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On typical degenerate convex surfaces

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Abstract Various properties are given concerning geodesics on, and distance functions from points in, typical degenerate convex surfaces; i.e., surfaces obtained by gluing together two isometric copies of typical (in the sense of Baire category) convex bodies, by identifying the corresponding points of their boundaries.

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1 Introduction and statement of results

1.1 Introduction

In order to provide easy examples, degenerate convex surfaces (doubles of convex bodies) have been considered, for example, by Alexandrov [1] and, nowadays, by Shiohama and Tanaka [22]. The aim of this paper is to study typical such surfaces; we present properties of their geodesics and distance functions, particularly interesting because the faces of such surfaces are Euclidean.

A *convex body* in the Euclidean space $\mathbb{R}^d$ is a compact convex set with interior points. By a *convex surface of dimension d* we always mean a closed one; i.e., the boundary of a convex body in $\mathbb{R}^{d+1}$.

A *d-dimensional degenerate convex surface D* is the union of two isometric copies B and B' of a convex body $B_0 \subset \mathbb{R}^d$ $(d \geq 2)$, glued together along their boundary by identifying the points $x \in \mathrm{bd}B$ and $x' = \iota(x) \in \mathrm{bd}B'$, where $\iota : B \to B'$ is the isometry between B and B'. With some abuse of notation, we shall identify B and B_0

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when no confusion is possible. Call B and B' the *faces* of D, and D the *double* of B; the *ridge of* D is $\mathrm{rd}D = B \cap B'$. Thus, D is (seen as) limit in $\mathbb{R}^{d+1}$ of d-dimensional convex surfaces containing $\mathrm{rd}D$.

Unless otherwise stated, we shall assume the dimension d to be arbitrary.

The geometry of degenerate convex surfaces provides a bridge between the geometry of convex bodies and that of convex surfaces. It also provides examples of various metric properties on—topologically very simple—nondifferentiable, nonnegatively curved Alexandrov spaces (see [4] for the precise definition).

Denote by $\mathcal{K}$ the space of all convex bodies in $\mathbb{R}^d$, by $\mathcal{S}$ the set of their boundaries and by $\mathcal{D}$ the space of all d-dimensional degenerate convex surfaces.

Endowed with the usual Pompeiu–Hausdorff metric δ, the spaces $\mathcal{K}$, $\mathcal{S}$ and $\mathcal{D}$ are Baire. Each element of $\mathcal{K}$, $\mathcal{S}$ or $\mathcal{D}$, taken with the natural metric (see Sect. 1.2), is itself a Baire space. In any Baire space *most* (or *typical*) elements means "all except those in a set of first category".

We shall often mention, as term of comparison, results about typical convex surfaces. It seems necessary to point out, from the very beginning of this paper, that no such result refers—at least not directly—to degenerate surfaces. This is because the former are smooth (i.e., of differentiability class $\mathcal{C}^1$) and strictly convex (see [16] or [7]), while the latter neither smooth nor strictly convex. For properties of typical convex surfaces refer to the surveys [11,31] or, very close to our topic, [35].

1.2 Endpoints

For any two points x, y on a (possibly degenerate) convex surface D, $\rho(x, y)$ denotes the *intrinsic distance* between them, induced by the Euclidean distance, and ρ_x the *distance function from* x, $\rho_x(y) = \rho(x, y)$.

A *segment* between two distinct points is a shortest path joining them, and a *geodesic* is a curve which is locally a segment. With some abuse of terminology, we shall also call a segment or a geodesic the image set of such a curve, e.g., when talking about the intersection of geodesic arcs.

An *endpoint* of D is a point not interior to any segment. Of course, no such point exists on $\mathcal{C}^2$-differentiable surfaces.

Zamfirescu [29] proved that most points on a typical convex surface are endpoints, and asked if each point with infinite sectional curvature in every tangent direction is an endpoint. Our first result answers affirmatively a stronger form of this open problem, for typical degenerate convex surfaces.

Denote by $\gamma_i^\tau(x)$ and $\gamma_s^\tau(x)$ the *lower* and *upper curvatures* of the convex surface S at the point $x \in S$ in the tangent direction τ (see [3] p. 14 for the precise definition). Then [27] for most convex surfaces S, at most points $x \in S$, $\gamma_i^\tau(x) = 0$ and $\gamma_s^\tau(x) = \infty$, for any direction τ tangent to S at x.

Theorem 1 *If the point p in the ridge* $\mathrm{rd}D$ *of the typical degenerate convex surface D is interior to a segment, then $\gamma_i^\tau(p) = 0$ and $\gamma_s^\tau(p) < \infty$ hold for any direction τ tangent at p to* $\mathrm{rd}D$. *Consequently, most points of* $\mathrm{rd}D$ *are endpoints of D.*

1.3 Geodesics

We are concerned next with the existence of closed geodesics. A non trivial geodesic $G : I \subset \mathbb{R} \to D$ is *closed* if $I = \mathbb{R}$ and there exists $t > 0$ such that $G(t + s) = G(s)$ for any $s \in \mathbb{R}$; the smallest such t is the *period* of G.

The non-existence of closed geodesics on typical convex surfaces in $\mathbb{R}^3$ was proved by Gruber [10]. One might relate his result to the existence of residually-many endpoints on such surfaces.

Typical degenerate convex surfaces have much fewer endpoints. And, as we shall see in Sect. 3, there are 2-dimensional typical doubles D, and convex surfaces $S \subset \mathbb{R}^3$ arbitrarily close to D, such that each surface S contains a closed curve O most points of which are endpoints of S, and yet S has a simple closed geodesic crossing O.

Moreover, a classical result of Birkhoff [2] states that in any planar billiard table K there always exist trajectories of period n, for any integer $n \geq 2$.

A *billiard table* K is a convex body in $\mathbb{R}^d$, usually taken smooth (i.e., bd$K$ is of differentiability class $\mathcal{C}^1$). A *billiard ball* is a point which moves at unit velocity along a straight line inside K until it hits bd$K$, say at p, where it is reflected in the usual way (that is, the component of the velocity parallel to the exterior unit normal $n(p)$ of bd$K$ at p changes its sign). The curve described by a billiard ball is a *trajectory* in K.

Birkhoff also suggested the existence of a one-to-one correspondence between trajectories in a smooth billiard table K and geodesics on the double of K.

The mentioned theorem of Birkhoff is contrasted by the next result.

Theorem 2 *Most 2-dimensional degenerate convex surfaces contain no closed geodesics.*

We notice here that the family of all degenerate convex surfaces containing simple closed geodesics is dense in $\mathcal{D}$.

Let $G : I \subset \mathbb{R} \to D$ be a maximal geodesic on a typical double D; whether generically I is the real line, or a ray, or a (topologically closed) line-segment is clarified in the following.

Endow the sphere bundle $T_1 D$, associated with the degenerate convex surface D, with the topology induced by the distance

$$\delta_1((x, \tau), (y, \mu)) = \rho(x, y) + \rho_{S^{d-1}}(\tau, \mu),$$

where $\rho_{S^{d-1}}$ is the standard metric of S^{d-1}.

For $(x, \tau) \in T_1 D$, denote by $G(x, \tau)$ the maximal (with respect to inclusion) geodesic starting at x in direction τ; if there is no such geodesic put $G(x, \tau) = \{x\}$. Then $T_1 G(x, \tau) \subset T_1 D$ is the set of all pairs (y, μ) with $y \in G(x, \tau)$ and μ the direction of $G(x, \tau)$ at y.

Theorem 1 in [32] states that for most convex surfaces S the following holds: for any positive number r there exists a set T dense in $T_1 S$ such that, for any $(x, \tau) \in T$, there is a geodesic of length r, with midpoint x and with directions τ and $-\tau$ at x. Next result improves this statement in the framework of degenerate convex surfaces; it also contrasts, in some intriguing sense, the following result of Zamfirescu (Theorem

2 in [29]): on most convex surfaces, at each point, most tangent directions are *singular* (i.e., no segment starts in those directions).

Theorem 3 *For most* 2*-dimensional degenerate convex surfaces D, and most pairs (x, τ) in $T_1 D$, both $T_1 G(x, \tau)$ and $T_1 G(x, -\tau)$ are dense in $T_1 D$.*

Notice that the density of $T_1 G$ in $T_1 D$ implies the density of G in D, but not conversely.

The *phase space* ph$K$ of a billiard table K is the set $\mathrm{ph}K = \{(p, v) \in \mathrm{bd}K \times S^{d-1} : \langle v, n(p)\rangle < 0\}$.

We refer to the work of Gruber [9] for properties of trajectories in typical billiard tables; Theorem 5 therein states the density, in the phase space of a typical K, of the trajectories determined by most pairs $(p, v) \in \mathrm{ph}K$.

Choose $K \in \mathcal{K}$ and $(x, \tau) \in T_1 K$. The trajectory $T = T(x, \tau)$ in the billiard table K and the geodesic $G = G(x, \tau)$ on the double D_K of K may appear, at first glance, to correspond to each other via the isometries from K to the faces of D_K. This is indeed the case if bd$K$ is a polygon or of differentiability class $\mathcal{C}^2$, but it is false for most $K \in \mathcal{K}$. By Theorem 1, if $K \in \mathcal{K}$ is typical then no geodesic goes beyond most points of $\mathrm{bd}K = \mathrm{rd}D_K$, while all trajectories do. Theorem 2 underlines the difference, while Theorems 3 and 4 show some similarity.

With the price of the local length minimality, one might eliminate this difference by replacing geodesics with *quasigeodesics* (see [1] p. 373 for the definition). Indeed, each periodical trajectory of K yields a closed quasigeodesic on the double of K.

The first part of the next result parallels (and uses for its proof) Theorem 6 in [9]. The last part improves, in the framework of degenerate convex surfaces, the statement of Theorem 2 in [32], that on most convex surfaces there are non-self-intersecting geodesic arcs of arbitrary finite lengths.

The *positive orientation* of a planar convex curve is counter-clockwise.

Put $\Delta(\mathrm{rd}D, \varepsilon) = \{y \in D : \rho(y, \mathrm{rd}D) < \varepsilon\}$, for $\varepsilon > 0$.

