Every graph is a cut locus

Jin-ichi Itoh and Costin Vîlcu

September 11, 2013

Abstract We prove that every connected graph can be realized as the cut locus of some point on some riemannian surface S which, in some cases, has constant curvature. We also study the stability of such realisations, and their generic behaviour.

Math. Subj. Classification (2010): 53C22, 05C62

1 Introduction

Unless explicitly stated otherwise, by a riemannian manifold here we always mean a complete, compact and connected manifold without boundary. We shall work most of the time with surfaces (2-dimensional manifolds) S, and let M denote manifolds of arbitrary dimension d.

All graphs we consider in the following are finite, connected and may have loops and multiple edges. For the simplicity of our exposition, we see every graph G as a 1-dimensional simplicial complex. The *cyclic part* of G is the minimal (with respect to inclusion) subgraph G^{cp} of G, to which G is contractible; i.e., the minimal subgraph of G obtained by repeatedly contracting external edges, and for each vertex remaining of degree two (if any) merging its incident edges. A graph is called *cyclic* if it it equal to its cyclic part, and it is called of *constant order* if all its vertices have the same degree.

The notion of cut locus was introduced by H. Poincaré [20] in 1905, and gained since then an important place in global riemannian geometry. The *cut locus* $C(x)$ *of the point* x in the riemannian manifold M is the set of all extremities (different from x) of maximal (with respect to inclusion) *segments* (i.e., shortest geodesics) starting at x; for basic properties and equivalent definitions refer, for example, to [18] or [21].

For riemannian surfaces S is known that $C(x)$, if not a single point, is a local tree (i.e., each of its points z has a neighbourhood V in S such that the component $K_z(V)$ of z in $C(x) \cap V$ is a tree), even a tree if S is homeomorphic to the sphere. A *tree* is a set T any two points of which can be joined by a unique Jordan arc included in T. The *degree* of a point y of a local tree is the number of components of $K_y(V) \setminus \{y\}$ if V is chosen such that $K_y(V)$ is a tree. A tree is *finite* if it has finitely many points of degree ≥ 3, each of which has finite degree.

S. B. Myers [19] for $d = 2$, and M. Buchner [3] for general d, established that the cut locus of a real analytic riemannian manifold of dimension d is homeomorphic to a finite $(d-1)$-dimensional simplicial complex. For a class of Liouville manifolds, in particular for hyperellipsoids in the euclidean space $I\!R^d$, the cut locus is reduced to a disc of dimension of most $(d-1)$, see [10] and [11].

For riemannian metrics of S non-analytic, cut loci may be quite large sets. J. Hebda [6] showed, for any C^∞ metric on S, that the Hausdorff 1-measure of any compact subset of the cut locus of any point is finite. Independently and using different techniques, J. Itoh [9] proved the same result under the weaker assumption of C^2 metric. The differentiability of the metric cannot be lowered more; for example, the main result in [25] states that on most (in the sense of Baire category) convex surfaces (which are $\mathcal{C}^1 \setminus \mathcal{C}^2$), most points are extremities of any cut locus.

The problem of constructing a riemannian metric with preassigned cut locus on a given manifold also received a certain interest. H. Gluck and D. Singer [5] constructed a riemannian metric such that a non triangulable set, consisting of infinitely many arcs with a common extremity, becomes a cut locus. Another example of infinite length cut locus was provided by J. Hebda [7], while the case of a submanifold as preassigned cut locus was considered by L. Bérard-Bergery [1]. J. Itoh [8] showed that for any Morse function on a differentiable surface S, with only one critical point of index 0 and no saddle connection, there exists a riemannian metric on S with respect to which C_f, the union of all unstable manifolds of critical points of f with positive index, becomes a cut locus. All these results assume the manifold be given, and search for a metric with respect to which some subset of the manifold becomes a cut locus.

A different approach was considered in [13], where the authors showed that any combinatorial type of finite tree can be realized as a cut locus on some, initially unknown, doubly covered convex polygon.

Our results here give this approach much more generality, by showing (see Theorem 2.6) that every connected graph can be realized as a cut locus; i.e., there exist a riemannian surface $S_G = (S_G, h)$ and a point $x \in S_G$ such that $C(x)$ is isometric to G. This is a partial converse to Myers' theorem mentioned above. If moreover G is cyclic of constant order, then it can be realized on a surface of constant curvature (Theorem 3.1). At the end of this paper we show that –roughly speaking– stability is a generic property of cut locus realizations.

