z6}+C[4],$

$g10[t]\to -\frac{e^{-3tz6}z10}{3z6}+C[5], g3[t]\to \frac{e^{3tz6}z3}{3z6}+C[6], g5[t]\to \frac{e^{4tz6}z5}{4z6}+C[7],$

$g9[t]\to -\frac{e^{-3tz6}z9}{3z6}+C[8], g11[t]\to -\frac{e^{-4tz6}z11}{4z6}+C[9]\Big\}\Big\}$

We take the solutions of the previous system and then we deal with the remaining two equations.

$g8[t\_]:=-\frac{(e^{-2tz6}-1)(-2z1z10+z6z8)}{2z6^2}-\frac{2(e^{-3tz6}-1)z10z1}{3z6^2};$

$g1[t\_]:=\frac{(e^{tz6}-1)z1}{z6}$

$g2[t\_]:=\frac{(e^{2tz6}-1)z2}{2z6};$

$g7[t\_]:=-\frac{(e^{-tz6}-1)(z10z2+z6z7)}{z6^2}+\frac{2(e^{-3tz6}-1)z10z2}{6z6^2};$

$g10[t\_]:=-\frac{(e^{-3tz6}-1)z10}{3z6};$

$g3[t\_]:=\frac{(e^{3tz6}-1)z3}{3z6};$

$g5[t\_]:=\frac{(e^{4tz6}-1)z5}{4z6};$

$g9[t\_]:=-\frac{(e^{-3tz6}-1)z9}{3z6};$

$g11[t\_]:=-\frac{(e^{-4tz6}-1)z11}{4z6};$

DSolve $[\{e^{-3z6t}g2[t]g1'[t]-e^{-3z6t}g1[t]g2'[t]+e^{-3z6t}g4'[t]==z4,$

$(-g1[t]g2[t]-3g4[t])g10'[t]-4g5[t]g11'[t]+g12'[t]-g1[t]g7'[t]-$

$2g2[t]g8'[t]-3g3[t]g9'[t]==z12\}, \{g4[t], g12[t]\}, t]$

$\Big\{\Big\{g4[t]\to \frac{e^{tz6}\left(\left(3-3e^{tz6}+e^{2tz6}\right)z1z2+2e^{2tz6}z4z6\right)}{6z6^2}+C[1],$

$g12[t]\to \frac{1}{12z6^3}\,(e^{-4tz6}\,(3z11z5z6+12e^{3tz6}z1(z10z2+z6z7)-$

$6e^{2tz6}z2(2z1z10-z6z8)+2e^{tz6}(z1z10z2+2z3z6z9)-$

$2e^{4tz6}\left(z1\left(z10z2-6tz6^2z7\right)-2z6(z10z4(-1+3tz6)+3tz6\right.$

$(\mathrm{z11z5}+\mathrm{z12z6}+\mathrm{z2z8}+\mathrm{z3z9})))))-\frac{e^{-3tz6}\mathrm{z10}C[1]}{\mathrm{z6}}+C[2]\}\}$

DSolve $[\{e^{-3z6t}\mathrm{g2}[t]\mathrm{g1}'[t]-e^{-3z6t}\mathrm{g1}[t]\mathrm{g2}'[t]+e^{-3z6t}\mathrm{g4}'[t]==\mathrm{z4}\},\{\mathrm{g4}[t]\},t]$

$$\left\{\left\{\mathrm{g4}[t]\to\frac{e^{tz6}\left(\left(3-3e^{tz6}+e^{2tz6}\right)\mathrm{z1z2}+2e^{2tz6}\mathrm{z4z6}\right)}{6\mathrm{z6}^2}+C[1]\right\}\right\}$$

$$\frac{e^{tz6}\left(\left(3-3e^{tz6}+e^{2tz6}\right)\mathrm{z1z2}+2e^{2tz6}\mathrm{z4z6}\right)}{6\mathrm{z6}^2}/.\{t\to0\}$$

$$\frac{\mathrm{z1z2}+2\mathrm{z4z6}}{6\mathrm{z6}^2}$$

$\mathrm{g4}[\mathrm{t\_}]:=\frac{e^{tz6}\left(\left(3-3e^{tz6}+e^{2tz6}\right)\mathrm{z1z2}+2e^{2tz6}\mathrm{z4z6}\right)}{6\mathrm{z6}^2}-\frac{\mathrm{z1z2}+2\mathrm{z4z6}}{6\mathrm{z6}^2};$

DSolve $[\{(-\mathrm{g1}[t]\mathrm{g2}[t]-3\mathrm{g4}[t])\mathrm{g10}'[t]-4\mathrm{g5}[t]\mathrm{g11}'[t]+\mathrm{g12}'[t]-\mathrm{g1}[t]\mathrm{g7}'[t]-$

$2\mathrm{g2}[t]\mathrm{g8}'[t]-3\mathrm{g3}[t]\mathrm{g9}'[t]==\mathrm{z12}\},\{\mathrm{g12}[t]\},t]$

$\{\{\mathrm{g12}[t]\to\frac{1}{\mathrm{z6}^2}\left(\frac{1}{4}e^{-4tz6}\mathrm{z11z5}+\frac{e^{-tz6}\mathrm{z1}(\mathrm{z10z2}+\mathrm{z6z7})}{\mathrm{z6}}+\frac{e^{-2tz6}\mathrm{z2}(-2\mathrm{z1z10}+\mathrm{z6z8})}{2\mathrm{z6}}+\right.$

$tz6(\mathrm{z10z4}+\mathrm{z11z5}+\mathrm{z12z6}+\mathrm{z1z7}+\mathrm{z2z8}+\mathrm{z3z9})+$

$\left.\frac{e^{-3tz6}(\mathrm{z1z10z2}+\mathrm{z10z4z6}+\mathrm{z3z6z9})}{3\mathrm{z6}}\right)+C[1]\}\}$

Expand $[\frac{1}{\mathrm{z6}^2}\left(\frac{1}{4}e^{-4tz6}\mathrm{z11z5}+\frac{e^{-tz6}\mathrm{z1}(\mathrm{z10z2}+\mathrm{z6z7})}{\mathrm{z6}}+\frac{e^{-2tz6}\mathrm{z2}(-2\mathrm{z1z10}+\mathrm{z6z8})}{2\mathrm{z6}}+\right.$

$tz6(\mathrm{z10z4}+\mathrm{z11z5}+\mathrm{z12z6}+\mathrm{z1z7}+\mathrm{z2z8}+\mathrm{z3z9})+$

$\left.\frac{e^{-3tz6}(\mathrm{z1z10z2}+\mathrm{z10z4z6}+\mathrm{z3z6z9})}{3\mathrm{z6}}\right)/.t\to0]$

$$\frac{\mathrm{z1z10z2}}{3\mathrm{z6}^3}+\frac{\mathrm{z10z4}}{3\mathrm{z6}^2}+\frac{\mathrm{z11z5}}{4\mathrm{z6}^2}+\frac{\mathrm{z1z7}}{\mathrm{z6}^2}+\frac{\mathrm{z2z8}}{2\mathrm{z6}^2}+\frac{\mathrm{z3z9}}{3\mathrm{z6}^2}$$

Expand $[\frac{1}{\mathrm{z6}^2}\left(\frac{1}{4}e^{-4tz6}\mathrm{z11z5}+\frac{e^{-tz6}\mathrm{z1}(\mathrm{z10z2}+\mathrm{z6z7})}{\mathrm{z6}}+\frac{e^{-2tz6}\mathrm{z2}(-2\mathrm{z1z10}+\mathrm{z6z8})}{2\mathrm{z6}}+\right.$

$\left.tz6(\mathrm{z10z4}+\mathrm{z11z5}+\mathrm{z12z6}+\mathrm{z1z7}+\mathrm{z2z8}+\mathrm{z3z9})+\frac{e^{-3tz6}(\mathrm{z1z10z2}+\mathrm{z10z4z6}+\mathrm{z3z6z9})}{3\mathrm{z6}}\right)-$

$\left(\frac{\mathrm{z1z10z2}}{3\mathrm{z6}^3}+\frac{\mathrm{z10z4}}{3\mathrm{z6}^2}+\frac{\mathrm{z11z5}}{4\mathrm{z6}^2}+\frac{\mathrm{z1z7}}{\mathrm{z6}^2}+\frac{\mathrm{z2z8}}{2\mathrm{z6}^2}+\frac{\mathrm{z3z9}}{3\mathrm{z6}^2}\right)]$

$t\mathrm{z12}-\frac{\mathrm{z1z10z2}}{3\mathrm{z6}^3}+\frac{e^{-3tz6}\mathrm{z1z10z2}}{3\mathrm{z6}^3}-\frac{e^{-2tz6}\mathrm{z1z10z2}}{\mathrm{z6}^3}+\frac{e^{-tz6}\mathrm{z1z10z2}}{\mathrm{z6}^3}-\frac{\mathrm{z10z4}}{3\mathrm{z6}^2}+$

$\frac{e^{-3tz6}\mathrm{z10z4}}{3\mathrm{z6}^2}-\frac{\mathrm{z11z5}}{4\mathrm{z6}^2}+\frac{e^{-4tz6}\mathrm{z11z5}}{4\mathrm{z6}^2}+\frac{t\mathrm{z10z4}}{\mathrm{z6}}+\frac{t\mathrm{z11z5}}{\mathrm{z6}}-\frac{\mathrm{z1z7}}{\mathrm{z6}^2}+\frac{e^{-tz6}\mathrm{z1z7}}{\mathrm{z6}^2}+$

$\frac{t\mathrm{z1z7}}{\mathrm{z6}}-\frac{\mathrm{z2z8}}{2\mathrm{z6}^2}+\frac{e^{-2tz6}\mathrm{z2z8}}{2\mathrm{z6}^2}+\frac{t\mathrm{z2z8}}{\mathrm{z6}}-\frac{\mathrm{z3z9}}{3\mathrm{z6}^2}+\frac{e^{-3tz6}\mathrm{z3z9}}{3\mathrm{z6}^2}+\frac{t\mathrm{z3z9}}{\mathrm{z6}}$

$\mathrm{g12}[\mathrm{t\_}]:=t\mathrm{z12}-\frac{\mathrm{z1z10z2}}{3\mathrm{z6}^3}+\frac{e^{-3tz6}\mathrm{z1z10z2}}{3\mathrm{z6}^3}-\frac{e^{-2tz6}\mathrm{z1z10z2}}{\mathrm{z6}^3}+$

$\frac{e^{-tz6}\mathrm{z1z10z2}}{\mathrm{z6}^3}-\frac{\mathrm{z10z4}}{3\mathrm{z6}^2}+\frac{e^{-3tz6}\mathrm{z10z4}}{3\mathrm{z6}^2}-\frac{\mathrm{z11z5}}{4\mathrm{z6}^2}+\frac{e^{-4tz6}\mathrm{z11z5}}{4\mathrm{z6}^2}+\frac{t\mathrm{z10z4}}{\mathrm{z6}}+$

$$\mathbf{\frac{tz11z5}{z6} - \frac{z1z7}{z6^2} + \frac{e^{-tz6}z1z7}{z6^2} + \frac{tz1z7}{z6} - \frac{z2z8}{2z6^2} + \frac{e^{-2tz6}z2z8}{2z6^2} + \frac{tz2z8}{z6} - \frac{z3z9}{3z6^2} + \frac{e^{-3tz6}z3z9}{3z6^2} + \frac{tz3z9}{z6}}$$

Finally, we get the exponential.