Theorem 4 *For most* 2*-dimensional degenerate convex surfaces D and most pairs (x, τ) in $T_1 D$, for any $\varepsilon > 0$ and any integer $m > 0$, there are geodesic arcs $G_+, G_-, G' \subset G(x, \tau)$ such that G_+ circles m times in the positive direction in $\Delta(\mathrm{rd}D, \varepsilon)$, G_- circles m times in the negative direction in $\Delta(\mathrm{rd}D, \varepsilon)$, and G' is without self-intersections and of length larger than m.*

1.4 Cut loci

A *segment* between a point x and a closed set K not containing x is a segment from x to a point in K, not longer than any other such segment.

The *cut locus* $C(K)$ of the closed set $K \subset D$ is the set of all points $y \in D$ such that there is a segment from y to K not extendable as a segment beyond y.

The *multijoined locus* of K is the set $M(K)$ of all points $y \in D$ whose distance to K is realized by at least two segments to (not necessarily distinct) points in K.

Clearly, $C(K)$ includes both $M(K)$ and the set of all endpoints of D.

Cut loci have been studied for long time in Riemannian geometry (see, for example, [17] or [20]), and in the last years have been introduced for convex surfaces or Alexandrov spaces (see, for example, [18,22,38,39]).

Zamfirescu [29] showed that on most convex surfaces, any cut locus is residual and thus it has infinite length. Shiohama and Tanaka [22] also provided examples of 2-dimensional convex surfaces with non-rectifiable cut loci. Other examples of cut loci of infinite length were given by Gluck and Singer [6], and Hebda [12].

A recent result of Zamfirescu [39] proves, under very general hypotheses (see Lemma 14), a density property of $M(K)$ in a compact Alexandrov space.

This paper settles a new entry in the list of such examples; in contrast to the typical non-degenerate case, on typical doubles we have relatively few endpoints (by Theorem 1) and still very large cut loci.

Theorem 5 *For any closed set K interior to a face of a typical degenerate convex surface D, $M(K)$ is dense, and $C(K) \setminus M(K)$ is residual, in the opposite face.*

Theorem 5 says, in particular, that for any point x interior to a face of a typical double D, the set $C(x) \setminus M(x)$ contains most points of the opposite face. The segments from x to all points in $C(x) \setminus M(x)$ have mutually disjoint interiors and still, by Theorem 1, they cross the ridge of D at a set of first category in rd$D$.

Assume next $d = 2$. The set R_x of all points joined to the point x by at least three segments (i.e., the ramification points of $C(x)$) was studied by Zamfirescu [34], who proved that for each point x in a typical convex surface S, R_x is dense in S.

Theorem 6 *For any point x interior to a face of a typical 2-dimensional degenerate convex surface D, the set R_x is dense in the opposite face.*

1.5 Relative maxima

Among the points in $C(x)$, a special attention received the relative maxima of ρ_x. If $d = 2$ then, eventhough $C(x)$ can be residual on $S \in \mathcal{S}$, all relative maxima of ρ_x belong to some tree of $C(x)$ with at most three extremities [37].

It was also proven in [37] that in a certain open subset $\mathcal{S}_2 \subset \mathcal{S}$ of 2-dimensional convex surfaces, each typical element contains a point x with infinitely many relative maxima of ρ_x. For typical doubles of arbitrary dimension we have a stronger result. It is closely related to the main theorem in [30], stating that for most convex surfaces, most points in $\mathbb{R}^d$ lie on infinitely many normals.

Theorem 7 *For most points x on the ridge of a typical degenerate convex surface, the distance function ρ_x has infinitely many relative maxima.*

We propose here the following open question: can Theorem 7 be improved, at least for smaller sets of points x, to global maxima of ρ_x?

1.6 Farthest points

Let F_x denote the *set of farthest points from x*, i.e., global maxima of ρ_x. We shall usually write, when it is the case, $F_x = y$ instead of $F_x = \{y\}$.

This paper also contributes to the study of the farthest point sets, which Steinhaus had asked for (see Chap. A35 of [5]). Several of Steinhaus' questions have been answered for convex surfaces S in $\mathbb{R}^3$ by Zamfirescu; see for example [33,36,37], or the survey [24]. He proved that for any point x in S, any component of F_x is either a point or a Jordan arc, and gave examples of sets F_x homeomorphic to any compact subsets of the line [33]. For the case of Alexandrov surfaces, see [26]. It was also shown that for any convex surface $S \subset \mathbb{R}^3$, the farthest point mapping F is single-valued for most and—in the sense of measure—almost all points [37]. If moreover S is typical then most points $x \in S$ are joined to their unique farthest point by precisely three segments [34]. For doubles we have a similar result, but in arbitrary dimension.

Theorem 8 *On most degenerate convex surfaces D, for most points $x \in D$ there exists a unique farthest point, joined to x by precisely $d + 1$ segments.*

Rouyer [19] proved, for a compact manifold M endowed with a generic Riemannian metric, that a generic point x admits a unique farthest point. If moreover M is 2-dimensional, then x is joined to its unique farthest point by at most three segments.

A direct consequence of Theorem 8 is the next statement, of independent interest. It generalizes the main result in [28], about the spheres inscribed to a typical convex body, and—in some sense—contrasts a result of Gruber [8], that the unique ellipsoid of maximal volume inscribed to a d-dimensional typical convex body B has precisely $d(d + 3)/2$ contact points with bd$B$.

An ellipsoid Ell is said to be *inscribed to* a convex body B (or to bd$B$) if $Ell \subset B$ and card $(Ell \cap \text{bd}B) \geq 2$.

Theorem 9 *For most points x interior to a typical convex body $B \subset \mathbb{R}^d$, there exists a unique ellipsoid of revolution Ell inscribed to B, with a focus at x and of largest major axis. Ell touches* bd$B$ *at precisely $d+1$ points. Moreover, any open half-sphere of tangent directions at the second focus y of Ell contains the direction of some line yz, with $z \in Ell \cap \text{bd}B$.*

Notice that the ellipsoid inscribed to B and of largest possible major axis, obtained above if the point x moves freely in B, is a line-segment with extremities in bd$B$, by a direct consequence of Theorem 11.

The multivalued mapping F is called *injective* if $F_x \cap F_y = \emptyset$, for any distinct points $x, y \in D$; it is called *totally disconnected* if, for any point x, each component of F_x is a point.

The upper semicontinuity of F is a well-known fact. Motivated by a conjecture of Steinhaus (see Sect. A35 in [5]), aiming to characterize the spheres by the use of F, several classes of examples were recently provided, with the mapping F a(n involutive) homeomorphism or even an isometry [13,15,23,25]. Thus it appeared the open question of an alternative description for the set $\mathcal{S}_1 \subset \mathcal{S}$ of all surfaces with single-valued bijective F. By our Theorem 10, typical doubles do not belong to $\mathcal{S}_1$.

In $\mathbb{R}^3$, on a typical convex surface S there is no point x with an arc in F_x, but there exists an open set $\mathcal{S}_2$ of convex surfaces, most elements of which contain points x with card$F_x > 1$ [26]. It is an open conjecture of Zamfirescu [37], that $\mathcal{S}_2$ is dense in $\mathcal{S}$. The next theorem solves this conjecture for degenerate surfaces of arbitrary dimension.

The *diameter* of the subset M of the convex surface D is defined by $\mathrm{diam} M = \sup_{x,y \in M} \rho(x, y)$. Put $F_{\mathrm{rd} D} = \bigcup_{x \in \mathrm{rd} D} F_x$.

Theorem 10 *For any typical degenerate convex surface D, F is properly multivalued and $F_{\mathrm{rd} D}$ is a subset of first category in* rd$D$. *Any point y in $F_{\mathrm{rd} D}$ is an endpoint of D, and* $(\mathrm{diam} D)^{-1} \leq \gamma_i^\tau(y)$ *and* $\gamma_s^\tau(y) = \infty$, *for any direction τ tangent at y to* rd$D$. *If moreover $d = 2$ then F is injective and totally disconnected.*

Notice that, since most points in rd$D$ are endpoints of D and $F_{\mathrm{rd} D}$ is of first category in rd$D$, there are endpoints on rd$D$ which are not farthest points on D.

A detailed description of cut loci and farthest points for the doubles of convex n-gons and of d-dimensional simplices is given in [14].

Call *diametrally opposite* any two points $x, y \in D$ which verify $\rho(x, y) = \mathrm{diam} D$. For typical 2-dimensional convex surfaces, diametrally opposite points correspond to each other via F [24], but nothing seems to be known for dimension $d > 2$.

Theorem 11 *If the points x, y of a typical degenerate convex surface D are diametrally opposite then $x, y \in$* rd$D$ *and they correspond to each other via F, $F_x = y$ and $F_y = x$.*

A direct consequence of Theorem 11 is that, on most degenerate convex surfaces, any two diametrally opposite points are joined by precisely two segments, whose union is a simple closed quasigeodesic. This contrasts the following open problem of Zamfirescu [34]: is it true, for most convex surfaces, that the points realizing the diameter are joined by precisely five segments?

The remaining of the paper is devoted to the proofs of our results.

For $B \in \mathcal{K}$ and $S \in \mathcal{S}$ we sometimes denote by D_B the double of B and use $D_S = D_{\mathrm{conv} S}$. Let λG stand for the length of the curve G, and $[xv]$ for the line-segment determined by the points $x, v \subset \mathbb{R}^d$. We denote by $\Delta(p, r)$ the open intrinsic ball of radius r centered at p, and by $\Theta(p, r)$ the extrinsic one.

2 Proof of Theorem 1

Let us start with some lemmas we shall use later.

Lemma 1 [16,7] *Most convex surfaces are strictly convex and of differentiability class $\mathcal{C}^1$, but not $\mathcal{C}^2$.*

The following result is known for a long time. For $d = 3$ it is a particular case of the Cohn–Vossen–Herglotz–Pogorelov theorem, and for $d > 3$ it was proved by Sen′kin [21].