In a forthcoming paper [15] we are concerned about the orientability of the surfaces S_G realizing the graph G as a cut locus.

2 Every graph is a cut locus

Recall that *a segment between a point x and a closed set K* not containing x is a segment from x to a point in K, not longer than any other such segment; the *cut locus $C(K)$ of the closed set $K \subset S$* is the set of all points $y \in S$ such that there is a segment from y to K not extendable as a segment beyond y.

A graph is *metric* if each of its edges is endowed with a positive number, called *length*.

Definition 2.1 *Let G be a graph. A G-strip is a topological surface P_G with boundary, such that:*
(i) the boundary of P_G is homeomorphic to a circle, and
(ii) P_G contains (a graph isometric to) G and is contractible to G

A riemannian G-strip is a G-strip P_G endowed with a riemannian metric such that the cut locus of $\mathrm{bd}(P_G)$ in P_G is precisely G.

If the graph G is metric, we ask in addition that the induced metric on G by the metric of P_G coincides to the original metric of G.

Basic examples show that a topological surface with boundary is not contractible to each graph it contains.

Definition 2.2 *We say that a graph (or a metric graph) G can be realized as a cut locus if there exist a riemannian surface $S_G = (S_G, h)$ and a point x in S_G such that G is isometric to $C(x)$.*

A. D. Weinstein (Proposition C in [26]) proved the following.

Lemma 2.3 *Let M be a d-dimensional riemannian manifold and D an d-disc embedded in M. There exists a new metric on M agreeing with the original metric on a neighbourhood of $M \setminus (\text{interior of } D)$ such that, for some point p in D, the exponential mapping at p is a diffeomorphism of the unit disc about the origin in the tangent space at p to M, onto D.*

Proposition 2.4 *The following statements are equivalent:*
i) the metric graph G can be realized as a cut locus;
ii) there exists a G-strip;
iii) there exists a riemannian G-strip.

Proof: *(i) → (ii)* Consider a point x on a riemannian surface (S, g), and a segment $\gamma : [0, l_\gamma] \to S$ parametrized by arclength, with $\gamma(0) = x$ and $\gamma(l_\gamma) \in C(x)$. For $\varepsilon > 0$ strictly smaller than the injectivity radius $\mathrm{inj}(x)$ at x, the point $\gamma(l_\gamma - \varepsilon)$ is well defined because $\mathrm{inj}(x) \leq l_\gamma$. Since $S \setminus C(x)$ is constractible to x along geodesic segments, and thus homeomorphic to an open disk, the union over all γs of those points $\gamma(l_\gamma - \varepsilon)$ is homeomorphic to the unit circle.

(ii) → (iii) An explicit construction of a riemannian G-strip from a given G-strip was provided by the first author in [8].

(iii) → (i) A. D. Weinstein's result above (Lemma 2.3) shows that, given a riemannian G-strip P_G, one can glue it to a disk to obtain a surface S_G, and there exists a metric g on S_G agreeing with the original metric on P_G, and a point x in S_G with $C(x) = G$. □

We need one more result, well known in the graph theory.

Lemma 2.5 *For every graph with m edges, n vertices, and q generating cycles holds $q = m - n + 1$.*

Theorem 2.6 *Every metric graph can be realized as a cut locus.*

Proof: By Proposition 2.4, it suffices to provide, for every metric graph G, at least one G-strip.

We notice first that we can reduce our problem to the cyclic part G^{cp} of G. Assume $G \setminus G^{cp}$ consists of finitely many finite trees, say $T_1, T_2, ..., T_m$. Since every tree T has a "leaf"-type T-strip, one can attach (in a natural way) all the T_i-strips to a G^{cp}-strip to obtain a G-strip.

We proceed by induction over the number k of generating cycles of G.

For $k=0$ and $G=G^{cp}$ the strip is elementary.

For $k=1$ and $G=G^{cp}$ our strip is the flat compact Möbius band.

Assume now that there exist strips for all graphs with k generating cycles, for some $k \geq 1$.

Let $G_{k+1} = G_{k+1}^{cp}$ be a metric graph with $k+1$ generating cycles, and e an edge of G_{k+1} in some generating cycle of G_{k+1}.