{g1[t], g2[t], g3[t], g4[t], g5[t], z6t, g7[t], g8[t], g9[t], g10[t], g11[t], g12[t]}

$$\left\{\frac{(-1+e^{tz6})z1}{z6}, \frac{(-1+e^{2tz6})z2}{2z6}, \frac{(-1+e^{3tz6})z3}{3z6}, -\frac{z1z2+2z4z6}{6z6^2} + \frac{e^{tz6}\left((3-3e^{tz6}+e^{2tz6})z1z2+2e^{2tz6}z4z6\right)}{6z6^2}, \frac{(-1+e^{4tz6})z5}{4z6}, tz6, \frac{(-1+e^{-3tz6})z10z2}{3z6^2} - \frac{(-1+e^{-tz6})(z10z2+z6z7)}{z6^2}, -\frac{2(-1+e^{-3tz6})z1z10}{3z6^2} - \frac{(-1+e^{-2tz6})(-2z1z10+z6z8)}{2z6^2}, -\frac{(-1+e^{-3tz6})z9}{3z6}, -\frac{(-1+e^{-3tz6})z10}{3z6}, -\frac{(-1+e^{-4tz6})z11}{4z6}, tz12 - \frac{z1z10z2}{3z6^3} + \frac{e^{-3tz6}z1z10z2}{3z6^3} - \frac{e^{-2tz6}z1z10z2}{z6^3} + \frac{e^{-tz6}z1z10z2}{z6^3} - \frac{z10z4}{3z6^2} + \frac{e^{-3tz6}z10z4}{3z6^2} - \frac{z11z5}{4z6^2} + \frac{e^{-4tz6}z11z5}{4z6^2} + \frac{tz10z4}{z6} + \frac{tz11z5}{z6} - \frac{z1z7}{z6^2} + \frac{e^{-tz6}z1z7}{z6^2} + \frac{tz1z7}{z6} - \frac{z2z8}{2z6^2} + \frac{e^{-2tz6}z2z8}{2z6^2} + \frac{tz2z8}{z6} - \frac{z3z9}{3z6^2} + \frac{e^{-3tz6}z3z9}{3z6^2} + \frac{tz3z9}{z6}\right\}$$

Expand[g12′[t]]

$$z12 - \frac{e^{-3tz6}z1z10z2}{z6^2} + \frac{2e^{-2tz6}z1z10z2}{z6^2} - \frac{e^{-tz6}z1z10z2}{z6^2} + \frac{z10z4}{z6} - \frac{e^{-3tz6}z10z4}{z6} + \frac{z11z5}{z6} - \frac{e^{-4tz6}z11z5}{z6} + \frac{z1z7}{z6} - \frac{e^{-tz6}z1z7}{z6} + \frac{z2z8}{z6} - \frac{e^{-2tz6}z2z8}{z6} + \frac{z3z9}{z6} - \frac{e^{-3tz6}z3z9}{z6}$$

We also need to find the exponential when $z_6 = 0$. We take the limit at $z_6 = 0$ of previous map.

$$\mathbf{Limit[\left\{\frac{(-1+e^{tz6})z1}{z6}, \frac{(-1+e^{2tz6})z2}{2z6}, \frac{(-1+e^{3tz6})z3}{3z6}, -\frac{z1z2+2z4z6}{6z6^2} + \frac{e^{tz6}\left((3-3e^{tz6}+e^{2tz6})z1z2+2e^{2tz6}z4z6\right)}{6z6^2}, \frac{(-1+e^{4tz6})z5}{4z6}, tz6, \frac{(-1+e^{-3tz6})z10z2}{3z6^2} - \frac{(-1+e^{-tz6})(z10z2+z6z7)}{z6^2}, -\frac{2(-1+e^{-3tz6})z1z10}{3z6^2} - \frac{(-1+e^{-2tz6})(-2z1z10+z6z8)}{2z6^2}, -\frac{(-1+e^{-3tz6})z9}{3z6}, -\frac{(-1+e^{-3tz6})z10}{3z6}, -\frac{(-1+e^{-4tz6})z11}{4z6},}$$

$$\mathbf{tz12 - \frac{z1z10z2}{3z6^3} + \frac{e^{-3tz6}z1z10z2}{3z6^3} - \frac{e^{-2tz6}z1z10z2}{z6^3} + \frac{e^{-tz6}z1z10z2}{z6^3} - \frac{z10z4}{3z6^2} + \frac{e^{-3tz6}z10z4}{3z6^2} - \frac{z11z5}{4z6^2} + \frac{e^{-4tz6}z11z5}{4z6^2} + \frac{tz10z4}{z6} + \frac{tz11z5}{z6} - \frac{z1z7}{z6^2} + \frac{e^{-tz6}z1z7}{z6^2} + \frac{tz1z7}{z6} - \frac{z2z8}{2z6^2} + \frac{e^{-2tz6}z2z8}{2z6^2} + \frac{tz2z8}{z6} - \frac{z3z9}{3z6^2} +}$$

$$\frac{e^{-3tz6}\text{z3z9}}{3\text{z6}^2}+\frac{t\text{z3z9}}{\text{z6}}\Big\}, \text{z6}\to 0\Big]$$

$$\{t\text{z1}, t\text{z2}, t\text{z3}, t\text{z4}, t\text{z5}, 0, t(t\text{z10z2}+\text{z7}), t(-t\text{z1z10}+\text{z8}), t\text{z9}, t\text{z10}, t\text{z11},$$
$$\tfrac{1}{6}t(6\text{z12}+t(-2t\text{z1z10z2}+9\text{z10z4}+12\text{z11z5}+3\text{z1z7}+6\text{z2z8}+9\text{z3z9}))\}$$

Expand[{tz1, tz2, tz3, tz4, tz5, 0, t(tz10z2 + z7), t(−tz1z10 + z8), tz9, tz10, tz11,
$\frac{1}{6}t$(6z12 + t(−2tz1z10z2 + 9z10z4 + 12z11z5 + 3z1z7 + 6z2z8 + 9z3z9))}]

$$\{t\text{z1}, t\text{z2}, t\text{z3}, t\text{z4}, t\text{z5}, 0, t^2\text{z10z2}+t\text{z7}, -t^2\text{z1z10}+t\text{z8}, t\text{z9}, t\text{z10}, t\text{z11},$$
$$t\text{z12}-\tfrac{1}{3}t^3\text{z1z10z2}+\tfrac{3}{2}t^2\text{z10z4}+2t^2\text{z11z5}+\tfrac{1}{2}t^2\text{z1z7}+t^2\text{z2z8}+\tfrac{3}{2}t^2\text{z3z9}\}$$

{tz1, tz2, tz3, tz4, tz5, 0, t^2z10z2 + tz7, −t^2z1z10 + tz8, tz9,
tz10, tz11, tz12 − $\frac{1}{3}t^3$z1z10z2 + $\frac{3}{2}t^2$z10z4 + 2t^2z11z5 + $\frac{1}{2}t^2$z1z7+
t^2z2z8 + $\frac{3}{2}t^2$z3z9} /.{t → 1}

$$\{\text{z1}, \text{z2}, \text{z3}, \text{z4}, \text{z5}, 0, \text{z10z2}+\text{z7}, -\text{z1z10}+\text{z8}, \text{z9}, \text{z10},$$
$$\text{z11}, \text{z12}-\frac{\text{z1z10z2}}{3}+\frac{3\text{z10z4}}{2}+2\text{z11z5}+\frac{\text{z1z7}}{2}+\text{z2z8}+\frac{3\text{z3z9}}{2}\}$$

$$\Big\{\frac{(-1+e^{tz6})\text{z1}}{\text{z6}}, \frac{(-1+e^{2tz6})\text{z2}}{2\text{z6}}, \frac{(-1+e^{3tz6})\text{z3}}{3\text{z6}},$$
$$-\frac{\text{z1z2}+2\text{z4z6}}{6\text{z6}^2}+\frac{e^{tz6}\left((3-3e^{tz6}+e^{2tz6})\text{z1z2}+2e^{2tz6}\text{z4z6}\right)}{6\text{z6}^2},$$
$$\frac{(-1+e^{4tz6})\text{z5}}{4\text{z6}}, t\text{z6}, \frac{(-1+e^{-3tz6})\text{z10z2}}{3\text{z6}^2}-\frac{(-1+e^{-tz6})(\text{z10z2}+\text{z6z7})}{\text{z6}^2},$$
$$-\frac{2(-1+e^{-3tz6})\text{z1z10}}{3\text{z6}^2}-\frac{(-1+e^{-2tz6})(-2\text{z1z10}+\text{z6z8})}{2\text{z6}^2}, -\frac{(-1+e^{-3tz6})\text{z9}}{3\text{z6}},$$
$$-\frac{(-1+e^{-3tz6})\text{z10}}{3\text{z6}}, -\frac{(-1+e^{-4tz6})\text{z11}}{4\text{z6}},$$
$$t\text{z12}-\frac{\text{z1z10z2}}{3\text{z6}^3}+\frac{e^{-3tz6}\text{z1z10z2}}{3\text{z6}^3}-\frac{e^{-2tz6}\text{z1z10z2}}{\text{z6}^3}+\frac{e^{-tz6}\text{z1z10z2}}{\text{z6}^3}-$$
$$\frac{\text{z10z4}}{3\text{z6}^2}+\frac{e^{-3tz6}\text{z10z4}}{3\text{z6}^2}-\frac{\text{z11z5}}{4\text{z6}^2}+\frac{e^{-4tz6}\text{z11z5}}{4\text{z6}^2}+\frac{t\text{z10z4}}{\text{z6}}+\frac{t\text{z11z5}}{\text{z6}}-$$
$$\frac{\text{z1z7}}{\text{z6}^2}+\frac{e^{-tz6}\text{z1z7}}{\text{z6}^2}+\frac{t\text{z1z7}}{\text{z6}}-\frac{\text{z2z8}}{2\text{z6}^2}+\frac{e^{-2tz6}\text{z2z8}}{2\text{z6}^2}+\frac{t\text{z2z8}}{\text{z6}}-\frac{\text{z3z9}}{3\text{z6}^2}+$$
$$\frac{e^{-3tz6}\text{z3z9}}{3\text{z6}^2}+\frac{t\text{z3z9}}{\text{z6}}\Big\} /.\{t\to 1\}$$