Lemma 2 *Let D, D′ be convex bodies in $\mathbb{R}^d$, whose boundaries S and S′ are of differentiability class $\mathcal{C}^1$ ($d \geq 3$). If S and S′ are isometric in the induced intrinsic metrics then D and D′ are isometric.*

Lemma 3 *A typical convex surface in $\mathcal{S}$ is the boundary of a typical convex body in $\mathcal{K}$, which corresponds to a typical double in $\mathcal{D}$, and conversely.*

Proof The set of all convex surfaces not isometric to a given one is easily seen to be open and dense in $\mathcal{S}$; similarly, for convex bodies in $\mathcal{K}$. This, together with Lemmas 1 and 2, completes the proof.

Lemma 4 [31] *For most convex surfaces S,*
(i) *at each point $x \in S$, $\gamma_i^\tau(x) = 0$ or $\gamma_s^\tau(x) = \infty$ for any tangent direction τ at x;*
(ii) *at most points $x \in S$, $\gamma_i^\tau(x) = 0$ and $\gamma_s^\tau(x) = \infty$ for any tangent direction τ at x.*

Let x be a point interior to $B \in \mathcal{K}$, D the double of B and $B' = \iota(B)$. Denote by $\mathcal{E}_{B,x}$ the family of all (hyper)ellipsoids of revolution with a focus at x and inscribed to B, and by y_E the point in B' corresponding to the second focus of $E \in \mathcal{E}_{B,x}$ through the isometry $\iota : B \to B'$. This defines a (very useful in the sequence) mapping $\Psi : \mathcal{E}_{B,x} \to \mathrm{int}B'$, by $\Psi(E) = y_E$.

Lemma 5 *For any $E \in \mathcal{E}_{B,x}$, each point v in $E \cap \mathrm{bd}B$ provides a segment $\Gamma = [xv] \cup [vy_E]$ from x to y_E, and conversely, each segment Γ from x to y_E provides a point in $E \cap \mathrm{bd}B$, given by $\Gamma \cap \mathrm{rd}D$.*

Proof Consider points $x, z \in \mathrm{int}B$ and $y \in \mathrm{int}B'$ with $y = \iota(z)$. Let $Ell_{xz} \subset \mathbb{R}^d$ be the ellipsoid of revolution with the foci at x and z and the sum of the focal radii equal to $\rho(x, y)$. Then $Ell_{xz} \setminus B = \emptyset$, for otherwise one could easily find points $v \in (\mathrm{int}(\mathrm{conv}Ell_{xz})) \cap \mathrm{bd}B$, for which the length of the path $[xv] \cup [vy] \subset D$ from x to y is shorter than $\rho(x, y)$, which is not possible. So $Ell_{xz} \subset B$.

Since x and y lie on different faces of D, any segment Γ joining them consists of two line-segments, say $\Gamma = [xv] \cup [vy]$, where $\{v\} = \Gamma \cap \mathrm{rd}D$. Therefore, since $Ell_{xz} \subset B$, $v \in Ell_{xz} \cap \mathrm{bd}B$.

Conversely, if $E \in \mathcal{E}_{B,x}$ then each point $v \in (\mathrm{bd}B) \setminus E$ verifies

$$||x - v|| + ||v - y_E|| > \rho(x, y_E),$$

and the conclusion follows.

Proof of Theorem 1 By Lemmas 3 and 1, the ridge of a typical double D is isometric to a strictly convex surface, hence it does not contain line-segments.

Let Γ be a segment from some point $x \in \mathrm{int}B$ to some point $y \in \mathrm{int}B'$, $z = \iota^{-1}(y) \in B$, and $Ell_{xz} \subset \mathbb{R}^d$ be the ellipsoid of revolution with foci at x and z and the sum of the focal radii equal to $\rho(x, y)$. Then Ell_{xz} is inscribed to B, by Lemma 5.

Since $\mathrm{bd}B$ is tangent to Ell_{xz} and $B \supset Ell_{xz}$, $\mathrm{bd}B$ has finite curvatures in all tangent directions at each contact point p with Ell_{xz}. So for any direction τ tangent at p to $\mathrm{bd}B = \mathrm{rd}D$, $\gamma_s^\tau(p) < \infty$ and $\gamma_i^\tau(p) = 0$, by *i)* of Lemma 4.

Consequently, by (ii) of Lemma 4, most points of $\mathrm{rd}D$ are not interior to any segment joining points on different faces of D. Since $\mathrm{bd}B$ contains no line-segments, most points of $\mathrm{rd}D$ are endpoints.

3 Proof of Theorem 2

The next result complements Theorem 1 and, together with the example following it, justifies the remark accompanying Theorem 2. Its proof follows the line used to prove

Theorem 1 in [29], and will be given elsewhere [15]. Consider next $\mathbb{R}^d \equiv \mathbb{R}^d \times \{0\} \subset \mathbb{R}^{d+1}$.

Lemma 6 [15] *For any compact right cylinder C in $\mathbb{R}^{d+1}$ over a typical convex body $B \subset \mathbb{R}^d$, most points of bd$B$ are endpoints of C.*

Example There exist 2-dimensional typical degenerate convex surfaces D, and convex surfaces $S \subset \mathbb{R}^3$ arbitrarily close to D, such that each surface S has a closed curve O most points of which are endpoints of S, and also has a simple closed geodesic crossing O.

Proof Consider a right cylinder $C \subset \mathbb{R}^3$ over a typical planar convex body B. Then, by Lemma 6, most points of bd$B$ are endpoints of C. Let B' denote the base of C opposite to B. We can choose points $x \in B$ and $y \in C \setminus (B \cup B')$ such that the angle α made by bd$B$ with some segment Γ from x to y is arbitrarily small.

Put $\{z\} = \Gamma \cap \text{bd}B$ and $\Gamma_B = \Gamma \cap B$. If α is small enough then there is a unique line T tangent to bd$B$, parallel to Γ_B and at smallest distance to Γ_B. Cut B along the normal N to bd$B$ at $T \cap \text{bd}B$, and keep the part $B_{1/2}$ containing the point z. Of course, we may assume $x \in B_{1/2}$.

Let L be the line parallel to N and tangent to bd$B_{1/2}$, $L \neq N$, and denote by M the normal to bd$B_{1/2}$ at $L \cap \text{bd}B_{1/2}$. Cut $B_{1/2}$ along M and keep the part $B_{1/4}$ containing the point z. Once again, we may assume $x \in B_{1/4}$.

Denote by B_1 the convex body obtained from $B_{1/4}$ by symmetries with respect to the lines M and N. Let $v \in B_1$ be the point symmetric to z with respect to M, and l the distance in $O = \text{bd}B_1$ from z to v.

Let C_1 be a right cylinder over B_1, of height $h = l \tan \alpha$.

Then, by the choice of B, most points of O are endpoints of C_1.

Observe that the maximal geodesic G of C_1 starting at x and including Γ_B is simple and closed. Indeed, it is orthogonal to N and, by the choice of h, is invariant with respect to the central symmetry of C_1.

Notice finally that for any $\varepsilon > 0$ we can construct a cylinder C_1 as above, and find a typical double $D \in \mathcal{D}$ at distance less than ε to C_1. The example is complete.

The argument below is similar to that proving the main result in [10]; for the reader's convenience, we indicate next the main steps of the proof and the necessary slight modifications in order to obtain our Theorem 2. We shall mostly use the same notation as in [10], to refer easier to [10] for details and further definitions.

Proof of Theorem 2 All geodesics below are considered with standard parametrizations $p : T \to D$, where $T = \mathbb{R}/\mathbb{Z}$. Define, for any integer $k \geq 1$,

$$\mathcal{A}_k = \{D \in \mathcal{D} : D \text{ contains a closed geodesic } G \text{ with properties } i) - v) \text{ below}\}.$$

(i) Any subarc H of G defined on a closed interval in T of length $\leq 1/k$ is a segment.
(ii) $1/k \leq \lambda G \leq k$.
(iii) There exists a number $\alpha \in T$ such that $\rho(p(\sigma), p(\alpha)) \geq 1/k$ for all $\sigma \in T$ with $|\sigma - \alpha|_T > 1/k$.

(iv) There are at most k values of the parameter corresponding to multiple points of G, say $0 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_e < 1$, where $0 \leq e \leq k$ and $|\sigma_l - \sigma_m|_T \geq 1/k$ for $l \neq m$. Distinct multiple points have distance $\geq 1/k$ in the sense of ρ.

(v) Each component of $D \setminus G$ contains an open intrinsic disk of radius $1/k$.

Then (see Proposition 1 in [10])

$$\{D \in \mathcal{D} : D \text{ contains a closed geodesic}\} \subset \bigcup_{k=1}^{\infty} \mathcal{A}_k,$$

and it suffices to show that the sets $\mathcal{A}_k$ are nowhere dense.

The proof that $\mathcal{A}_k$ is closed in $\mathcal{D}$ is the same as for Proposition 2 in [10].

In order to see that $\mathcal{A}_k$ has empty interior in $\mathcal{D}$, assume there exists a double $D_P \in \text{int}\mathcal{A}_k$ of the convex hull of an n-gon P, and consider countably many hyperplanes H_q given by

$$\{\omega \in \mathbb{R}^n : i_0\pi = i_1\omega^{(1)} + \cdots + i_n\omega^{(n)}\},$$

where $i_0, \ldots, i_n$ are integers and $i_1, \ldots, i_n$ are not all equal.

By Lemma 4 in [10], we may choose an n-gon Q close to P such that the double D_Q belongs to $\mathcal{A}_k$, and the curvature vector $\omega_{D_Q} = (\omega^1_{D_Q}, \ldots, \omega^n_{D_Q})$ is not contained in any of the hyperplanes H_q, where $\omega^i_{D_Q}$ is the curvature of D_Q at its vertex v_i.

Of course, it is impossible to assume, as in [10], that any geodesic disk of radius $1/k$ in D_Q contains at least one vertex of D_Q; but this assumption will not be necessary in our framework.