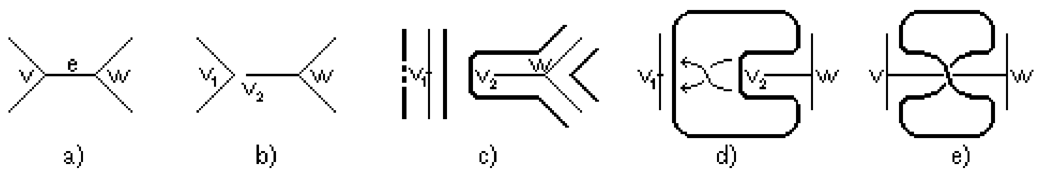

Figure 1: Induction reduction: edge e joins distinct vertices $v \neq w$.

Detach e from G_{k+1} at one extremity, say v; Figure 1(a)-(b) presents the case when e joins distinct vertices $v \neq w$, while Figure 2(a)-(b) presents the case $v = w$. Denote by G_k the resulting metric graph, and by v_1, v_2 the images of v in G_k.

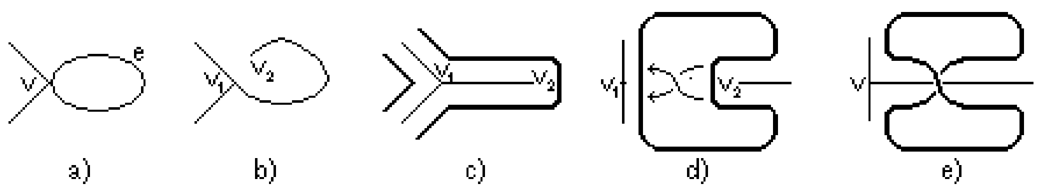

Figure 2: Induction reduction: edge e is a loop at v.

Since G_k has one vertex more than G_{k+1}, it has k generating cycles (see Lemma 2.5), and by the induction assumption there exists a G_k-strip P_{G_k} (see Figures 1(c) and 2(c)). Consider a planar representation of the boundary of P_{G_k} as a simple closed curve (illustrated in Figures 1(d) and 2(d)), and attach to it a switched e-strip, see Figures 1(e) and 2(e), to obtain a G_{k+1}-strip. $\square$

Remark 2.7 *Our results are also related to the cycle double cover conjecture, proposed by G. Szekeres [24] and P. Seymour [23], which states that every bridgeless graph G has a cycle double cover; i.e., a collection of cycles in G such that each edge is contained in exactly two of the cycles.*

The equivalent statement of this conjecture in terms of graph embeddings is known as the circular embedding conjecture: every biconnected graph has *a circular embedding onto a manifold; i.e., every face of the embedding is a simple cycle in the graph.*

A stronger version of the circular embedding conjecture asks about circular embeddings on orientable manifolds; in terms of cycle double covers, this is equivalent to the conjecture that there exists a cycle double cover, and an orientation for each of the cycles in the cover, such that for every edge e the two cycles that cover e are oriented in opposite directions through e [17].

In the language of graph theory, Theorem 2.6 shows that, for every connected graph G, there exists a 2-cell embedding with just one face, onto some surface S_G. In particular, for every such G there exists a closed path D in G containing all edges of G precisely twice, no edge in the cyclic part of G having consecutive appearances in D.

Question 2.8 *Several open questions naturally arise from Theorem 2.6.*

i) Can the metric of the surfaces S_G, realizing G as a cut locus, be chosen analytic? See the result of S. B. Myers [19] mentined in the introduction.

ii) Cut loci on riemannian surfaces may be quite large sets, see the introduction. Can Theorem 2.6 be extended to infinite graphs?

iii) Can Theorem 2.6 be extended to higher dimensions?

There usually are many strips on the same graph; we formalized this by two concepts [13].