$$\Big\{\frac{(-1+e^{\text{z6}})\text{z1}}{\text{z6}}, \frac{(-1+e^{2\text{z6}})\text{z2}}{2\text{z6}}, \frac{(-1+e^{3\text{z6}})\text{z3}}{3\text{z6}}, -\frac{\text{z1z2}+2\text{z4z6}}{6\text{z6}^2}+\frac{e^{\text{z6}}\left((3-3e^{\text{z6}}+e^{2\text{z6}})\text{z1z2}+2e^{2\text{z6}}\text{z4z6}\right)}{6\text{z6}^2},$$
$$\frac{(-1+e^{4\text{z6}})\text{z5}}{4\text{z6}}, \text{z6}, \frac{(-1+e^{-3\text{z6}})\text{z10z2}}{3\text{z6}^2}-\frac{(-1+e^{-\text{z6}})(\text{z10z2}+\text{z6z7})}{\text{z6}^2},$$
$$-\frac{2(-1+e^{-3\text{z6}})\text{z1z10}}{3\text{z6}^2}-\frac{(-1+e^{-2\text{z6}})(-2\text{z1z10}+\text{z6z8})}{2\text{z6}^2},$$

$-\frac{\left(-1+e^{-3z6}\right)z9}{3z6}, -\frac{\left(-1+e^{-3z6}\right)z10}{3z6}, -\frac{\left(-1+e^{-4z6}\right)z11}{4z6},$

$z12-\frac{z1z10z2}{3z6^3}+\frac{e^{-3z6}z1z10z2}{3z6^3}-\frac{e^{-2z6}z1z10z2}{z6^3}+\frac{e^{-z6}z1z10z2}{z6^3}-\frac{z10z4}{3z6^2}+$

$\frac{e^{-3z6}z10z4}{3z6^2}-\frac{z11z5}{4z6^2}+\frac{e^{-4z6}z11z5}{4z6^2}+\frac{z10z4}{z6}+\frac{z11z5}{z6}-\frac{z1z7}{z6^2}+\frac{e^{-z6}z1z7}{z6^2}+$

$\frac{z1z7}{z6}-\frac{z2z8}{2z6^2}+\frac{e^{-2z6}z2z8}{2z6^2}+\frac{z2z8}{z6}-\frac{z3z9}{3z6^2}+\frac{e^{-3z6}z3z9}{3z6^2}+\frac{z3z9}{z6}\Big\}$

$\mathbf{\{z1, z2, z3, z4, z5, 0, z10z2 + z7, -z1z10 + z8, z9, z10, z11,}$

$\mathbf{z12 - \frac{z1z10z2}{3} + \frac{3z10z4}{2} + 2z11z5 + \frac{z1z7}{2} + z2z8 + \frac{3z3z9}{2}\}}\ /.$

$\mathbf{\{z7 \to z1, z8 \to z2, z9 \to z3, z10 \to z4, z11 \to z5, z12 \to 0\}}$

$\{z1, z2, z3, z4, z5, 0, z1 + z2z4, z2 - z1z4,$

$z3, z4, z5, \frac{z1^2}{2} + z2^2 + \frac{3z3^2}{2} - \frac{z1z2z4}{3} + \frac{3z4^2}{2} + 2z5^2\Big\}$

$\mathbf{Adjointofg = \{\{e^t, 0, 0, 0, 0, -x1, 0, 0, 0, 0, 0, 0\},}$

$\mathbf{\{0, e^{2t}, 0, 0, 0, -2x2, 0, 0, 0, 0, 0, 0\}, \{0, 0, e^{3t}, 0, 0, -3x3, 0, 0, 0, 0, 0, 0\},}$

$\mathbf{\{-2e^t x2, 2e^{2t} x1, 0, e^{3t}, 0, -x1x2 - 3x4, 0, 0, 0, 0, 0, 0\},}$

$\mathbf{\{0, 0, 0, 0, e^{4t}, -4x5, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0\},}$

$\mathbf{\{0, -2e^{2t} x10, 0, 0, 0, 4x10x2 + x7, e^{-t}, 0, 0, 2e^{-3t} x2, 0, 0\},}$

$\mathbf{\{2e^t x10, 0, 0, 0, 0, -2x1x10 + 2x8, 0, e^{-2t}, 0, -2e^{-3t} x1, 0, 0\},}$

$\mathbf{\{0, 0, 0, 0, 0, 3x9, 0, 0, e^{-3t}, 0, 0, 0\}, \{0, 0, 0, 0, 0, 3x10, 0, 0, 0, e^{-3t}, 0, 0\},}$

$\mathbf{\{0, 0, 0, 0, 0, 4x11, 0, 0, 0, 0, e^{-4t}, 0\},}$

$\mathbf{\{6e^t x10x2 - e^t x7, -6e^{2t} x1x10 - 2e^{2t} x8, -3e^{3t} x9, -3e^{3t} x10, -4e^{4t} x11,}$

$\mathbf{3x1x10x2 + 9x10x4 + 16x11x5 + x1x7 + 4x2x8 + 9x3x9, e^{-t} x1, 2e^{-2t} x2,}$

$\mathbf{3e^{-3t} x3, -e^{-3t} x1x2 + 3e^{-3t} x4, 4e^{-4t} x5, 1\}\};}$

$\mathbf{\{\{e^{-t}, 0, 0, 2e^{-3t} x2, 0, 0, 0, -2e^{2t} x10, 0, 0, 0, 4x10x2 + x7\},}$

$\mathbf{\{0, e^{-2t}, 0, -2e^{-3t} x1, 0, 0, 2e^t x10, 0, 0, 0, 0, -2x1x10 + 2x8\},}$

$\mathbf{\{0, 0, e^{-3t}, 0, 0, 0, 0, 0, 0, 0, 0, 3x9\}, \{0, 0, 0, e^{-3t}, 0, 0, 0, 0, 0, 0, 0, 3x10\},}$

$\mathbf{\{0, 0, 0, 0, e^{-4t}, 0, 0, 0, 0, 0, 0, 4x11\},}$

$\mathbf{\{e^{-t} x1, 2e^{-2t} x2, 3e^{-3t} x3, -e^{-3t} x1x2 + 3e^{-3t} x4, 4e^{-4t} x5, 1,}$

$\mathbf{6e^{t}x10x2 - e^{t}x7, -6e^{2t}x1x10 - 2e^{2t}x8, -3e^{3t}x9, -3e^{3t}x10, -4e^{4t}x11,}$

$\mathbf{3x1x10x2 + 9x10x4 + 16x11x5 + x1x7 + 4x2x8 + 9x3x9\},}$

$\mathbf{\{0, 0, 0, 0, 0, 0, e^{t}, 0, 0, 0, 0, -x1\}, \{0, 0, 0, 0, 0, 0, 0, e^{2t}, 0, 0, 0, -2x2\},}$

$\mathbf{\{0, 0, 0, 0, 0, 0, 0, 0, e^{3t}, 0, 0, -3x3\},}$

$\mathbf{\{0, 0, 0, 0, 0, 0, -2e^{t}x2, 2e^{2t}x1, 0, e^{3t}, 0, -x1x2 - 3x4\},}$

$\mathbf{\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, e^{4t}, -4x5\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}\} /. \{t \to 0\}}$

$\{\{1, 0, 0, 2x2, 0, 0, 0, -2x10, 0, 0, 0, 4x10x2 + x7\},$

$\{0, 1, 0, -2x1, 0, 0, 2x10, 0, 0, 0, 0, -2x1x10 + 2x8\},$

$\{0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 3x9\}, \{0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 3x10\},$

$\{0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 4x11\},$

$\{x1, 2x2, 3x3, -x1x2 + 3x4, 4x5, 1, 6x10x2 - x7, -6x1x10 - 2x8, -3x9,$

$-3x10, -4x11, 3x1x10x2 + 9x10x4 + 16x11x5 + x1x7 + 4x2x8 + 9x3x9\},$

$\{0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -x1\}, \{0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -2x2\},$

$\{0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -3x3\}, \{0, 0, 0, 0, 0, 0, -2x2, 2x1, 0, 1, 0, -x1x2 - 3x4\},$

$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -4x5\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}\}$

$\mathbf{\{\{1, 0, 0, 2x2, 0, 0, 0, -2x10, 0, 0, 0, 4x10x2 + x7\},}$

$\mathbf{\{0, 1, 0, -2x1, 0, 0, 2x10, 0, 0, 0, 0, -2x1x10 + 2x8\},}$

$\mathbf{\{0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 3x9\}, \{0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 3x10\},}$

$\mathbf{\{0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 4x11\},}$

$\mathbf{\{x1, 2x2, 3x3, -x1x2 + 3x4, 4x5, 1, 6x10x2 - x7, -6x1x10 - 2x8,}$

$\mathbf{-3x9, -3x10, -4x11, 3x1x10x2 + 9x10x4 + 16x11x5 + x1x7 + 4x2x8 + 9x3x9\},}$

$\mathbf{\{0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -x1\}, \{0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -2x2\},}$