By the definition of $\mathcal{A}_k$, there exists a closed geodesic G on D_Q satisfying the assumptions (i) through (v). Let C_l be the components of $D_Q \setminus G$, $l = 1, \ldots, f$; each C_l is simply connected. Let G_l be the closed geodesic polygon bounding C_l, oriented in such a way that C_l is on the left hand side of G_l. Denote by α_{lm} the angles (measured in D_Q) of C_l at the multiple points of G contained in G_l, $m = 1, \ldots, n_l$.

Then (see [10]) one can assign integers $i(C_l)$ to the C_l's, not all of them equal, such that the sum $i(C_l) + i(C_m)$ is some constant $i(p)$ depending on the multiple point p, for any components C_l and C_m opposite with respect to p.

In addition to the proof in [10], here we need to assume that no $i(C_l)$ is zero. This is possible by adding, if necessary, the same (sufficiently large) positive integer to all $i(C_l)$'s.

The Gauss–Bonnet theorem implies now (see [10] for details)

$$\sum_{l=1}^{f} i(C_l)\omega(C_l) = \sum_{l=1}^{f} i(C_l)(2\chi_l - n_l)\pi + \sum_{l=1}^{f}\sum_{m=1}^{n_l} i(C_l)\alpha_{lm}.$$

It follows, just as in [10], that the right hand side of the preceding equality is an integer multiple of π, say $i_0\pi$.

Notice that $\omega(C_l) = \sum_{v_i \in C_l} \omega^i_{D_Q}$. Then, since D_Q is degenerate, some—but certainly not all—of $\omega(C_l)$'s might be zero. Since $i(C_l)$ were assumed all different from zero, the equality above shows that the n-tuple ω_{D_Q} belongs to one of the hyperplanes H_q. This contradicts the choice of ω_{D_Q}, and ends the proof.

4 Proof of Theorem 3

A straightforward improvement of the next result will be essential for the following proofs.

Lemma 7 [32] *Let G_P be a geodesic arc on a 2-dimensional polytopal convex surface P. If $S \in \mathcal{S}$ and $S \to P$ then there exist geodesic arcs $G_S \subset S$ such that $G_S \to G_P$.*

Lemma 8 *Let S, S_0 be 2-dimensional convex surfaces, and let G_0 be a geodesic arc on S_0. If $S \to S_0$ then there exist geodesic arcs $G_S \subset S$ such that $G_S \to G_0$.*

Proof Write Lemma 7 as follows

$$\forall \varepsilon > 0\ \exists \eta^* > 0 : \delta(P, S) < \eta^*/2 \Rightarrow \exists G_S \subset S \text{ s.t. } \delta(G_S, G_P) < \varepsilon/2. \quad (1)$$

Here, δ stands for the Pompeiu–Hausdorff distance between surfaces in $\mathcal{S}$, and also between closed subsets of $\mathbb{R}^d$:

$$\delta(G_S, G_P) < \varepsilon/2 \Leftrightarrow \begin{cases} \forall x_S \in G_S\ \exists x_P \in G_P \text{ s.t. } \|x_S - x_P\| < \varepsilon/2, \\ \forall x_P \in G_P\ \exists x_S \in G_S \text{ s.t. } \|x_S - x_P\| < \varepsilon/2. \end{cases}$$

The statement in Lemma 7 also holds when changing the places of P and S; the proof is similar to its proof in [32] and will therefore be omitted. Thus, if $S_0 \in \mathcal{S}$ and $G_0 \subset S_0$ are given, with G_0 a geodesic arc of S_0, and if $P \to S_0$, with P convex polytopal surfaces, then

$$\forall \varepsilon > 0\ \exists \eta^+ > 0 : \delta(S_0, P) < \eta^+/2 \Rightarrow \exists G_P \subset P \text{ s.t. } \delta(G_0, G_P) < \varepsilon/2, \quad (2)$$

where G_P is a geodesic arc on P. Together, (1) and (2) imply

$$\begin{aligned} \forall \varepsilon > 0\ \exists \eta &= \min\{\eta^+, \eta^*\} : \delta(S_0, S) < \delta(S_0, P) + \delta(P, S) < \eta \\ &\Rightarrow \exists G_S \subset S \text{ s.t. } \delta(G_0, G_S) < \delta(G_0, G_P) + \delta(G_S, G_P) < \varepsilon \end{aligned}$$

and we are done.

In the following, we shall implicitly assume the geodesic arcs $G = G(s)$, $s \in [0, L]$, and $G_0 = G_0(t)$, $t \in [0, L_0]$, to be parametrized in terms of arclength. Then $T_1 G \to T_1 G_0$ means $L \to L_0$ and $(G(s), \tau_{G(s)}) \to (G_0(t), \mu_{G_0(t)})$ as $s \to t$, for any $t \in [0, L_0]$; here, $\tau_{G(s)}$ and $\mu_{G_0(t)}$ are the directions of G and G_0 at $G(s)$ and $G_0(t)$, respectively.

With this convention, the next statement is a simple consequence of the fact that each component of $G \setminus \mathrm{rd} D$ is a line-segment, in particular it has constant direction, for any geodesic arc G on any 2-dimensional degenerate convex surface D.

Lemma 9 *Let D, $D_0 \in \mathcal{D}$ be 2-dimensional and $G \subset D$, $G_0 \subset D_0$ be geodesic arcs. If $D \to D_0$ and $G \to G_0$ then $T_1 G \to T_1 G_0$.*

Lemma 10 [41] *There is a set $\mathcal{P}$ of polygons which is dense in the set of all planar convex curves, such that for each $P \in \mathcal{P}$ there is a billiard trajectory in P which approaches any point x of P and any direction τ at x arbitrarily closely.*

A geodesic arc G of D is called an *ε-net* if $T_1 D \setminus T_1 G$ contains no open ball of radius ε.

Lemma 11 *Most 2-dimensional degenerate convex surfaces have an ε-net for any $\varepsilon > 0$.*

Proof Define, for any natural $q \geq 1$,

$$\mathcal{A}_q = \{D \in \mathcal{D} \ : \ \text{there is no } q^{-1}\text{−net in } T_1 D\},$$

and observe that a surface with no ε-net, for some $\varepsilon > 0$, necessarily belongs to $\bigcup_{q \geq 1} \mathcal{A}_q$.

We show next that $\mathcal{A}_q$ is closed in $\mathcal{D}$. Assume the contrary be true, and consider a sequence of doubles $D_n \in \mathcal{A}_q$ convergent to a double $D \in \mathcal{D} \setminus \mathcal{A}_q$. Then there exists a geodesic arc $G \subset D$ such that the maximal radius of a ball (if any) included in $T_1 D \setminus T_1 G$ is $r < q^{-1}$. Take $\varepsilon > 0$ such that $r + \varepsilon < q^{-1}$. Since $D_n \to D$, Lemmas 8 and 9 provide geodesic arcs $G_n \subset D_n$ such that $T_1 G_n \to T_1 G$. Then there exists $\eta > 0$ such that $\delta(D, D_n) < \eta$ and $|s - t| < \eta$ imply

$$\delta_1((G(s), \tau_{G(s)}), (G_n(t), \mu_{G_n(t)})) < \varepsilon,$$

where we assumed n large enough to assure $G_n(t) \in \text{int}B \cap \text{int}B_n$, with B and B_n faces of D and D_n, respectively. Thereby, the maximal radius of a ball included in $T_1 D_n \setminus T_1 G_n$ is less that $r + \varepsilon < q^{-1}$, and a contradiction is obtained.

Note that $\mathcal{A}_q$ has empty interior in $\mathcal{D}$. Indeed, if not then take an open set $\mathcal{O} \subset \mathcal{A}_q$ and a polygonal double $D_P \in \mathcal{O}$ corresponding to some polygon P as in Lemma 10. Denote by G_P a geodesic of D_P corresponding to a trajectory in P described by Lemma 10. A straightforward verification shows that $T_1 G_P$ is dense in $T_1 D_P$. Now, Lemmas 8 and 9 together with the previous argument show that, if $D_n \in \mathcal{O}$ is close enough to D_P, then $D_n \notin \mathcal{A}_q$, in contradiction to the choice of $\mathcal{O}$.

Since $\mathcal{A}_q$ is closed and has empty interior in $\mathcal{D}$, $\bigcup_{q \geq 1} \mathcal{A}_q$ is of first category and the proof is complete.

Proof of Theorem 3 Consider a surface $D \in \mathcal{D}$ with the property given by Lemma 11.

Let $\{x_n : n \in \mathbb{N}\}$ be dense in $\mathbb{R}^2$, and $\{\sigma_n : n \in \mathbb{N}\}$ be dense in S^1. Define

$$A = \{(x, \tau) \in T_1 D : \text{cl}\,(T_1 G(x, \tau)) \neq T_1 D\},$$
$$A_{m,p,q} = \{(x, \tau) \in T_1 D : \Delta((x_m, \sigma_p), q^{-1}) \subset T_1 D \setminus \text{cl}\,(T_1 G(x, \tau))\}.$$

Then clearly

$$A \subset \bigcup_{m,p,q\geq 1} A_{m,p,q}.$$

Observe first that $A_{m,p,q}$ is closed in $T_1 D$. Indeed, if $A_{m,p,q}$ were not closed then there would be a sequence of pairs $(y_n, \tau_n) \in A_{m,p,q}$ convergent to a pair $(y, \tau) \in T_1 D \setminus A_{m,p,q}$. By Lemmas 8 and 9, $T_1 G(y_n, \tau_n)$ would converge to $T_1 G(y, \tau)$.

Since $(y, \tau) \notin A_{m,p,q}$, $T_1 G(y, \tau) \cap \Delta((x_m, \sigma_p), q^{-1}) \neq \emptyset$. Then there exist $s, t > 0$ such that $y = G(s)$, $\tau = \tau_{G(s)}$ and, for n large enough,

$$\delta_1((G_n(t), \tau_{G_n(t)}), (y, \tau)) < q^{-1} - \delta_1((y, \tau), (x_m, \sigma_p)),$$

whereby

$$\delta_1((G_n(t), \tau_{G_n(t)}), (x_m, \sigma_p)) < q^{-1}$$

and consequently

$$T_1 G(y_n, \tau_n) \cap \Delta((x_m, \sigma_p), q^{-1}) \neq \emptyset,$$

in contradiction to the choice of (y_n, τ_n) in $A_{m,p,q}$.