Definition 2.9 *A* cut locus structure *(shortly, a CL-structure) on the graph G is a strip on the cyclic part G^{cp} of G.*

Definition 2.10 *Consider, for a point x on a riemannian surface (S, g) and for $0 < \varepsilon < \mathrm{inj}(x)$, the $C(x)$-strip obtained as the union, over all segments γ starting at x and parametrized by arclength, of the points $\gamma(l_\gamma - \varepsilon)$ (see Prop. 2.4). We call the CL-structure constructed in this way the* cut locus natural structure *defined by x, and denote it by $CLNS(x)$, or by $CLNS(x, g)$ if to point out (the dependence on) the metric g.*

With these definitions, Theorem 2.6, Proposition 2.4 and Lemma 2.3 can rephrased as that *each graph posess at least one CL-structure, and each CL-structure can be realized in a natural way*. They also allow us, whenever we consider surfaces realizing the graph G as a cut locus, to actually think about CL-structures on G.

The next two sections are related to the following.

Question 2.11 *What can be said about the riemannian surface S if, for every point x in S, $CLNS(x)$ does not depend on x?*

3 Constant curvature realizations

In this short section we present a direct way to realize some graphs as cut loci, different from that provided by Theorem 2.6.

Theorem 3.1 *Every CL-structure on a graph of constant order can be realized on a surface of constant curvature.*

Proof: Denote by G a cyclic graph of constant order k, and by $\mathcal{C}$ a CL-structure on G.

If G is a point then the unique CL-structure on G can be realized as $CLNS(x)$ for any point x on the unit 2-dimensional sphere.

Assume now that G is a cycle. Then again we have a unique CL-structure on G, and it can be realized as $CLNS(x)$ for any point x on the standard projective plane.

Consider now a graph G with $q \geq 2$ generating cycles; by Lemma 2.5, we get $m \geq 2$.

For $m = 2$, let $F_{2m} = F_4$ denote the square in the Euclidean plane Π.

For $m = 3$, let $F_{2m} = F_6$ denote the regular hexagon in Π.

For $m \geq 4$, consider a regular $2m$-gon $F_{2m} = \bar{z}_1 ... \bar{z}_{2m}$ in the hyperbolic plane $\mathbb{H}^2$ of constant curvature -1, such that $\angle \bar{z}_i \bar{z}_{i+1} \bar{z}_{i+2} = 2\pi/k$ (all indices are taken (mod $2m$)).

We view now the CL-structure $\mathcal{C}$ on G as a closed path D in G containing all edges of G precisely twice, hence every vertex of G appears precisely k times in D.

We identify now the path D with (the boundary of) F_{2m}, such that each image in D of an edge of G corresponds to precisely one edge in F_{2m}, each image in D of a vertex of G corresponds to precisely one vertex in F_{2m}, and the order of edges and vertices along D is preserved. It remains to identify, for every edge e in G, its two images in F_{2m}, to obtain a differentiable surface S_G of constant curvature -1. By construction, the natural cut locus structure of the image x in S_G of the center of F_{2m} is precisely $\mathcal{C}$. $\square$

Remark 3.2 *With a similar proof, one can show than every CL-structure on an arbitrary graph can be realized on a surface of constant curvature with* *at most $(n-p)$-singular points (i.e., on an Alexandrov surface with curvature bounded below, see [22] for the definition). Here, p is the number of vertices in G of maximal degree.*

Example 3.3 *The complete graphs K_r, the multipartite graphs $K_{p_1,...,p_r}$, as well as the graph of Petersen, can be realized as cut loci in constant curvature ($r, p_1, ..., p_r \in \mathbb{N}$).*

4 Stability

In this section we propose a notion of stability for cut locus structures, while in the next section we show that –roughly speaking– stability is a generic property of CL-structures. For our goal, we need to further investigate cut loci.

The cyclic part of the cut locus was introduced and first studied by J. Itoh and T. Zamfirescu [16].

Proposition 4.1 *The cyclic part of the cut locus depends continuously on the point.*

Proof: The upper semi-continuity of cut loci, as the reference point varies on the riemannian surface S, is well known and follows from the upper semi-continuity of geodesic segments.

The number q of generating cycles in the cyclic part of a cut locus does not depend on the point in S, hence it is constant on S. Therefore, if a new edge would appear in the cyclic part of a cut locus, it would produce a new cycle, hence another edge would have to disappear (see Lemma 2.5), contradicting the upper semi-continuity. □

Definition 4.2 *Consider a CL-structure $\mathcal{C}$ on the graph G, a riemannian surface (S, g) and a point $x \in S$. $\mathcal{C}$ is called* stable *with respect to x in S if*
(i) $CLNS(x) = \mathcal{C}$, and
(ii) there exists a neighbourhood of x in S, for all points y of which holds $CLNS(y) = \mathcal{C}$.