$\mathbf{\{0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -3x3\}, \{0, 0, 0, 0, 0, 0, -2x2, 2x1, 0, 1, 0, -x1x2 - 3x4\},}$

$\mathbf{\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -4x5\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}\}/.}$

$\mathbf{\{x7 \to y1, x8 \to y2, x9 \to y3, x10 \to y4, x11 \to y5, x12 \to y6\}}$

$\{\{1, 0, 0, 2x2, 0, 0, 0, -2y4, 0, 0, 0, y1 + 4x2y4\},$

$\{0, 1, 0, -2\text{x1}, 0, 0, 2\text{y4}, 0, 0, 0, 0, 2\text{y2} - 2\text{x1y4}\}$,

$\{0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 3\text{y3}\}$,

$\{0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 3\text{y4}\}, \{0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 4\text{y5}\}$,

$\{\text{x1}, 2\text{x2}, 3\text{x3}, -\text{x1x2} + 3\text{x4}, 4\text{x5}, 1, -\text{y1} + 6\text{x2y4}, -2\text{y2} - 6\text{x1y4}, -3\text{y3},$

$-3\text{y4}, -4\text{y5}, \text{x1y1} + 4\text{x2y2} + 9\text{x3y3} + 3\text{x1x2y4} + 9\text{x4y4} + 16\text{x5y5}\}$,

$\{0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -\text{x1}\}, \{0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -2\text{x2}\}$,

$\{0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -3\text{x3}\}$,

$\{0, 0, 0, 0, 0, 0, -2\text{x2}, 2\text{x1}, 0, 1, 0, -\text{x1x2} - 3\text{x4}\}$,

$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -4\text{x5}\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}\}$

We compute $\mathrm{Ad}^*_{G_\delta} a_- \cdot L$. *We take* $L = \{\text{l1}, \text{l2}, \text{l3}, \text{l4}, \text{l5}, \text{l6}, \text{l7}, \text{l8}, \text{l8}, \text{l9}, \text{l10}, \text{l11}, \text{l12}\}$

$\boldsymbol{\{\{1, 0, 0, 2\text{x2}, 0, 0, 0, -2\text{y4}, 0, 0, 0, \text{y1} + 4\text{x2y4}\},}$

$\boldsymbol{\{0, 1, 0, -2\text{x1}, 0, 0, 2\text{y4}, 0, 0, 0, 0, 2\text{y2} - 2\text{x1y4}\},}$

$\boldsymbol{\{0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 3\text{y3}\}, \{0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 3\text{y4}\},}$

$\boldsymbol{\{0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 4\text{y5}\},}$

$\boldsymbol{\{\text{x1}, 2\text{x2}, 3\text{x3}, -\text{x1x2} + 3\text{x4}, 4\text{x5}, 1, -\text{y1} + 6\text{x2y4}, -2\text{y2} - 6\text{x1y4},}$

$\boldsymbol{-3\text{y3}, -3\text{y4}, -4\text{y5}, \text{x1y1} + 4\text{x2y2} + 9\text{x3y3} + 3\text{x1x2y4} + 9\text{x4y4} + 16\text{x5y5}\},}$

$\boldsymbol{\{0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, -\text{x1}\}, \{0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, -2\text{x2}\},}$

$\boldsymbol{\{0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, -3\text{x3}\},}$

$\boldsymbol{\{0, 0, 0, 0, 0, 0, -2\text{x2}, 2\text{x1}, 0, 1, 0, -\text{x1x2} - 3\text{x4}\},}$

$\boldsymbol{\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, -4\text{x5}\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}\}.}$

$\boldsymbol{\{\text{l1}, \text{l2}, \text{l3}, \text{l4}, \text{l5}, \text{l6}, \text{l7}, \text{l8}, \text{l9}, \text{l10}, \text{l11}, \text{l12}\}}$

$\{\text{l1} + 2\text{l4x2} - 2\text{l8y4} + \text{l12}(\text{y1} + 4\text{x2y4}),$

$\text{l2} - 2\text{l4x1} + 2\text{l7y4} + \text{l12}(2\text{y2} - 2\text{x1y4}), \text{l3} + 3\text{l12y3}, \text{l4} + 3\text{l12y4},$

$\text{l5} + 4\text{l12y5}, \text{l6} + \text{l1x1} + 2\text{l2x2} + 3\text{l3x3} + \text{l4}(-\text{x1x2} + 3\text{x4}) +$

$4\text{l5x5} - 3\text{l9y3} - 3\text{l10y4} + \text{l8}(-2\text{y2} - 6\text{x1y4}) + \text{l7}(-\text{y1} + 6\text{x2y4}) -$

$4\text{l11y5} + \text{l12}(\text{x1y1} + 4\text{x2y2} + 9\text{x3y3} + 3\text{x1x2y4} + 9\text{x4y4} + 16\text{x5y5}),$

$$l7 - l12x1, l8 - 2l12x2, l9 - 3l12x3,$$

$$l10 + 2l8x1 - 2l7x2 + l12(-x1x2 - 3x4), l11 - 4l12x5, l12\}$$

Chapter 8

Appendix C

In this chapter we find all ad-invariant bilinear symmetric form on our 6-dimensional Lie algebra. This checks the claim from chapters 4 and 5 that there is no ad-invariant non-degenerate symmetric bilinear form on $\mathfrak{g}_0$.

```
c = Array[w, {6, 6, 6}];
Do[w[i, j, k] = 0, {i, 6}, {j, 6}, {k, 6}]; w[1, 2, 4] = 2; w[2, 1, 4] = −2;
Do[w[6, i, i] = i, {i, 3}];
Do[w[i, 6, i] = −i, {i, 3}];
w[6, 4, 4] = 3; w[6, 5, 5] = 4;
w[4, 6, 4] = −3;
w[5, 6, 5] = −4;

adform = Array[b, {6, 6}];
adcondition = Array[adb, {6, 6, 6}];
Do[adb[i, j, k] = Sum[c[[i, j, l]]b[l, k] − b[i, l]c[[j, k, l]], {l, 1, 6}], {i, 1, 6}, {j, 1, 6},
{k, 1, 6}];

Solve[adcondition == 0]
```

{{b[1, 1] → 0, b[1, 2] → 0, b[1, 3] → 0, b[1, 4] → 0, b[1, 5] → 0, b[1, 6] → 0, b[2, 1] → 0, b[2, 2] → 0, b[2, 3] → 0,

b[2, 4] → 0, b[2, 5] → 0, b[2, 6] → 0, b[3, 1] → 0, b[3, 2] → 0, b[3, 3] → 0, b[3, 4] → 0, b[3, 5] → 0, b[3, 6] → 0,

$b[4,1] \to 0, b[4,2] \to 0, b[4,3] \to 0, b[4,4] \to 0, b[4,5] \to 0, b[4,6] \to 0, b[5,1] \to 0, b[5,2] \to 0, b[5,3] \to 0,$

$b[5,4] \to 0, b[5,5] \to 0, b[5,6] \to 0, b[6,1] \to 0, b[6,2] \to 0, b[6,3] \to 0, b[6,4] \to 0, b[6,5] \to 0\}\}$

MatrixForm[adform]/.

$\{b[1,1] \to 0, b[1,2] \to 0, b[1,3] \to 0, b[1,4] \to 0, b[1,5] \to 0, b[1,6] \to 0, b[2,1] \to 0,$
$b[2,2] \to 0, b[2,3] \to 0, b[2,4] \to 0, b[2,5] \to 0, b[2,6] \to 0, b[3,1] \to 0, b[3,2] \to 0,$
$b[3,3] \to 0, b[3,4] \to 0, b[3,5] \to 0, b[3,6] \to 0, b[4,1] \to 0, b[4,2] \to 0, b[4,3] \to 0,$
$b[4,4] \to 0, b[4,5] \to 0, b[4,6] \to 0, b[5,1] \to 0, b[5,2] \to 0, b[5,3] \to 0, b[5,4] \to 0,$
$b[5,5] \to 0, b[5,6] \to 0, b[6,1] \to 0, b[6,2] \to 0, b[6,3] \to 0, b[6,4] \to 0, b[6,5] \to 0\}$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b[6,6] \end{pmatrix}$$

Chapter 9

Appendix D

This chapter is to support the statements from Lemma 5.5.1 and Remark 5.11.1. Notice that Corollary 5.5.2 is proven directly without the results from this chapter. We also give the computation of the spectral curve.

```
c = Array[w, {6, 6, 6}];
Do[w[i, j, k] = 0, {i, 6}, {j, 6}, {k, 6}]; w[1, 2, 4] = 2; w[2, 1, 4] = −2;
Do[w[6, i, i] = i, {i, 3}]; Do[w[i, 6, i] = −i, {i, 3}];
w[6, 4, 4] = 3; w[4, 6, 4] = −3; w[6, 5, 5] = 4; w[5, 6, 5] = −4;
dc = Array[d, {12, 12, 12}];
Do[d[i, j, k] = 0, {i, 1, 12}, {j, 1, 12}, {k, 1, 12}];
Do[d[i, j, k] = w[i, j, k], {i, 1, 6}, {j, 1, 6}, {k, 1, 6}]
Do[d[i + 6, j + 6, k + 6] = 0, {i, 1, 6}, {j, 1, 6}, {k, 1, 6}]
Do[d[i, j + 6, k + 6] = −w[i, k, j], {i, 1, 6}, {j, 1, 6}, {k, 1, 6}];
Do[d[i, j + 6, k] = 0, {i, 1, 6}, {j, 1, 6}, {k, 1, 6}];
Do[d[j + 6, i, k + 6] = w[i, k, j], {i, 1, 6}, {j, 1, 6}, {k, 1, 6}];
Do[d[j + 6, i, k] = 0, {i, 1, 6}, {j, 1, 6}, {k, 1, 6}];
p = Array[x, {12}];
ad = Array[y, {12, 12}];
Do[y[k, j] = Sum[x[i]d[i, j, k], {i, 1, 12}], {j, 1, 12}, {k, 1, 12}]
```

The following gives $\mathrm{Tr}(\mathrm{ad})^n$ *for n from* 1 *to* 100. *The output is omitted.*

```
R = ad; tra = Array[trac, {100}]; tra[[1]] = Tr[R];
Do[R = R.ad ; trac[n + 1] = Tr[R]; , {n, 1, 100}];
```

We find the Ker(ad).