Assume now that the set $A_{m,p,q} \subset T_1 D$ has interior points, and consider a ball Δ^* of radius $r < q^{-1}$ included in $A_{m,p,q}$. By Lemma 11, there exists an ε-net G^* of $T_1 D$, for some fixed $\varepsilon < r$. Since $\varepsilon < r < q^{-1}$,

$$T_1 G^* \cap \Delta^* \neq \emptyset \quad \text{and} \quad T_1 G^* \cap \Delta((x_m, \sigma_p), q^{-1}) \neq \emptyset.$$

Then, for $(x, \tau) \in T_1 G^* \cap \Delta^*$, there is a geodesic arc $G \subset G(x, \tau) \cap G^*$ and $(y, \mu) \in T_1 G \cap \Delta((x_m, \sigma_p), q^{-1})$. This implies $(x, \tau) \in A_{m,p,q}$ and $T_1 G(x, \tau) \cap \Delta((x_m, \sigma_p), q^{-1}) \neq \emptyset$, contradicting the definition of $A_{m,p,q}$.

Since $A_{m,p,q}$ is closed and has empty interior, it is nowhere dense in D, and $A \subset \cup_{m,p,q\geq 1} A_{m,p,q}$ is of first category. Therefore, the set

$$A_- = \{(x, \tau) \in T_1 D : \text{cl} T_1 G(x, -\tau) \neq T_1 D\}$$

is also of first category, as well as $A \cup A_-$. Then $\complement(A \cup A_-) = \complement A \cap \complement A_-$ contains most elements of D, and the proof is complete.

5 Proof of Theorem 4

For $K \in \mathcal{K}$, define $K_\varepsilon = \{x \in K : \text{cl}\Theta(x, \varepsilon) \not\subset K\}$.

Lemma 12 [9] *Most billiard tables K in $\mathbb{R}^2$ have the following property: for most $(p, v) \in \text{ph}K$ the trajectory in K starting at p in direction v circles, in a certain* *period of time, m times in the positive direction in* K_ε *and, a later period of time, m times in the negative direction, for any* $\varepsilon > 0$ *and any integer* $m > 0$.

Lemma 13 *On most* 2*-dimensional degenerate convex surfaces there are geodesic arcs without self-intersections, of arbitrary finite lengths.*

Proof The argument below is similar to that proving Theorem 2 in [32]; for the sake of completeness, we indicate next the main steps of the proof and refer to [32] for details.

Let $\mathcal{A}_n$ denote the set of all $D \in \mathcal{D}$ admitting only geodesic arcs without self-intersections of length at most n.

In order to show that $\mathcal{A}_n$ is nowhere dense, let $\mathcal{O} \subset \mathcal{D}$ be open and choose $D \in \mathcal{O}$ typical. Let $x, y \in \mathrm{rd}D$ realize the diameter of D (see Theorem 11). Then there are precisely two segments from x to y, orthogonal to $\mathrm{rd}D$ at both x and y, whose union G decompose D into two pieces. Cut along G and insert (glue along the cuts) the union R of two rectangles of length $\rho(x, y)$ and width ε, with ε conveniently small. The resulting degenerate convex surface D_ε belongs to $\mathcal{O}$ and contains, inside R, a geodesic arc without self-intersections and of length $> n$. By Lemma 8, any surface D^* close enough to D_ε will also contain a geodesic arc without self-intersections and of length $> n$, which proves that $D^* \notin \mathcal{A}_n$ and ends the proof.

Proof of Theorem 4 Consider $K \in \mathcal{K}$ as in Lemma 12, and such that its double D has the property given by Theorem 3. Fix $\varepsilon > 0$ and an integer $m > 0$.

Also consider $(p, v) \in \mathrm{ph}K$ such that the trajectory T starting at p in direction v circles, in a certain period of time, m times in the positive direction in $K_{\varepsilon/2}$.

Choose $(x, \tau) \in T_1 D$ such that $\mathrm{cl}(T_1 G(x, \tau)) = T_1 D$. Then there exists a sequence of pairs $(x_n, \tau_n) \in T_1 G(x, \tau)$ converging to (p, v). Let $z_n \in \mathrm{rd}D$ be the closest point to p where the maximal (with respect to inclusion) line-segment of $G(x_n, \tau_n)$ containing x_n cuts $\mathrm{rd}D$. Then $z_n \to p$.

Consider the trajectory T_n in K determined by (z_n, τ_n). The faces of D are isometric to K, so T_n and $G(z_n, \tau_n)$ corresponds to each other as long as $G(z_n, \tau_n)$ does not hit an endpoint, which is not the case because $G(z_n, \tau_n) \subset G(x, \tau)$ and by our choice of $G(x, \tau)$ according to Theorem 3. Since $(z_n, \tau_n) \to (p, v)$ and $\angle(v, n(p)) < 0$, $T_n \to T$ (see Lemma 1 in [9]). Therefore, in a certain period of time, T_n circles m times in the positive direction in K_ε, for n large enough. Consequently, there is a geodesic arc $G_+ \subset G(x, \tau)$ such that G_+ circles m times in positive direction in $\Delta(\mathrm{rd}D, \varepsilon)$.

The proof for the existence of G_- is similar.

For the last part of the statement, consider $D \in \mathcal{D}$ with the properties in Lemma 13 and Theorem 3. Take a geodesic arc G_0 of D without self-intersections, of length $\lambda G_0 > m$, joining the points $x_0, y_0 \in D$. Assume $(x_0, \tau_0) \in T_1 G_0$. Also take $(x, \tau) \in T_1 D$ such that $\mathrm{cl}(T_1 G(x, \tau)) = T_1 D$. Then there exists a sequence of pairs $(x_n, \tau_n) \in T_1 G(x, \tau)$ converging to (x_0, τ_0). Consequently, there exist points $y_n \in G(x, \tau)$, $y_n \to y$, and geodesic arcs $G_n \subset G(x_n, \tau_n)$ from x_n to y_n, $G_n \to G_0$, such that $\lambda G_n > m$. Since $G_n \to G_0$, G_n has no self-intersections if n is large enough.

6 Proofs of Theorems 5, 6 and 7

Once again, we start the section with a lemma.

Lemma 14 [39] *Let the compact Alexandrov space $\mathcal{A}$ be a d-dimensional topological manifold ($d \geq 2$), and A a subset of $\mathcal{A}$. Assume that the set of endpoints of $\mathcal{A}$ is dense in A (with respect to its relative topology), and K is a closed subset of a union U of components of $\mathcal{A} \setminus A$. If U is not dense in $\mathcal{A}$ then the multijoined locus $M(K)$ is dense in the interior of $\mathcal{A} \setminus U$.*

Proof of Theorem 5 By Theorem 1, the set of endpoints of D is dense in rd$D$, so we may apply Lemma 14 for $K \subset B$ and $A = \text{rd}D$, to get the density of $M(K)$ in B', where B and B' are the faces of D.

Next we easily adapt the proof of Theorem 2 in [38] to show that $C(K) \setminus M(K)$ is residual in B'.

Let E_m be the set of those points $z \in B'$ interior to a segment from K to $C(K)$, whose length from z to $C(K)$ is at least $1/m$. Then E_m is nowhere dense in B'.

To see this, take $y \in M(K)$. Suppose there exists a sequence of points $z_k \in E_m$ converging to y, and consider a compact neighbourhood V of y, containing some ball $\Delta(y, \varepsilon)$. Then, for integers $m_0 > m$ such that $1/m_0 < \varepsilon/3$, and k_0 such that $\rho(z_k, y) < \varepsilon/3$ for each $k \geq k_0$, we have $z_k \in E_{m_0} \cap V$ for all $k \geq k_0$.

Denote by y_k the cut point of K along the segment joining it to z_k. Possibly passing to a subsequence, we may assume that $\{y_k\}_{k \geq k_0}$ converges; then there exists a subsequence of the corresponding sequence of segments from K to y_k's, which converges to a segment from K to y, of length

$$\lim_{k\to\infty} (\rho(K, z_k) + \rho(z_k, y_k)) \geq \lim_{k\to\infty} (\rho(K, z_k) + m_0^{-1}) = \rho(K, y) + m_0^{-1},$$

impossible. Thus, there exists an open neighbourhood of y in B' whose points are not in E_m, and so E_m is nowhere dense in B'.

Therefore, $C(K) = B' \setminus \cup_{m\geq 1} E_m$ contains most points of B'.

Let now G_m be the set of points in $C(K)$ joined to K by two segments at Pompeiu–Hausdorff distance at least $1/m$. We show that G_m is nowhere dense in B'.

Indeed, let $y \in C(K)$, and assume that there exists a sequence of points $y_k \in G_m$ which converges to y. Then, since the two sequences of segments from K to y_k converge to two segments from K to y, at Pompeiu–Hausdorff distance at least $1/m$, $y \in G_m$. Thus, G_m is closed and, since $\text{int}G_m \subset \text{int}C(K) = \emptyset$, G_m is nowhere dense in B'.

Therefore, $M(K) = \cup_{m\geq 1} G_m$ is of first category in B', and the proof is complete.

Lemma 15 [26] *Let S be a 2-dimensional convex surface, $x \in S$, and $J \subset C(x)$ be an arc each point of which is joined to x by precisely two segments. Let y_1, y_2 be the endpoints of J. Then the domain Δ bounded by the segments from x to y_1, y_2 and containing* int$J$ *verifies $\Delta \cap C(x) \subset J$.*

Proof of Theorem 6 Suppose there exists, for some point $x \in \text{int}B$, an arc $J \subset C(x)$ without ramification points. Possibly restricting to a subset, we may assume

$J \cap \mathrm{rd}D = \emptyset$. Then the domain Δ provided by Lemma 15 contains no endpoints. But the curve $\mathrm{rd}D$ separates x and J, so $\Delta \cap \mathrm{rd}D$ is an arc each point of which is interior to a segment, in contradiction to Theorem 1.