Definition 4.3 *The CL-structure $\mathcal{C}$ is called* globally stable *if it is stable on all surfaces where it can be realized as a CLNS.*

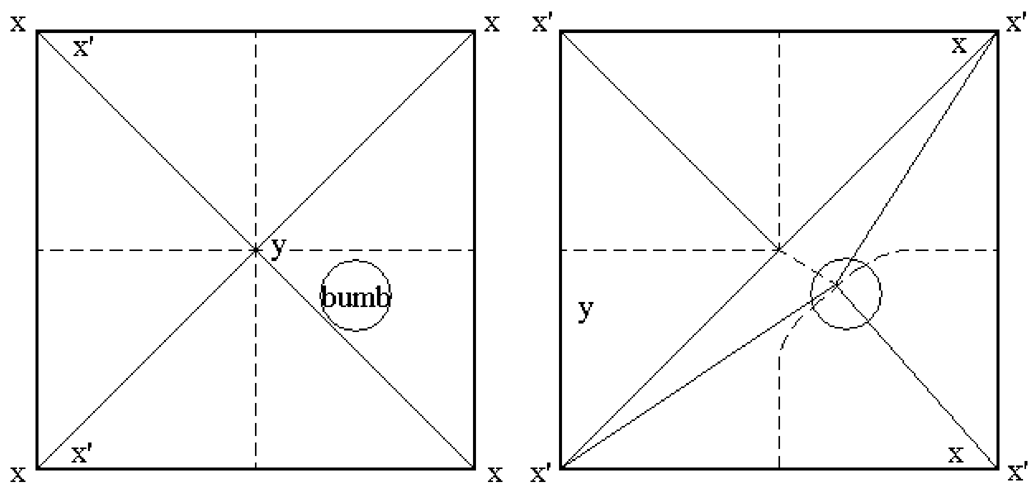

Figure 3: Unstable cut locus structure.

Remark 4.4 *Assume we have distinct pairs (S, x) and (S', x') of riemannian surfaces S, S' and points $x \in S$, $x' \in S'$ such that $CLNS(x) = CLNS(x') = \mathcal{C}$. If $\mathcal{C}$ is stable with respect to (S, x), it is not necessarily stable with respect to (S', x'), as the following example shows.*

Example 4.5 *i) Any CL-structure on a graph of constant order $k > 3$ is stable with respect to the natural realization given by Theorem 3.1.*

ii) We roughly explain here how to produce unstable CL-structures from those stable CL-structures at (i).

Consider, for example, a square fundamental domain of a flat torus T with a bump, see Figure 3 left. The cut locus of the point $x \in T$ represented at the corners of the square, is the 4-graph with one vertex y, as indicated by the dashed line. The four segments from x to y are also indicated, and are not affected by the bump. We choose x such that one segment is tangent to the bump's boundary.

Now consider a point x' arbitrarily close to x: slightly move x to "the right", for example, to x', see Figure 3 right. There remain three segments from x' to y, those in the upper-left half-domain; they are all shorter than the geodesic joining x' to y that crosses the bump, so y is a vertex of degree three in $C(x')$. There is another vertex of degree three in $C(x')$, also indicated in the figure together with the segments joining it to x'. In this case, $C(x')$ is *a 3-graph with two vertices and two generating cycles. J. Itoh and T. Sakai describe into details a similar procedure, see Remark 2.7 in [12].*

In conclusion, the 4-graph with one vertex is not stable with respect to x in T.

Theorem 4.6 *A cut locus structure on the graph G is globally stable if and only if G is a 3-graph.*

Proof: Let $\mathcal{C}$ be a locus structure on G.

Assume first that G is a 3-graph; then its cyclic part is itself a 3-graph. Assume, moreover, that $\mathcal{C}$ is realized as $\mathcal{C} = CLNS(x)$, for some point x on some riemannian surface S. If the point x' in S is a close to x then, by Proposition 4.1, the cyclic parts of $C(x)$ and $C(x')$ are homeomorphic, and we obtain the conclusion.