Solve[ad == 0, {x[1], x[2], x[3], x[4], x[5], x[6], x[7], x[7], x[8], x[9], x[10], x[11], x[12]}]

Solve::svars : Equations may not give solutions for all solve variables. More...

$\{\{x[8] \to 0, x[1] \to 0, x[2] \to 0, x[6] \to 0, x[7] \to 0, x[3] \to 0, x[4] \to 0, x[5] \to 0,$

$x[9] \to 0, x[10] \to 0, x[11] \to 0\}\}$

So Ker(ad) *is 1-dimensional and it is generated by* $Y_{12} = Y_6^*$. *This proves Corollary 5.4.1.*

Expand[ad]

$\{\{x[6], 0, 0, 0, 0, -x[1], 0, 0, 0, 0, 0, 0\},$

$\{0, 2x[6], 0, 0, 0, -2x[2], 0, 0, 0, 0, 0, 0\}, \{0, 0, 3x[6], 0, 0, -3x[3], 0, 0, 0, 0, 0, 0\},$

$\{-2x[2], 2x[1], 0, 3x[6], 0, -3x[4], 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 4x[6], -4x[5], 0, 0, 0, 0, 0, 0\},$

$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, -2x[10], 0, 0, 0, x[7], -x[6], 0, 0, 2x[2], 0, 0\},$

$\{2x[10], 0, 0, 0, 0, 2x[8], 0, -2x[6], 0, -2x[1], 0, 0\},$

$\{0, 0, 0, 0, 0, 3x[9], 0, 0, -3x[6], 0, 0, 0\},$

$\{0, 0, 0, 0, 0, 3x[10], 0, 0, 0, -3x[6], 0, 0\}, \{0, 0, 0, 0, 0, 4x[11], 0, 0, 0, 0, -4x[6], 0\},$

$\{-x[7], -2x[8], -3x[9], -3x[10], -4x[11], 0, x[1], 2x[2], 3x[3], 3x[4], 4x[5], 0\}\}$

Now we get the characteristic curve.

Factor[Expand[Det[ad − *a*IdentityMatrix[12]]]]

$a^2(a-4x[6])(a-3x[6])^2(a-2x[6])(a-x[6])(a+x[6])(a+2x[6])(a+3x[6])^2(a+4x[6])$

The eigenvalues of ad a given below.

Eingev = Eigenvectors[ad]

$\{\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1\},$

$\left\{\frac{x[1]}{x[11]}, \frac{x[2]}{x[11]}, \frac{x[3]}{x[11]}, \frac{x[4]}{x[11]}, \frac{x[5]}{x[11]}, \frac{x[6]}{x[11]}, \frac{x[7]}{x[11]}, \frac{x[8]}{x[11]}, \frac{x[9]}{x[11]}, \frac{x[10]}{x[11]}, 1, 0\right\},$

$\left\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{x[6]}{x[5]}, 1\right\}, \left\{0, 0, 0, 0, 0, 0, 0, 0, -\frac{x[6]}{x[3]}, 0, 0, 1\right\},$

$\left\{0, 0, 0, 0, 0, 0, -\frac{x[2]}{x[6]}, \frac{2x[1]}{x[6]}, -\frac{x[1]x[2]+x[4]x[6]}{x[3]x[6]}, 1, 0, 0\right\},$

$\left\{0, 0, 0, 0, 0, 0, 0, -\frac{x[6]}{x[2]}, 0, 0, 0, 1\right\}, \left\{0, 0, 0, 0, 0, 0, -\frac{x[6]}{x[1]}, 0, 0, 0, 0, 1\right\},$

$\left\{-\frac{3x[6]^2}{3x[6]x[7]+5x[2]x[10]}, 0, 0, -\frac{3x[2]x[6]}{3x[6]x[7]+5x[2]x[10]}, 0, 0, 0, -\frac{2x[6]x[10]}{3x[6]x[7]+5x[2]x[10]}, 0, 0, 0, 1\right\}$,
$\left\{0, -\frac{3x[6]^2}{3x[6]x[8]-8x[1]x[10]}, 0, -\frac{6x[1]x[6]}{-3x[6]x[8]+8x[1]x[10]}, 0, 0, \frac{2x[6]x[10]}{3x[6]x[8]-8x[1]x[10]}, 0, 0, 0, 0, 1\right\}$,
$\left\{0, 0, -\frac{x[6]}{x[9]}, 0, 0, 0, 0, 0, 0, 0, 0, 1\right\}, \left\{0, 0, -\frac{x[10]}{x[9]}, 1, 0, 0, 0, 0, 0, 0, 0, 0\right\}$,
$\left\{0, 0, 0, 0, -\frac{x[6]}{x[11]}, 0, 0, 0, 0, 0, 0, 1\right\}\Big\}$

The space of eigenvectors is 12*-dimensional in the case* $x[6] \neq 0$*, i.e.* $z_6 \neq 0$ *in Section 5.7 notation.*

MatrixRank[Eingev]

12

Det[Eingev]

$$-\frac{9x[6]^{11}}{x[1]x[2]x[3]x[5]x[9](-3x[6]x[8]+8x[1]x[10])(3x[6]x[7]+5x[2]x[10])x[11]^2}$$

The space of eigenvectors is 8*-dimensional in the case* $x[6] = 0$*, i.e.* $z_6 = 0$ *in Section 5.7 notation.*

Eigenvectors[ad/.x[6] → 0]

$\Big\{\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1\}, \left\{0, 0, \frac{4x[5]}{3x[9]}, 0, 0, 0, 0, 0, 0, 0, 1, 0\right\}$,
$\left\{\frac{x[1]}{x[10]}, \frac{x[2]}{x[10]}, -\frac{x[1]x[7]+2x[2]x[8]-3x[4]x[10]}{3x[9]x[10]}, 0, 0, 0, 0, 0, 0, 1, 0, 0\right\}$,
$\left\{0, 0, \frac{x[3]}{x[9]}, 0, 0, 0, 0, 0, 1, 0, 0, 0\right\}, \left\{0, 0, \frac{2x[2]}{3x[9]}, 0, 0, 0, 0, 1, 0, 0, 0, 0\right\}$,
$\left\{0, 0, \frac{x[1]}{3x[9]}, 0, 0, 0, 1, 0, 0, 0, 0, 0\right\}, \left\{0, 0, -\frac{4x[11]}{3x[9]}, 0, 1, 0, 0, 0, 0, 0, 0, 0\right\}$,
$\left\{0, 0, -\frac{x[10]}{x[9]}, 1, 0, 0, 0, 0, 0, 0, 0, 0\right\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$,
$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$,
$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}\Big\}$

Chapter 10

Appendix E

In this section we compute the Lie algebra cohomology $H^2(\mathfrak{g}, \mathfrak{g})$ *in a five-dimensional case. First, we set the Lie algebra structure constants* $c[i, j, k]$ *of the Lie algebra* $\mathfrak{g}$, *i.e.* $[Y_i, Y_j] = \sum_{k=1}^{5} c[i, j, k] Y_k$ *for any* $i,\ j,\ k \in \{1, 2, \ldots, 5\}$.

```
c = Array[t, {5, 5, 5}];
Do[t[i, j, k] = 0, {i, 5}, {j, 5}, {k, 5}]; t[1, 2, 4] = 2; t[2, 1, 4] = −2;
Do[t[5, i, i] = i, {i, 3}]; Do[t[i, 5, i] = −i, {i, 3}];
t[5, 4, 4] = 3;  t[4, 5, 4] = −3;
```

Let $f : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *be an alternating map given in coordinates by*

$$f_3[Y_i, Y_j, Y_k] = \sum_{s=1}^{5} f3[i, j, k, s] Y_s.$$

```
f = Array[f3, {5, 5, 5, 5}];
Do[mm = Sort[{i1, i2, i3}];
f3[i1, i2, i3, i4] = Signature[{i1, i2, i3}]f3[mm[[1]], mm[[2]], mm[[3]], i4],
{i1, 1, 5}, {i2, 1, 5}, {i3, 1, 5}, {i4, 1, 5}];
```

Now, we define the coboundary operator $d_3 : C^3(\mathfrak{g}, \mathfrak{g}) \to C^4(\mathfrak{g}, \mathfrak{g})$. *Recall that* $d_3 f_3 : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *with* $f_3 \in C^3(\mathfrak{g}, \mathfrak{g})$. *We do all computations in coordinates. Let* $d3f4[i1, i2, i3, i4, s]$ *be given by*

$$d_3 f_3 \left[Y_{i_1}, Y_{i_2}, Y_{i_3}, Y_{i_4}\right] = \sum_{s=1}^{5} d3f4[i1, i2, i3, i4, s] Y_s.$$

```
d3f4 = Array[t5, {5, 5, 5, 5, 5}];
Do[t5[i1, i2, i3, i4, s] = Sum[c[[i1, l, s]]f3[i2, i3, i4, l], {l, 1, 5}]−
Sum[c[[i2, l, s]]f3[i1, i3, i4, l], {l, 1, 5}] + Sum[c[[i3, l, s]]f3[i1, i2, i4, l], {l, 1, 5}]−
Sum[c[[i4, l, s]]f3[i1, i2, i3, l], {l, 1, 5}] − Sum[c[[i1, i2, l]]f3[l, i3, i4, s], {l, 1, 5}]+
Sum[c[[i1, i3, l]]f3[l, i2, i4, s], {l, 1, 5}] − Sum[c[[i1, i4, l]]f3[l, i2, i3, s], {l, 1, 5}]−
Sum[c[[i2, i3, l]]f3[l, i1, i4, s], {l, 1, 5}] + Sum[c[[i2, i4, l]]f3[l, i1, i3, s], {l, 1, 5}]−
Sum[c[[i3, i4, l]]f3[l, i1, i2, s], {l, 1, 5}], {i1, 1, 5}, {i2, 1, 5}, {i3, 1, 5}, {i4, 1, 5}, {s, 1, 5}];
zz1 = Solve[d3f4 == 0]
```

Solve::svars : Equations may not give solutions for all solve variables. More...