Since R_x is dense in $C(x)$, and $C(x)$ is dense in B', the conclusion follows.

The proof of Lemma 16 appears inside the proof of the main theorem in [30].

Lemma 16 *For any typical convex surface $S \subset \mathbb{R}^d$, any set O open in $\mathbb{R}^d$, any point y_0 in O, and any natural number k, there exist a normal N to S, a point $y \in N \cap O$ arbitrarily close to y_0, and a Euclidean ball around y for each point v of which the function $f : S \to \mathbb{R}$, given by $f(w) = ||v - w||$, has at least k relative maxima.*

Proof of Theorem 7 Consider a typical d-dimensional degenerate convex surface D whose ridge S has the property in Lemma 16 (by Lemma 3, S is a typical convex surface of dimension $d - 1$).

Let $\mathcal{N}_x$ denote the set of normals to S through the point x. For any natural number n define

$$A = \{x \in S : \mathcal{N}_x \text{ is finite}\},$$
$$A_n = \{x \in S : \mathrm{card}\mathcal{N}_x \leq n\}.$$

Then we clearly have

$$A \subset \bigcup_{n \geq 1} A_n.$$

Next we show that A_n is nowhere dense, for any $n \geq 1$. For, assume $A_n \neq \emptyset$ and consider a point $y_0 \in A_n$ and an open set $O \subset \mathbb{R}^d$ around y_0. By Lemma 16, there exist a normal N to S, a point $y \in N \cap O$ arbitrarily close to y_0, and a Euclidean ball $\Theta(y, \varepsilon)$ around y, for each point v of which the function $f : S \to \mathbb{R}$, given by $f(w) = ||v - w||$, has at least $n + 1$ relative maxima. Observe that we may take $y \in N \cap O \cap S$, and consider only points $v \in \Theta(y, \varepsilon) \cap S$.

Since A_n is nowhere dense, A is of first category and the proof is complete.

7 Proofs of Theorems 8 and 9

The next lemmas will be necessary for the proof of Theorem 8.

Lemma 17 *Let $D, D_n \in \mathcal{D}$ and $x, y \in D$, $x_n, y_n \in D_n$ with $y \in D \setminus \mathrm{rd}D$. Let $\Gamma, \Gamma' \subset D$ be segments from y to x and $\Gamma_n, \Gamma'_n \subset D_n$ be segments from y_n to x_n. If $D_n \to D$, $x_n \to x$, $y_n \to y$, $\Gamma_n \to \Gamma$ and $\Gamma'_n \to \Gamma'$ then the angle between Γ_n and Γ'_n at y_n converges to the angle between Γ and Γ' at y.*

Proof Suppose first that the points x, y belong to the same face of D. Then, since $y \notin \mathrm{rd}D$, $\Gamma = \Gamma'$ and now $\Gamma_n \to \Gamma$, $\Gamma'_n \to \Gamma'$ directly imply the conclusion.

Suppose now that x, y belong to opposite faces of D. Put $\{z\} = \Gamma \cap \mathrm{rd}D$, $\{z'\} = \Gamma' \cap \mathrm{rd}D$. We can assume, for n sufficiently large, that x_n, y_n belong to opposite faces of D_n. Define $\{z_n\} = \Gamma_n \cap \mathrm{rd}D_n$, $\{z'_n\} = \Gamma'_n \cap \mathrm{rd}D_n$.

Then, since $x_n \to x$, $y_n \to y$, $\Gamma_n \to \Gamma$ and $\Gamma'_n \to \Gamma'$, we get $z_n \to z$, $z'_n \to z'$. Now, the convergence

$$\angle(\Gamma_n, \Gamma'_n) = \angle z_n y_n z'_n \to \angle z y z' = \angle(\Gamma, \Gamma')$$

simply becomes convergence of angles in the Euclidean space, and completes the proof.

A point $y \in S$ is called *critical with respect to* ρ_x, or simply *critical*, if for any direction τ at y there is a segment from y to x with direction τ' at y, such that $\angle(\tau, \tau') \leq \pi/2$; y is called *strictly critical* if $\angle(\tau, \tau') < \pi/2$.

Lemma 18 [40] *Let x, y be two points on a convex polyhedral surface P in $\mathbb{R}^{d+1}$. If y is a relative maximum of ρ_x but not a vertex of P, then y is a strictly critical point for ρ_x, and there are at least $d+1$ segments from x to y.*

We shall also use the following variant of Lemma 18.

Lemma 19 *Let x, y be two points on a convex surface S in $\mathbb{R}^d$. If the point y is a relative maximum of ρ_x then it is critical for ρ_x.*

Proof Assume $y \in S$ is a relative maximum, but not a critical point for ρ_x. Denote by S_{xy} the set of all segments from y to x. Then there exists a direction τ at y making angles $> \pi/2$ with any segment $\Gamma \in S_{xy}$. Since S_{xy} is closed, there exists $\varepsilon_1 > 0$ such that $\min_{\Gamma \in S_{xy}} \angle(\tau, \Gamma) > \pi/2 + \varepsilon_1$ still holds.

Consider a segment Γ^* starting at y in a direction μ sufficiently close to τ in order to have, for any $\Gamma \in S_{xy}$, $\angle(\Gamma^*, \Gamma) > \pi/2 + \varepsilon_2$, for some $\varepsilon_2 > 0$. Then there exists $\varepsilon_3 > 0$ such that

$$-\cos \min_{\Gamma \in S_{xy}} \angle(\Gamma^*, \Gamma) > \varepsilon_3.$$

Also consider points $y_n \in \Gamma^*$, $y_n \to y$. The first variation formula (Theorem 3.5 in [18]) gives now

$$\begin{aligned}\rho(x, y_n) &= \rho(x, y) - \rho(y, y_n) \cos \min\nolimits_{\Gamma \in S_{xy}} \angle(\Gamma^*, \Gamma) + o(\rho(y, y_n))\\ &> \rho(x, y) + \rho(y, y_n)[\varepsilon_3 + \rho(y, y_n)^{-1} o(\rho(y, y_n))]\\ &> \rho(x, y)\end{aligned}$$

for n sufficiently large, and a contradiction is obtained.

The following reciprocal of Lemma 19 is of some independent interest.

Lemma 20 *If x, y are points in a convex surface $S \subset \mathbb{R}^d$, and y is a strictly critical point for ρ_x, then y is a strict relative maximum for ρ_x.*

Proof Denote by $\mathcal{T}$ the set of all directions at y of segments from y to x; by the hypothesis, no closed half-sphere of S^{d-1} contains $\mathcal{T}$. Notice that

$$\exists \varepsilon > 0 \;\; \forall \mu \in S^{d-1} \;\; \exists \tau \in \mathcal{T} \;\; \angle(\mu, \tau) < \pi/2 - \varepsilon.$$

Suppose this is false and take $\varepsilon_n \to 0$. So it exists $\mu_n \in S^{d-1}$ such that, for all $\tau \in \mathcal{T}$, $\angle(\mu_n, \tau) \geq \pi/2 - \varepsilon_n$. Consider a limit direction μ of $\{\mu_n\}_n$. Then for any $\tau \in \mathcal{T}$ holds $\angle(\mu, \tau) \geq \pi/2$, which provides an closed half-sphere of S^{d-1} (centered at τ) containing $\mathcal{T}$, and a contradiction is obtained.

Denote by V_y the closed intrinsic ball around y of radius $2l\cos(\pi/2 - \varepsilon)$, where $l = \rho(x, y)$. We have to show that $\rho_x(y) > \rho_x(z)$ holds for all $z \in V_y$.

For any point $z \in V_y \setminus \{y\}$ and any segment Γ_{yz} from y to z in direction τ_z at y, there exists some segment Γ_{yx} from y to x in direction τ_x at y such that $\alpha = \angle(\tau_z, \tau_x) < \pi/2 - \varepsilon$. We get

$$\rho(y, z) \leq 2l\cos(\pi/2 - \varepsilon) < 2l\cos\alpha.$$

Consider the planar triangle $\bar{x}\bar{y}\bar{z}$ with $||\bar{x} - \bar{y}|| = l$, $||\bar{y} - \bar{z}|| = \rho(y, z)$ and the angle at $\bar{y}$ equal to α. We have $||\bar{y} - \bar{z}|| < 2l\cos\alpha$, hence the angle at $\bar{z}$ is larger than α and thus $||\bar{x} - \bar{z}|| < ||\bar{x} - \bar{y}||$.

By the convexity of the metric of S (see [1] or [3]), we have $\rho(x, z) \leq ||\bar{x} - \bar{z}||$, so we obtain $\rho(x, z) < \rho(x, y)$, i.e., y is a strict local maximum for ρ_x.

Lemma 21 *If every open half-sphere of $S^{d-1} \subset \mathbb{R}^d$ contains a point of the set M then* $\mathrm{card}M \geq d + 1$.

Proof Any d points or fewer in $\mathbb{R}^d$ lie in a hyperplane, the intersection of which with S^{d-1} is contained in a closed half-sphere.

Lemma 22 *Let $D_0 \in \mathcal{D}$ and $x_0 \in D_0 \setminus \mathrm{rd}D_0$. Then there exist $D \in \mathcal{D}$, $D \to D_0$ and $x \in D$, $x \to x_0$ such that* $\mathrm{card}F_x = 1$ *and there are precisely $d + 1$ segments from x to its farthest point on D.*

Proof We shall see that any neighbourhood $\mathcal{O}$ of D_0 in $\mathcal{D}$ contains a polyhedral surface $D \in \mathcal{D}$ with the desired properties. Moreover, we may keep $x = x_0$.

Denote by S_{xy} (respectively G_{xy}) the set of all segments (respectively simple geodesic arcs) from y to the point x, and let $S_x = \cup_{y \in F_x} S_{xy}$.

We start by choosing a (degenerate) polyhedral approximation D_P of D_0 in $\mathcal{O}$, where $P = \mathrm{rd}D$, with a face $B = \mathrm{conv}P$ containing the point $x = x_0$ and such that the unit tangent cone at each of its vertices is close to S^{d-2}.