Assume now that G has a vertex y of degree strictly larger than 3, and consider a point x in the riemannian surface S such that $\mathcal{C} = CLNS(x)$. Then, by "putting" a bump tangent to one of the segment from x to y (i.e., modifying the metric on S accordingly) we obtain a new metric on S with respect to which we still have $\mathcal{C} = CLNS(x)$, but we have points x' arbitrarily close to x such that $CLNS(x') \neq \mathcal{C}$, see Example 4.5 or Theorem 5.2. $\square$

The following is, in some sense, opposite to Question 2.11.

Question 4.7 *How many stable CL-structures can exist on a given surface?*

5 Generic behaviour

We shall make use of the main result in [4], given in the following as a lemma. For, denote by $\mathcal{G}$ the *space of all Riemannian metrics on the surface* S; i.e., it is viewed as the space of sections of the bundle of positive definite symmetric matrices over S, endowed with the $\mathcal{C}^\infty$ Whitney topology [4].

Recall that a metric g on the surface S is called *cut locus stable* [4] if for any h close to g there is a diffeomorphism ϕ of the surface, depending continuously on h, such that $\phi(C(x,g)) = C(x,h)$; here, $C(x,g)$ denotes the cut locus of x with respect to g.

Lemma 5.1 [4] *For every point x in S there exists a set $\mathcal{B}_x$ of $C(x)$ stable metrics on S, open and dense in $\mathcal{G}$. Moreover, for any g in $\mathcal{B}_x$, every ramification point of the cut locus of x with respect to g is joined to x by precisely three segments.*

Theorem 5.2 *There exists an open and dense set $\mathcal{B} \subset \mathcal{G}$, every metric of which is stable with respect to the cyclic part C^{cp} of the cut loci, and realizes C^{cp} as a 3-graph.*

Proof: By Lemma 5.1 and Theorem 4.1, every $C(x)$ stable metric is also $C^{cp}(y)$ stable, for all points y close to x. Therefore, for every point x in S, there exists a neighbourhood $U_x \subset S$ all points y of which have the same set $\mathcal{B}_x$ of $C^{cp}(y)$ stable metrics (given by Lemma 5.1), open and dense in $\mathcal{G}$.

From the covering of S with the open sets U_x we can extract a finite subcovering, say $\{U_i\}_{i=1,\dots,f} = \{U_{x_i}\}_{i=1,\dots,f}$. It has the property that the sets of metrics $\mathcal{B}_i = \mathcal{B}_{x_i}$ given by Lemma 5.1 are constant on each U_i. Therefore, the set $\bigcap \mathcal{B}_i$ is still open and dense in $\mathcal{G}$, and consists of C^{cp} stable metrics. $\square$

The final result of this section follows directly from Theorem 4.6 and Theorem 5.2.

Theorem 5.3 *There exists an open and dense set $\mathcal{O} \subset S \times \mathcal{G}$, for every pair (x, g) of which the naturally defined cut locus structure $CLNS(x, g)$ is cubic and locally constant.*

The following result is well-known.

Lemma 5.4 *Every graph can be obtained from some cubic graph by edge contractions.*

Remark 5.5 *Passing from a stable CL-structure to another stable CL-structure is realized via a non-stable CL-structure, one that –in particular– lives on a non-cubic graph (see Theorem 5.3 4.6 and Lemma 5.4). This is realized in the first step by contracting one or several edge-strip(s), and in the second step by an operation, that we can think about as a "blowing up" all vertices of degree larger than 3 to trees of order 3. (A formal description of this is given in [14].)*

Remark 5.6 *Non-isometric surfaces realizing the same graph G as a cut locus (Theorem 2.6) are homeomorphic to each other, since topologically they can be distinguished only by their genus, which is a function on the number of generating cycles of G. Therefore, all distinct CL-structures on G "live" on homeomorphic surfaces. On the other hand, Theorem 5.3 shows in particular that "*equivalent*" CL-structures on G (the precise definition is given in [14]) can be realized on non-isometric surfaces.*

Acknowledgement This work was begun during C. Vîlcu's stay at Kumamoto University, supported by JSPS; its completition at IMAR Bucharest was partially supported by the grant PN II Idei 1187.