$\{\{f3[1,2,4,3] \to 0, f3[1,2,4,5] \to 0, f3[1,3,4,2] \to 0, f3[1,3,4,5] \to 0,$

$f3[1,3,4,3] \to \frac{3}{2}f3[1,2,4,2], f3[1,4,5,5] \to -2f3[1,2,4,2],$

$f3[2,3,4,1] \to 0, f3[1,2,3,5] \to -\frac{1}{3}f3[3,4,5,5],$

$f3[1,3,4,1] \to \frac{1}{6}f3[3,4,5,5], f3[2,3,4,2] \to \frac{1}{3}f3[3,4,5,5], f3[2,3,4,5] \to 0,$

$f3[1,2,4,4] \to f3[1,2,3,3] - \frac{2}{3}f3[1,4,5,1] - \frac{2}{3}f3[2,4,5,2] + \frac{2}{3}f3[3,4,5,3],$

$f3[1,2,5,5] \to f3[1,2,3,3] + \frac{2}{3}f3[3,4,5,3], f3[2,3,4,3] \to -3f3[1,2,4,1],$

$f3[2,4,5,5] \to 5f3[1,2,4,1], f3[2,3,4,4] \to 3f3[1,2,3,1] + \frac{8}{5}f3[3,4,5,1],$

$f3[2,3,5,5] \to 5f3[1,2,3,1] + 2f3[3,4,5,1],$

$f3[1,3,4,4] \to -\frac{3}{2}f3[1,2,3,2] - \frac{5}{4}f3[3,4,5,2],$

$f3[1,3,5,5] \to -2f3[1,2,3,2] - f3[3,4,5,2],$

$f3[1,2,3,4] \to -\frac{2}{3}f3[1,3,5,1] - \frac{2}{3}f3[2,3,5,2] - \frac{2}{3}f3[3,4,5,4]\}\}$

Remark 10.0.2. *The previous system reduces to 20 equations linearly independent. So* $\dim(\mathrm{Ker} d_3) = \dim(C^3(\mathfrak{g},\mathfrak{g})) - 20 = \frac{5(5!)}{(3!)(2!)} - 20 = 50 - 20 = 30.$

Let $f_2 : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *be an alternating map, and let* $f_2[Y_{i_1}, Y_{i_2}]$ *be the coordinates of* $f2$.

$$f_2[Y_{i_1}, Y_{i_2}] = \sum_{i_3=1}^{5} f2[i1, i2, i3] Y_{i_3}$$

g = Array[f2, {5, 5, 5}];

Do[mmm = Sort[{i1, i2}];

f2[i1, i2, i3] = Signature[{i1, i2}]f2[mmm[[1]], mmm[[2]], i3],

{i1, 1, 5}, {i2, 1, 5}, {i3, 1, 5}]

We define the coboundary operator $d_2 : C^2(\mathfrak{g}, \mathfrak{g}) \to C^3(\mathfrak{g}, \mathfrak{g})$. *Let* $f_2 \in C^2(\mathfrak{g}, \mathfrak{g})$. *The coordinates* $d2f3[i1, i2, i3, s]$ *of*

$$d_2 f_2 : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

are given by

$$d_2 f\left[Y_{i_1 1}, Y_{i_2}, Y_{i_3}\right] = \sum_{s=1}^{5} d2f3[\mathrm{i1}, \mathrm{i2}, \mathrm{i3}, s] Y_s$$

d2f3 = Array[t4, {5, 5, 5, 5}];

Do[t4[i1, i2, i3, s] = Sum[c[[i1, l, s]]f2[i2, i3, l], {l, 1, 5}]−

Sum[c[[i2, l, s]]f2[i1, i3, l], {l, 1, 5}] + Sum[c[[i3, l, s]]f2[i1, i2, l], {l, 1, 5}]−

Sum[c[[i1, i2, l]]f2[l, i3, s], {l, 1, 5}] + Sum[c[[i1, i3, l]]f2[l, i2, s], {l, 1, 5}]−

Sum[c[[i2, i3, l]]f2[l, i1, s], {l, 1, 5}], {i1, 1, 5}, {i2, 1, 5}, {i3, 1, 5}, {s, 1, 5}]

Solve[d2f3 == 0]

Solve::svars : Equations may not give solutions for all solve variables. More...

$\{\{\mathrm{f2}[1,3,2] \to 0, \mathrm{f2}[1,3,5] \to 0, \mathrm{f2}[1,4,2] \to 0,$

$\mathrm{f2}[1,4,3] \to 0, \mathrm{f2}[1,4,5] \to 0, \mathrm{f2}[1,3,3] \to \frac{3}{2}\mathrm{f2}[1,2,2] + 3\mathrm{f2}[4,5,2],$

$\mathrm{f2}[1,5,5] \to \frac{1}{2}\mathrm{f2}[1,2,2] + \mathrm{f2}[4,5,2], \mathrm{f2}[2,3,1] \to 0, \mathrm{f2}[2,3,5] \to 0,$

$\mathrm{f2}[2,4,1] \to 0, \mathrm{f2}[1,2,5] \to -\frac{2}{3}\mathrm{f2}[4,5,5], \mathrm{f2}[1,4,1] \to -\frac{1}{3}\mathrm{f2}[4,5,5],$

$\mathrm{f2}[2,4,2] \to -\frac{2}{3}\mathrm{f2}[4,5,5], \mathrm{f2}[2,4,3] \to 0, \mathrm{f2}[2,4,5] \to 0,$

$\mathrm{f2}[2,3,3] \to \mathrm{f2}[2,4,4] + \mathrm{f2}[4,5,1], \mathrm{f2}[2,5,5] \to \frac{2}{3}\mathrm{f2}[2,4,4] + \frac{2}{3}\mathrm{f2}[4,5,1],$

$\mathrm{f2}[3,4,1] \to 0, \mathrm{f2}[3,4,2] \to 0, \mathrm{f2}[3,4,3] \to -\mathrm{f2}[4,5,5],$

$f2[1,3,1] \to -\frac{1}{3}f2[3,5,5], f2[2,3,2] \to -\frac{2}{3}f2[3,5,5], f2[3,4,4] \to f2[3,5,5],$

$f2[3,4,5] \to 0, f2[2,3,4] \to -f2[3,5,1], f2[1,3,4] \to 2f2[3,5,2],$

$f2[1,2,1] \to -\frac{1}{3}f2[2,4,4] - \frac{4}{3}f2[4,5,1], f2[1,4,4] \to \frac{3}{2}f2[1,2,2] + 5f2[4,5,2],$

$f2[4,5,3] \to 0, f2[1,5,1] \to -f2[2,5,2] + f2[4,5,4]\}\}$

Remark 10.0.3. *The previous system reduces to 30 linearly independent equations and*

$$\begin{aligned} \dim \mathrm{Im}\ d_2 &= \dim \left(C^2(\mathfrak{g},\mathfrak{g})/\mathrm{Ker} d_2\right) \\ &= \dim C^2(\mathfrak{g},\mathfrak{g}) - \dim \mathrm{Ker} d_2 \\ &= 5\binom{5}{2} - (5\binom{5}{2} - 30) = 30. \end{aligned}$$

Remark 10.0.4. *Since* $\dim(\mathrm{Ker} d_3) = \dim(\mathrm{Im} d_2)$*, it follows that* $\mathrm{Ker}\ d_3 = \mathrm{Im}\ d_2$*, so* $H^3(\mathfrak{g},\mathfrak{g}) = 0$.

Let $f3 = f \in$ *Ker* d_3 *, i.e. in the notation of this appendix* $f3$ *is in zz1.*

$\mathbf{f3[1,2,4,3] = 0;}$

$\mathbf{f3[1,2,4,5] = 0;}$

$\mathbf{f3[1,3,4,2] = 0;}$

$\mathbf{f3[1,3,4,5] = 0;}$

$\mathbf{f3[1,3,4,3] = \frac{3}{2}f3[1,2,4,2];}$

$\mathbf{f3[1,4,5,5] = -2f3[1,2,4,2];}$

$\mathbf{f3[2,3,4,1] = 0;}$

$\mathbf{f3[1,2,3,5] = -\frac{1}{3}f3[3,4,5,5];}$

$\mathbf{f3[1,3,4,1] = \frac{1}{6}f3[3,4,5,5];}$

$\mathbf{f3[2,3,4,2] = \frac{1}{3}f3[3,4,5,5];}$

$\mathbf{f3[2,3,4,5] = 0;}$

$\mathbf{f3[1,2,4,4] = f3[1,2,3,3] - \frac{2}{3}f3[1,4,5,1] - \frac{2}{3}f3[2,4,5,2] + \frac{2}{3}f3[3,4,5,3];}$

$\mathrm{f3}[1,2,5,5] = \mathrm{f3}[1,2,3,3] + \frac{2}{3}\mathrm{f3}[3,4,5,3];$

$\mathrm{f3}[2,3,4,3] = -3\mathrm{f3}[1,2,4,1];$

$\mathrm{f3}[2,4,5,5] = 5\mathrm{f3}[1,2,4,1];$

$\mathrm{f3}[2,3,4,4] = 3\mathrm{f3}[1,2,3,1] + \frac{8}{5}\mathrm{f3}[3,4,5,1];$

$\mathrm{f3}[2,3,5,5] = 5\mathrm{f3}[1,2,3,1] + 2\mathrm{f3}[3,4,5,1];$

$\mathrm{f3}[1,3,4,4] = -\frac{3}{2}\mathrm{f3}[1,2,3,2] - \frac{5}{4}\mathrm{f3}[3,4,5,2];$

$\mathrm{f3}[1,3,5,5] = -2\mathrm{f3}[1,2,3,2] - \mathrm{f3}[3,4,5,2];$

$\mathrm{f3}[1,2,3,4] = -\frac{2}{3}\mathrm{f3}[1,3,5,1] - \frac{2}{3}\mathrm{f3}[2,3,5,2] - \frac{2}{3}\mathrm{f3}[3,4,5,4];$