For any point $y \in D_P$ in F_x, there are at least two segments from x to y, so y is not a vertex of $B' = \iota(B)$ and thus $F_x \subset \mathrm{int}B'$. Then, by Lemma 18, there are at least $d + 1$ segments from y to x, whose directions at y are not all of them contained in a closed half-sphere of S^{d-1}.

Assume F_x has at least two points, at least one of which is joined to x by more than $d + 1$ segments.

First, we find a polyhedral approximation of D_P in $\mathcal{O}$ with $\mathrm{card}F_x = 1$. The idea is, roughly speaking, to cut small parts of B such that to keep only one maximal (with respect to the major axis) ellipsoid of revolution inscribed to P with a focus at x, the second focus corresponding to the unique point in F_x (see Lemma 5). Figure 1 illustrates this idea for $d = 2$.

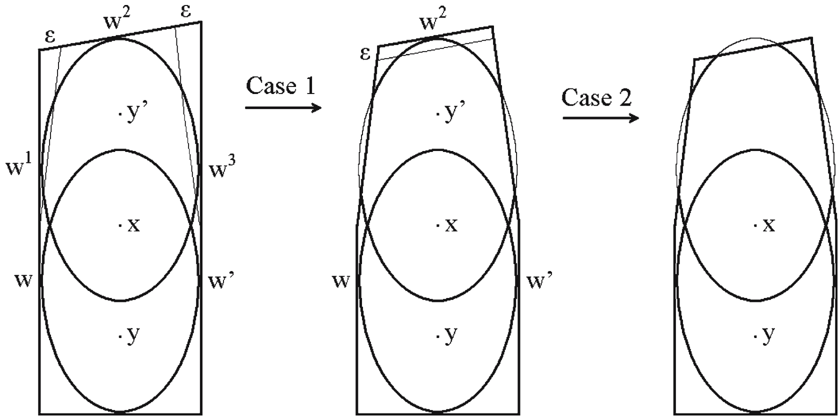

Fig. 1 Approximation with a unique farthest point from x $(d = 2)$

The segments in S_x do not meet each other except at (one or both of) their extremities. Of course, each of them crosses only one face of P, in a point interior to that face, so S_x and F_x are finite sets. Define, for $u \in F_x$,

$$W_u = \bigcup_{\Gamma \in S_{xu}} \Gamma \cap P,$$

and notice that any two points in W_u belong to different faces of P.

Choose a point y in F_x. For each $w \in W_y$, denote by E_w the face of P containing w. Let

$$W = \bigcup_{u \in F_x \setminus \{y\}} W_u;$$

because S_x is finite, $W \cup W_y$ is also finite.

Case (i) We consider first the points $w \in W_y$ such that $E_w \cap W \neq \emptyset$.

Choose a $(d-2)$-dimensional polyhedral convex surface O_w in E_w homothetic to $\mathrm{bd}E_w$, separating w from the points in $E_w \cap W$.

Also choose a hyperplane $H_{w,\varepsilon}$ in $\mathbb{R}^d$ parallel to E_w and separating E_w from $(W \cup W_y) \setminus E_w$, say at distance ε to E_w. Let $\mathbb{R}^d_-(w, \varepsilon)$ denote the closed half-space bounded by $H_{w,\varepsilon}$ disjoint to w, and put

$$P_{w,\varepsilon} = P \cap \mathbb{R}^d_-(w, \varepsilon), \qquad B_\varepsilon = \mathrm{conv}(P_{w,\varepsilon} \cup O_w).$$

Then the segments joining x to y on D_P remain geodesic arcs of length $\rho(x, y)$ on D_{B_ε} and, for ε small enough, each segment joining y to x on D_{B_ε} coincides to a segment on D_P. Indeed, a segment Γ_ε joining y to x on D_{B_ε} and not coinciding to a segment on D_P is close to a geodesic arc $G \in G_{xy} \setminus S_{xy}$, hence of length closer to λG than $\rho(x, y)$, a contradiction. Consequently, y is a farthest point from x on D_{B_ε}. Moreover, the segments on D_{B_ε} corresponding to segments in S_x through the

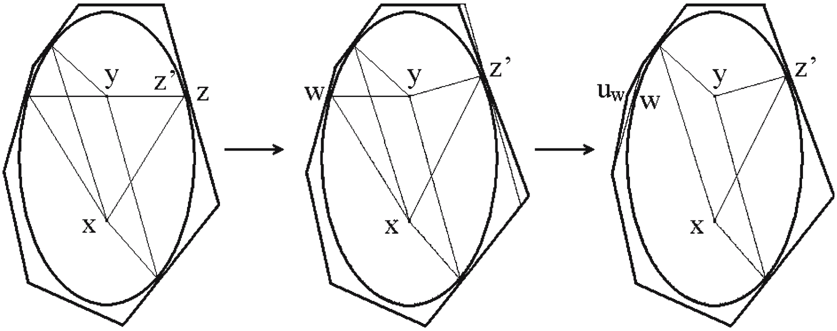

Fig. 2 Approximation with $d+1=3$ segments to the farthest point from x

points in $E_w \cap W$ have smaller length than the length of their correspondents (which is $\rho(x, y) = \rho(x, F_x)$).

Case (ii) We can assume now that for any $w \in W_y$, $E_w \cap W = \emptyset$.

For each point $w_j \in W$ denote by $H_{j,\varepsilon}$ the hyperplane in $\mathbb{R}^d$ parallel to the face E_j of P that w_j belongs to, at distance ε to E_j, and separating w_j from W_y. Let $\mathbb{R}^d_-(j, \varepsilon)$ be the closed half-space disjoint to w_j bounded by $H_{j,\varepsilon}$, and put

$$B_{j,\varepsilon} = \text{conv}(P \cap \mathbb{R}^d_-(j, \varepsilon)),$$

hence $B_{j,\varepsilon} \supset (W \setminus E_j) \cup W_y$.

In this case too, the argument used at *Case* (i) shows that the segments joining x to y on $D_{B_{j,\varepsilon}}$ are precisely those on D_P and, since the segments on $D_{B_{j,\varepsilon}}$ corresponding to the segments in S_x through $W \cap E_j$ have length smaller than $\rho(x, y)$, y is a farthest point from x on $D_{B_{j,\varepsilon}}$.

Because S_x is a finite set, after finitely many procedures as in *Case* (i) and *Case* (ii), we obtain a polyhedral double D_ε where $F_x = y$ and the segments from x to y coincide to those on D_P.

Clearly, $D_\varepsilon \to D_P$ as $\varepsilon \to 0$, so for ε small enough we still have $D_\varepsilon \in \mathcal{O}$.

Rename $D_\varepsilon = D_P$, where $P = \text{rd}D_\varepsilon$. We find now an approximation of D_P in $\mathcal{O}$ with $\text{card}S_{xy} = d + 1$. Figure 2 illustrates the idea of approximation for $d = 2$.

Denote by $\mathcal{T}(D_P)$ the set of all directions at y of segments on D_P from y to x.

Assume that the set $\mathcal{E}(D_P)$ of all equations expressing linear dependences of at most d directions in $\mathcal{T}(D_P)$ is non-void (otherwise the proof is simpler). Consider an equation in $\mathcal{E}(D_P)$, say

$$\sum_{i=1}^{k} \alpha_i \tau_i = 0, \quad \alpha_i \neq 0 \;\; \forall i = 1, \ldots, k \leq d. \tag{3}$$

Also consider, in the space $\mathbb{R}^d$ containing P, the ellipsoid of revolution Ell_{xy} with foci at x and y and the sum of focal radii equal to $\rho(x, y)$. By Lemma 5, conv$P \supset Ell_{xy}$, P is tangent to Ell_{xy}, and the points in $Ell_{xy} \cap P$ are precisely the intersection of the segments in S_{xy} with the faces of P.

Take a point $z \in Ell_{xy} \cap P$ corresponding to the segment starting at y in direction τ_1, and denote by E_z the face of P containing z. Slightly move the point z to $z' \in Ell_{xy}$, denote by H' the hyperplane tangent to Ell_{xy} at z', by A_z the union of the hyperplanes spanned by the faces of P incident to E_z, and by E' the convex subset of H' determined by A_z. Take the convex hull of the union of E' with the faces of P not incident to E_z, and denote it by $B_{z'}$.

On D_P we have $F_x = y$, hence (the images of) any two points in W_y belong to different faces of P. Consequently, if z' is close enough to z then

$$W_y \setminus \{z\} \subset (\mathrm{bd} B_{z'}) \cap P.$$

By the choice of z', the Eq. (3) is no longer satisfied on the double $D_{B_{z'}}$ of $B_{z'}$. Let τ' be the direction at y of the segment joining y to z'. Of course, z' can be taken such that τ' appears in no equation in $\mathcal{E}(D_{B_{z'}})$, so $\mathrm{card}\mathcal{E}(D_{B_{z'}}) < \mathrm{card}\mathcal{E}(D_P)$.

Moreover, z' can be chosen such that y is still a strictly critical point for ρ_x on $D_{B_{z'}}$.

By Lemma 20, there exists $r > 0$ such that y is a strict maximum for the restriction of ρ_x to the intrinsic closed ball V_y of radius $r > 0$ centered at y. Now, if $z' \to z$ then $D_{B_{z'}} \to D_P$ and the set $F_x^{z'}$ of farthest points from x in $D_{B_{z'}}$ converges to $y = F_x$, so from some moment on we have $F_x^{z'} \subset V_y$, which implies $F_x^{z'} = y$.

Moreover, the segments from x to y on $D_{B_{z'}}$ coincide to those on D_P except for the one through z, which now passes through z' (see Lemma 5).

If z' is close enough to z then clearly $D_{B_{z'}} \in \mathcal{O}$.

After—if necessary—such small perturbations of finitely many points in $Ell_{xy} \cap P$, we obtain a polyhedral approximation $D_R \in \mathcal{O}$ of D_0, with a face R containing the point x, and such that $F_x = y$ and $\mathcal{E}(D_R) = \emptyset$.