References

[1] L. Bérard-Bergery, *Quelques exemples de variétés riemanniennes où toutes les géodésiques issues d'un point sont fermées et de même longueur, suivis de quelques résultats sur leur topologie*, Ann. Inst. Fourier (Grenoble) **27** (1977), 231-249

[2] M. A. Buchner, *Stability of the cut locus in dimensions less than or equal to* 6, Invent. Math. **43** (1977), 199-231

[3] M. A. Buchner, *Simplicial structure of the real analytic cut locus*, Proc. Amer. Math. Soc. **64** (1977), 118-121

[4] M. A. Buchner, *The structure of the cut locus in dimension less than or equal to six*, Compos. Math. **37** (1978), 103-119

[5] H. Gluck and D. Singer, *Scattering of geodesic fields I* and *II*, Ann. Math. **108** (1978), 347-372, and **110** (1979), 205-225

[6] J. Hebda, *Metric structure of cut loci in surfaces and Ambrose's problem*, J. Differ. Geom. **40** (1994), 621-642

[7] J. Hebda, *Cut loci of submanifolds in space forms and in the geometries of Möbius and Lie*, Geom. Dedicata **55** (1995), 75-93

[8] J. Itoh, *Some considerations on the cut locus of a riemannian manifold*, in *Geometry of geodesics and related topics (Tokyo, 1982)*, 29-46, Adv. Stud. Pure Math., 3, North-Holland, Amsterdam, 1984

[9] J. Itoh, *The length of a cut locus on a surface and Ambrose's problem*, J. Differ. Geom. **43** (1996), 642-651

[10] J. Itoh and K. Kiyohara, *The cut loci and the conjugate loci on ellipsoids*, Manuscripta Math. **114** (2004), 247-264

[11] J. Itoh and K. Kiyohara, *The cut loci on ellipsoids and certain Liouville manifolds*, Asian J. Math., to appear

[12] J. Itoh and T. Sakai, *Cut loci and distance functions*, Math. J. Okayama Univ. **49** (2007), 65-92

[13] J. Itoh and C. Vîlcu, *Farthest points and cut loci on some degenerate convex surfaces*, J. Geom. **80** (2004), 106-120

[14] J. Itoh and C. Vîlcu, *Cut locus structures on graphs*, Discrete Math., to appear; arXiv:1103.1758

[15] J. Itoh and C. Vîlcu, *Orientable realizations of cut locus structures*, arXiv:1103.3136

[16] J. Itoh and T. Zamfirescu, *On the length of the cut locus on surfaces*, Rend. Circ. Mat. Palermo, Serie II, Suppl. 70 (2002) 53-58

[17] F. Jaeger, *A survey of the cycle double cover conjecture*, Ann. Discrete Math. **27** (1985), 1-12

[18] S. Kobayashi, *On conjugate and cut loci*, Global differential geometry, MAA Stud. Math. **27** (1989), 140-169

[19] S. B. Myers, *Connections between differential geometry and topology I* and *II*, Duke Math. J. **1** (1935), 376-391, and **2** (1936), 95-102

[20] H. Poincaré, *Sur les lignes géodésiques des surfaces convexes*, Trans. Amer. Math. Soc. **6** (1905), 237-274

[21] T. Sakai, *Riemannian Geometry*, Translation of Mathematical Monographs 149, Amer. Math. Soc. 1996

[22] K. Shiohama and M. Tanaka, *Cut loci and distance spheres on Alexandrov surfaces*, Actes de la Table Ronde de Géométrie Différentielle (Luminy, 1992), Sém. Congr., vol. 1, Soc. Math. France, 1996, 531-559

[23] P. D. Seymour, *Disjoint paths in graphs*, Discrete Math. **29** (1980), 293-309

[24] G. Szekeres, *Polyhedral decomposition of cubic graphs*, Bull. Austral. Math. Soc. 8 (1973), 367-387

[25] T. Zamfirescu, *Many endpoints and few interior points of geodesics*, Invent. Math. **69** (1982) 253-257

[26] A. D. Weinstein, *The cut locus and conjugate locus of a riemannian manifold*, Ann. Math. **87** (1968), 29-41

Jin-ichi Itoh
Faculty of Education, Kumamoto University
Kumamoto 860-8555, JAPAN
j-itoh@gpo.kumamoto-u.ac.jp

Costin Vîlcu
Institute of Mathematics “Simion Stoilow” of the Romanian Academy
P.O. Box 1-764, Bucharest 014700, ROMANIA
Costin.Vilcu@imar.ro