$\mathrm{zz2} = \mathrm{Solve}[f == \mathrm{d2f3}]$

The solution of the equation $d_2 f_2 = f_3$ *is given bellow.*

$\{\{\mathrm{f2}[2,3,5] \to -\frac{1}{5}\mathrm{f3}[2,3,5,5],$

$\mathrm{f2}[3,4,1] \to -\frac{1}{5}\mathrm{f3}[3,4,5,1],$

$\mathrm{f2}[1,3,5] \to -\frac{1}{4}\mathrm{f3}[1,3,5,5],$

$\mathrm{f2}[3,4,2] \to -\frac{1}{4}\mathrm{f3}[3,4,5,2],$

$\mathrm{f2}[1,2,5] \to -\frac{2}{3}\mathrm{f2}[4,5,5] - \frac{1}{3}\mathrm{f3}[1,2,5,5],$

$\mathrm{f2}[3,4,3] \to -\mathrm{f2}[4,5,5] - \frac{1}{3}\mathrm{f3}[3,4,5,3],$

$\mathrm{f2}[1,3,1] \to -\frac{1}{3}\mathrm{f2}[3,5,5] - \frac{1}{3}\mathrm{f3}[1,3,5,1],$

$\mathrm{f2}[2,3,2] \to -\frac{2}{3}\mathrm{f2}[3,5,5] - \frac{1}{3}\mathrm{f3}[2,3,5,2],$

$\mathrm{f2}[3,4,4] \to \mathrm{f2}[3,5,5] - \frac{1}{3}\mathrm{f3}[3,4,5,4],$

$\mathrm{f2}[3,4,5] \to -\frac{1}{6}\mathrm{f3}[3,4,5,5],$

$\mathrm{f2}[2,4,5] \to -\frac{1}{5}\mathrm{f3}[2,4,5,5],$

$\mathrm{f2}[1,4,5] \to -\frac{1}{4}\mathrm{f3}[1,4,5,5],$

$\mathrm{f2}[1,4,1] \to -\frac{1}{3}\mathrm{f2}[4,5,5] - \frac{1}{3}\mathrm{f3}[1,4,5,1],$

$\mathrm{f2}[2,4,2] \to -\frac{2}{3}\mathrm{f2}[4,5,5] - \frac{1}{3}\mathrm{f3}[2,4,5,2],$

$\mathrm{f2}[1,2,1] \to -\frac{4}{3}\mathrm{f2}[2,3,3] + \mathrm{f2}[2,4,4] - \frac{1}{2}\mathrm{f3}[1,2,5,1] - \frac{2}{3}\mathrm{f3}[2,3,5,3]+$

$\frac{1}{2}\mathrm{f3}[2,4,5,4], \mathrm{f2}[2,5,5] \to \frac{2}{3}\mathrm{f2}[2,3,3] + \frac{1}{3}\mathrm{f3}[2,3,5,3],$

$\mathbf{f2[4,5,1] \to f2[2,3,3] - f2[2,4,4] + \frac{1}{2}f3[2,3,5,3] - \frac{1}{2}f3[2,4,5,4],}$

$\mathbf{f2[1,2,2] \to \frac{5}{3}f2[1,3,3] - f2[1,4,4] - f3[1,2,5,2] + \frac{5}{3}f3[1,3,5,3] - f3[1,4,5,4],}$

$\mathbf{f2[1,5,5] \to \frac{1}{3}f2[1,3,3] + \frac{1}{3}f3[1,3,5,3],}$

$\mathbf{f2[4,5,2] \to -\frac{1}{2}f2[1,3,3] + \frac{1}{2}f2[1,4,4] - \frac{1}{2}f3[1,3,5,3] + \frac{1}{2}f3[1,4,5,4],}$

$\mathbf{f2[4,5,3] \to -\frac{1}{2}f3[1,2,5,3],}$

$\mathbf{f2[1,5,1] \to -f2[2,5,2] + f2[4,5,4] + \frac{1}{2}f3[1,2,5,4],}$

$\mathbf{f2[1,3,2] \to -\frac{1}{2}f3[1,3,5,2],}$

$\mathbf{f2[1,3,4] \to 2f2[3,5,2] - f3[1,3,5,4],}$

$\mathbf{f2[1,4,2] \to -\frac{1}{2}f3[1,4,5,2],}$

$\mathbf{f2[1,4,3] \to -f3[1,4,5,3],}$

$\mathbf{f2[2,3,1] \to -\frac{1}{4}f3[2,3,5,1],}$

$\mathbf{f2[2,3,4] \to -f2[3,5,1] - \frac{1}{2}f3[2,3,5,4],}$

$\mathbf{f2[2,4,1] \to -\frac{1}{4}f3[2,4,5,1],}$

$\mathbf{f2[2,4,3] \to -\frac{1}{2}f3[2,4,5,3]\}\}};$

$Z^2(\mathfrak{g},\mathfrak{g}) = \{f : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g} \,|\, f \text{ is alternating and } (d_2 f) = 0\} = \text{Ker } d_2$

$\dim(\text{Ker } d_2) = 5\binom{5}{2} - 30 = 20$ (by Remark 10.0.3). So $Z^2(\mathfrak{g},\mathfrak{g}) \neq 0$.

Now, we compute $H^2(\mathfrak{g},\mathfrak{g}) = \text{Ker } d_2/\text{ Im} d_1$. Recall that

$$B^1(\mathfrak{g},\mathfrak{g}) = \text{Im} d_1 = \{f_2 : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g} \,|\, f_2 = d_1 f_1 \text{for some } f_1 : \mathfrak{g} \to \mathfrak{g}\}$$

$$(d_1 f_1)(x,y) = [x, f_1(y)] - [y, f_1(x)] - f_1([x,y])$$

$$(d_1 f_1)(Y_i, Y_j) = \text{d1f2}[i,j,s]Y_s$$

We define coboundary operator $d_1 : C^1(\mathfrak{g},\mathfrak{g}) \to C^2(\mathfrak{g},\mathfrak{g})$. *Let* $f_1 \in C^1(\mathfrak{g},\mathfrak{g})$ *and let* d1f2[i1, i2, i3, s] be the coordinates of

$$d_1 f_1 : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, \text{ i.e.}$$

$$d_1 f\left[Y_{i_1}, Y_{i_2}\right] = \sum_{s=1}^{5} \text{d1f2}[\text{i1}, \text{i2}, s] Y_s$$

h = Array[f1, {5, 5}];

d1f2 = Array[t3, {5, 5, 5}];

Do[t3[i1, i2, s] = Sum[c[[i1, l, s]]h[[i2, l]], {l, 1, 5}] − Sum[c[[i2, l, s]]h[[i1, l]], {l, 1, 5}]

−Sum[c[[i1, i2, l]]h[[l, s]], {l, 1, 5}], {i1, 1, 5}, {i2, 1, 5}, {s, 1, 5}]

Solve[d1f2==0]

Solve::svars : Equations may not give solutions for all solve variables. More...

{{f1[1, 2] → 0, f1[1, 3] → 0, f1[1, 5] → 0, f1[2, 1] → 0,

f1[2, 3] → 0, f1[2, 5] → 0, f1[3, 1] → 0, f1[3, 2] → 0, f1[3, 5] → 0,

f1[4, 1] → 0, f1[4, 2] → 0, f1[4, 3] → 0, f1[1, 1] → −f1[2, 2] + f1[4, 4],

f1[4, 5] → 0, f1[2, 4] → −2f1[5, 1], f1[1, 4] → f1[5, 2], f1[5, 5] → 0}}

Remark 10.0.5.

$$H^2(\mathfrak{g}, \mathfrak{g}) \neq 0.$$

Proof.

$$\dim\left(\text{Im} d_1\right) = \dim\left(C^1(\mathfrak{g}, \mathfrak{g})/\text{Ker} d_1\right) = 25 - (25 - 17) = 17.$$

$$\dim\left(\text{Ker} d_2\right) = 5\binom{5}{2} - 30 = 20$$

(see Remark 10.0.3).

So $\dim\left(\text{Im} d_1\right) < \dim\left(\text{Ker } d_2\right)$. Therefore $H^2(\mathfrak{g}, \mathfrak{g}) \neq 0$ □

We define F_2 *as* gp(= fp2)

$F_1 \circ F_1$ *as* bracketf3 (=t7),

$F_1 \circ F_2 + F_2 \circ F_1$ *as* bracketf3p (=t8), and

$F_2 \circ F_2$ *as* bracketf3pp(=t9)

```
gp = Array[fp2, {5, 5, 5}];
Do[mmm = Sort[{i1, i2}];
fp2[i1, i2, i3] = Signature[{i1, i2}]fp2[mmm[[1]], mmm[[2]], i3],
{i1, 1, 5}, {i2, 1, 5}, {i3, 1, 5}]
d2f3p = Array[tp4, {5, 5, 5, 5}];
Do[tp4[i1, i2, i3, s] = Sum[c[[i1, l, s]]fp2[i2, i3, l], {l, 1, 5}]−
Sum[c[[i2, l, s]]fp2[i1, i3, l], {l, 1, 5}] + Sum[c[[i3, l, s]]fp2[i1, i2, l], {l, 1, 5}]−
Sum[c[[i1, i2, l]]fp2[l, i3, s], {l, 1, 5}] + Sum[c[[i1, i3, l]]fp2[l, i2, s], {l, 1, 5}]−
Sum[c[[i2, i3, l]]fp2[l, i1, s], {l, 1, 5}], {i1, 1, 5}, {i2, 1, 5}, {i3, 1, 5}, {s, 1, 5}]
bracketf3 = Array[t7, {5, 5, 5, 5}];
Do[t7[i1, i2, i3, s] = Sum[f2[i1, i2, i4]f2[i4, i3, s], {i4, 1, 5}]+
Sum[f2[i2, i3, i4]f2[i4, i1, s], {i4, 1, 5}] + Sum[f2[i3, i1, i4]f2[i4, i2, s], {i4, 1, 5}],
{i1, 1, 5}, {i2, 1, 5}, {i3, 1, 5}, {s, 1, 5}];
bracketf3p = Array[t8, {5, 5, 5, 5}];
Do[t8[i1, i2, i3, s] = Sum[f2p[i1, i2, i4]f2[i4, i3, s], {i4, 1, 5}]+
Sum[f2p[i2, i3, i4]f2[i4, i1, s], {i4, 1, 5}] + Sum[f2p[i3, i1, i4]f2[i4, i2, s], {i4, 1, 5}]+
Sum[f2[i1, i2, i4]f2p[i4, i3, s], {i4, 1, 5}] + Sum[f2[i2, i3, i4]f2p[i4, i1, s], {i4, 1, 5}]+
Sum[f2[i3, i1, i4]f2p[i4, i2, s], {i4, 1, 5}], {i1, 1, 5}, {i2, 1, 5}, {i3, 1, 5}, {s, 1, 5}];
bracketf3pp = Array[t9, {5, 5, 5, 5}];
Do[t9[i1, i2, i3, s] = Sum[f2p[i1, i2, i4]f2p[i4, i3, s], {i4, 1, 5}]+
Sum[f2p[i2, i3, i4]f2p[i4, i1, s], {i4, 1, 5}] + Sum[f2p[i3, i1, i4]f2p[i4, i2, s], {i4, 1, 5}],
{i1, 1, 5}, {i2, 1, 5}, {i3, 1, 5}, {s, 1, 5}];
```

Now, we try to find whether there are quadratic deformations, i.e.

$$F_3 = F_4 = \ldots = 0. \tag{10.0.1}$$

Solve[$d2f3$ == 0&&d2f3p == bracketf3&&bracketf3p == 0&&bracketf3pp == 0]

Here, we try to find an $F_1 \in Z^2$ such that $F_1 \circ F_1 = d_2 F_2$, $F_1 \circ F_2 + F_2 \circ F_1 = 0$ and $F_2 \circ F_2 = 0$. *If such* F_1, F_2 *exist, then we may take*

$$F_3 = F_4 = \ldots = 0$$

$$F_1\left[Y_{i_1}, Y_{i_2}\right] = \mathrm{f2[i1, i2, i3]} Y_{i_3}$$

$$F_2\left[Y_{i_1}, Y_{i_2}\right] = \mathrm{f2p[i1, i2, i3]} Y_{i_3}$$

p = Array[x, {5}]; ad = Array[y, {5, 5}];

Do[y[j, k] = Sum[x[i](c[[i, j, k]] + tf2[i, j, k] + t^2fp2[i, j, k]), {i, 1, 5}],

{j, 1, 5}, {k, 1, 5}]

Now, we find an invariant bilinear form $B(= b)$.

B = Array[b, {5, 5}]; invB = Array[invb, {5, 5, 5}];

Do[invb[i, j, k] = Sum[(c[[i, j, s]] + tf2[i, j, s])b[s, k] + (c[[i, k, s]] + tf2[i, k, s])b[j, s],

{s, 1, 5}], {i, 1, 5}, {j, 1, 5}, {k, 1, 5}];

Solve[invB == 0]

The output is omitted.

Solve[d2f3 == 0&&d2f3p==bracketf3&&bracketf3p == 0&&bracketf3pp == 0&&

bracketf3==d2f3p

The Mathematics cannot not get a solution (it runs forever or give a out of memory message). A good idea would be to solve a subsystem, then substitute into the remaining equations, but this doesn't work for Mathematics either. The following equation gives the solutions of the simple subsystem.

Solve[d2f3 == 0&&bracketf3 == 0]

{{f2[1, 5, 1] → −f2[2, 5, 2] + f2[4, 5, 4], f2[2, 3, 3] → 0, f2[1, 2, 1] → 0,

f2[2, 4, 4] → 0, f2[1, 3, 3] → 0, f2[1, 4, 4] → 0, f2[1, 2, 2] → 0, f2[3, 4, 4] → 0,

$f2[3,4,3] \to 0, f2[1,3,1] \to 0, f2[1,2,3] \to 0, f2[4,5,3] \to 0, f2[3,5,1] \to 0,$
$f2[2,3,2] \to 0, f2[3,5,2] \to 0, f2[1,4,1] \to 0, f2[2,5,5] \to 0, f2[2,3,4] \to 0,$
$f2[2,4,2] \to 0, f2[4,5,1] \to 0, f2[4,5,2] \to 0, f2[1,5,5] \to 0, f2[2,3,1] \to 0,$
$f2[2,4,3] \to 0, f2[1,3,4] \to 0, f2[2,4,1] \to 0, f2[3,5,5] \to 0, f2[1,4,3] \to 0,$
$f2[1,2,5] \to 0, f2[2,4,5] \to 0, f2[1,3,2] \to 0, f2[4,5,5] \to 0, f2[1,4,2] \to 0,$
$f2[2,3,5] \to 0, f2[1,4,5] \to 0, f2[1,3,5] \to 0, f2[3,4,5] \to 0, f2[3,4,2] \to 0,$
$f2[3,4,1] \to 0\},$

... another 13 solutions here...

$\Big\{f2[3,5,3] \to \frac{1}{2}\left(-3f2[1,2,4] - \frac{2f2[2,5,4]f2[3,5,5]}{f2[2,3,4]}\right),$
$f2[1,5,1] \to \frac{1}{6}\left(-3f2[1,2,4] - \frac{2f2[2,5,4]f2[3,5,5]}{f2[2,3,4]}\right),$
$f2[4,5,4] \to \frac{1}{2}\left(-3f2[1,2,4] - \frac{2f2[2,5,4]f2[3,5,5]}{f2[2,3,4]}\right),$
$f2[2,5,2] \to \frac{1}{3}\left(-3f2[1,2,4] - \frac{2f2[2,5,4]f2[3,5,5]}{f2[2,3,4]}\right), f2[2,3,3] \to 0,$
$f2[1,2,1] \to 0, f2[2,4,4] \to 0, f2[1,3,3] \to 0, f2[1,4,4] \to 0, f2[2,5,3] \to 0,$
$f2[2,5,1] \to 0, f2[1,2,2] \to 0, f2[3,4,4] \to f2[3,5,5], f2[1,5,4] \to 0,$
$f2[1,5,3] \to 0, f2[3,4,3] \to 0, f2[1,5,2] \to 0, f2[1,3,1] \to -\frac{1}{3}f2[3,5,5],$
$f2[1,2,3] \to 0, f2[4,5,3] \to 0, f2[3,5,1] \to -f2[2,3,4], f2[2,3,2] \to -\frac{2}{3}f2[3,5,5],$
$f2[3,5,2] \to 0, f2[1,4,1] \to 0, f2[2,5,5] \to 0, f2[2,4,2] \to 0, f2[4,5,1] \to 0,$
$f2[4,5,2] \to 0, f2[1,5,5] \to 0, f2[2,3,1] \to 0, f2[2,4,3] \to 0, f2[1,3,4] \to 0,$
$f2[2,4,1] \to 0, f2[1,4,3] \to 0, f2[1,2,5] \to 0, f2[2,4,5] \to 0, f2[1,3,2] \to 0,$
$f2[4,5,5] \to 0, f2[1,4,2] \to 0, f2[2,3,5] \to 0, f2[1,4,5] \to 0, f2[1,3,5] \to 0,$
$f2[3,4,5] \to 0, f2[3,4,2] \to 0, f2[3,4,1] \to 0\}\}$

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Publications

[1] Gabriel Bădiţoiu and Steven Rosenberg, *Feynman diagrams and Lax pair equations*, arXiv:math-ph/0611014.

[2] Gabriel Bădiţoiu, *Semi-Riemannian submersions with totally geodesic fibres*, Tohoku Mathematical Journal (2) **56** (2004), no. 2, 179–204. MR 2053318 (2005a:53115)

[3] Gabriel Bădiţoiu, *Semi-Riemannian submersions with totally umbilic fibres and warped products*, Mathematical Reports (Bucureşti) **6(56)** (2004), no. 1, 1–7. MR 2068392 (2005d:53110)

[4] Gabriel Bădiţoiu, Klaus Buchner, and Stere Ianuş, *Some remarkable connections and semi-Riemannian submersions*, Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie. (Nouvelle Série) **41(89)** (1998), no. 3, 153–169. MR 1880200 (2002k:53035)

[5] Gabriel Baditoiu, Richard H. Jr. Escobales, and Stere Ianus, *A cohomology* $(p+1)$ *form canonically associated with certain codimension-q foliations on a Riemannian manifold*, Tokyo Journal of Mathematics **29** (2006), no. 1, 247–270.

[6] Gabriel Bădiţoiu and Stere Ianuş, *Semi-Riemannian submersions from real and complex pseudo-hyperbolic spaces*, Differential Geometry and its Applications **16** (2002), no. 1, 79–94. MR 1877586 (2003h:53095)

[7] Gabriel Bădiţoiu and Stere Ianuş, *Semi-Riemannian submersions with totally umbilic fibres*, Rendiconti del Circolo Matematico di Palermo **51** (2002), no. 2, 249–276. MR 1916929 (2003f:53124)

Talks

June 2000 — A 45-minute talk at the Conference on Foliations: Geometry and Dynamics, Warsaw 2000: *Semi-Riemannian submersions from real and complex pseudo-hyperbolic spaces*

December 4, 2006 — A talk in the Boston University Mathematical Physics Seminar: *Feynman diagrams and Lax pair equations*

Conferences and workshops

10 – 14 August 1998 — 7th International Conference on Differential Geometry

	and applications, satellite conference of ICM98, Brno, Czech Republic
12 – 30 April 1999	School on Differential Geometry, ICTP Trieste (Italy)
7 – 15 July 1999	Global Theory of Minimal Surfaces in Flat Spaces, Martina Franca (Italy)
25 April – 12 May 2000	School on Vanishing Theorems and Effective Results in Algebraic Geometry, International Center for Theoretical Physics - Trieste (Italy)
29 May – 9 June 2000	Conference on Foliations: Geometry and Dynamics, Warsaw (Poland)
20 May – 9 June 2001	School on Topology of High-Dimensional Manifolds, ICTP Trieste (Italy)
16 – 21 June 2003	Summer School on Invariants in Low-Dimensional Topology, Alfred Renyi Institute of Mathematics, Budapest (Hungary)

Teaching experience

Fall 1997 and Fall 1999	teaching assistant at the Faculty of Mathematics, University of Bucharest
Fall 2002 – Spring 2007	Teaching Assistant for the Department of Mathematics of Boston University since for the following courses: Elementary Statistics, Applied Mathematics for Social and Management Sciences, Calculus 2, Multivariate Calculus, Differential Equations.
Summer 2002, 2004 – 2006	Stand-alone instructor for the following Boston University courses: Differential Equations in Summer 2002, Multivariate Calculus in Summer 2004 and Summer 2005, Calculus 2 in Summer 2006.

Boston, April 7, 2007