Now choose $d+1$ points in $Ell_{xy} \cap R$, say $z_1, \ldots, z_{d+1}$, such that any open half-sphere of S^{d-1} contains the direction at y of a segment from y to x through some z_i; their existence is guaranteed by Lemmas 18 and 21.

For each point $w \in Ell_{xy} \cap R \setminus \{z_1, \ldots, z_{d+1}\}$, take a point u_w on the normal to R at w and exterior to R. Put

$$Z = \mathrm{conv}(R \cup \{u_w : w \in Ell_{xy} \cap R \setminus \{z_1, \ldots, z_{d+1}\}\}).$$

If the points u_w are all close enough to R then the double D_Z of Z belongs to $\mathcal{O}$ and, on D_Z, the only segments from x to y are those through $z_1, \ldots, z_{d+1}$. This construction is possible because any two points in W_y belong to different faces of R.

The upper semi-continuity of F and Lemma 20 imply now (see the argument above) that $F_x^Z = y$ provided Z is close enough to R, where F_x^Z is the set of farthest points from x on D_Z. The proof is complete.

Apart the use of the preceding lemmas, the following argument is quite similar to that proving Theorem 2 in [34].

Proof of Theorem 8 Denote by S_{xy} the set of all segments from the point y in F_x to x, and let $S_x = \cup_{y \in F_x} S_{xy}$. For any surface $D \in \mathcal{D}$ and any natural number n define

$$\begin{aligned} A_0(D) &= \{x \in D : \mathrm{card} S_x < d+1\}, \\ A_n(D) &= \{x \in D : \text{there are } d+2 \text{ segments in } S_x \\ &\qquad \text{at mutual distances at least } n^{-1}\}, \\ B_n(D) &= \{x \in D : \mathrm{diam} F_x \geq n^{-1}\}. \end{aligned}$$

The sets $A_n(D)$ and $B_n(D)$ are clearly closed in D, for any n. Define, for $q, r \in \mathbb{N}$, $n, m \in \{0\} \cup \mathbb{N}$ and $z \in \mathbb{R}^d$ of rational coordinates,

$$\begin{aligned} \mathcal{A} &= \{D \in \mathcal{D} : \{x \in D : \mathrm{card} S_x \neq d+1\} \text{ is of 2nd category}\}, \\ \mathcal{A}_n &= \{D \in \mathcal{D} : A_n(D) \text{ is not nowhere dense}\}, \\ \mathcal{A}_{m,z,q} &= \{D \in \textstyle\bigcup_{n=0}^{\infty} \mathcal{A}_n : \Delta(z, q^{-1}) \subset A_m\}, \end{aligned}$$

and respectively

$$\begin{aligned} \mathcal{B} &= \{D \in \mathcal{D} : \{x \in D : \mathrm{diam} F_x \neq 0\} \text{ is of 2nd category}\}, \\ \mathcal{B}_r &= \{D \in \mathcal{D} : B_n(D) \text{ is not nowhere dense}\}, \\ \mathcal{B}_{r,z,q} &= \{D \in \textstyle\bigcup_{n=1}^{\infty} \mathcal{B}_n : \Delta(z, q^{-1}) \subset B_r\}. \end{aligned}$$

It suffices to prove that $\mathcal{A} \cup \mathcal{B}$ is of first category in $\mathcal{D}$. For, notice first that

$$\mathcal{A} \subset \bigcup_{n=0}^{\infty} \mathcal{A}_n \subset \bigcup_{m,z,q} \mathcal{A}_{m,z,q}$$

and

$$\mathcal{B} \subset \bigcup_{n=1}^{\infty} \mathcal{B}_n \subset \bigcup_{r,z,q} \mathcal{B}_{r,z,q},$$

because a closed subset of D which is not nowhere dense must contain some disk $\Delta(z, q^{-1})$.

We show next that

$$\mathcal{A}_{0,z,q} \cup \mathcal{A}_{m,z,q} \cup \mathcal{B}_{m,z,q}$$

is nowhere dense in $\mathcal{D}$. Let $\mathcal{O}$ be an open subset of $\mathcal{D}$, and suppose there exists $D_0 \in \mathcal{O} \cap (\mathcal{A}_{0,z,q} \cup \mathcal{A}_{m,z,q} \cup \mathcal{B}_{m,z,q})$. Take $x_0 \in \Delta(z, q^{-1}) \subset D_0$. By Lemma 22, we can choose a polyhedral approximation D_R of D_0, $D_R \in \mathcal{O}$, with a face R containing some point $x \in \Delta(z, q^{-1})$ such that x is close to x_0, $F_x = y$ and $\mathrm{card} S_x = d+1$.

Consider $D_n \in \mathcal{D}$ such that $D_n \to D_R$, $x_n \in D_n$ such that $x_n \to x$, and $y_n \in F_{x_n}$. Then $y_n \to y$, and any segment from x_n to y_n converges to some segment from x to $y \in F_x$. By Lemma 17, any angle at y_n between segments to x_n converges either to 0 or to the angle at y between some segments to x.

Because the directions at y of the segments to x of are not all contained in a closed half-sphere (see Lemma 18), the same happens at $y_n \in F_{x_n}$, for n sufficiently large, by Lemmas 17 and 19. Lemma 21 implies now the existence of at least $d+1$ segments from y_n to x_n, if D_n is close enough to D_R, so $\Delta(z, q^{-1})$ is not included in $A_0(D_n)$. Thus, there exists a ball around D_R in $\mathcal{D}$ disjoint from $\mathcal{A}_{0,z,q}$.

So, for n large enough, there are at least $d+1$ segments of D_n from x_n to each point y_n in F_{x_n}. Since any angle at y_n between segments to x_n converges either to 0 or to the angle at y between some segments to x, and $\mathrm{card}S_x = d+1$, if n is large enough then among any $d+2$ segments in $S_x \subset D_n$ there are two at distance at most $(m+1)^{-1} < m^{-1}$, so $\Delta(z, q^{-1})$ is not included in $A_m(D_n)$. Moreover, since $y_n \to y$, $\mathrm{diam}F_{x_n} \leq (m+1)^{-1} < m^{-1}$, so $\Delta(z, q^{-1})$ is neither included in $B_m(D_n)$. Therefore, there exists a ball around D_n in $\mathcal{D}$ disjoint to $\mathcal{A}_{m,z,q} \cup \mathcal{B}_{m,z,q}$, whereby $\mathcal{A}_{m,z,q} \cup \mathcal{B}_{m,z,q}$ is nowhere dense.

In conclusion, $\bigcup_{m,z,q}(\mathcal{A}_{m,z,q} \cup \mathcal{B}_{m,z,q})$ is of first category in $\mathcal{D}$, as well as $\mathcal{A} \cup \mathcal{B}$, and the proof is done.

Proof of Theorem 9 Let B be a typical planar convex body and x an interior point of B. Then the doubly covered convex surface D determined by B is also typical, by Lemma 3.

Assume Ell has the largest major axis among all ellipsoids of revolution with a focus at x and tangent to $\mathrm{bd}B$, and let z denote its second focus. Put $B' = \iota(B)$ and $y = \iota(z) = \Psi(Ell)$ (see Lemma 5). Then the length a of the major axis of Ell is equal to $\rho(x, y)$ whereby, since a is maximal, $y \in F_x$. By Theorem 8, if x is typical in D then it has a unique farthest point, joined to x by precisely $d+1$ segments, and the one-to-one correspondence (see Lemma 5 again) between the segments from x to $y = F_x$ and the points in $Ell \cap \mathrm{bd}B$ ends the proof.

8 Proofs of Theorems 10 and 11

Two more lemmas will be needed.

Lemma 23 [26] *Any point y in the convex surface $S \subset \mathbb{R}^3$ with $\lambda T_1 y = 2\pi$ is critical for at most one distance function.*

A *loop at* the point x in $S \subset \mathbb{R}^3$ is the union of two segments from some point $y \in S$ to x, which make an angle equal to π at y.

Lemma 24 [36] *Let S be a convex surface, $x \in S$, $y, z \in C(x)$ and J the arc joining y to z in $C(x)$. If $u \in J$ is a relative minimum of $\rho_x|_{\mathrm{int}J}$ then u is the midpoint of a loop Λ at x and, excepting the subarcs of Λ, no other segment connects x to u.*

Proof of Theorem 10 Take two points x, y in $\mathrm{rd}D$ such that $y \in F_x$. Then each face B of D (considered in $\mathbb{R}^d$) is interior to the closed disk O of radius $\rho(x, y)$ centered at x, and moreover $y \in O \cap B$.

Since B is interior and tangent to O, its boundary has strictly positive lower curvatures in all tangent direction at each contact point with O; so, for any $\tau \in T_1 y$,

$$\gamma_i^\tau(y) \geq \rho(x, y)^{-1} \geq (\mathrm{diam}D)^{-1} > 0.$$

Therefore, by (ii) of Lemma 4, the set $F_{\mathrm{rd}D}$ is of first category in $\mathrm{rd}D$. Moreover, by (i) of Lemma 4, $\gamma_i^\tau(y) = \infty$. Now, Theorem 1 shows that y is an endpoint of D.

Suppose F is single-valued, whence its upper semicontinuity is actually continuity. Then F_D is closed and, since F is not surjective, there is a small open ball U in $D \setminus F_D$. Clearly, $F_{D\setminus U}$ is included in $D \setminus U$. By Brouwer's fixed point theorem, $F|_{D\setminus U}$ has a fixed point, which is impossible.

Assume, for the rest of this proof, that $d = 2$.

The injectivity of the mapping F follows immediately from Lemma 23.

Suppose there exists a point $x \in D$ such that F_x contains an arc J of extremities y_1, y_2. Each point y interior to J is a relative minimum for $\rho_{x|J}$ so, by Lemma 24, y is the midpoint of a loop Λ at x, and no other segments connect x to y except those in Λ. Then each point of the domain Δ provided by Lemma 15 is interior to a geodesic, in contradiction to Theorem 5.

Proof of Theorem 11 Consider a point z interior to the face B of the typical degenerate convex surface D, $v \in C(z)$ interior to $B' = \iota(B)$ and $w = \iota^{-1}(v)$. Put $\{x, y\} = zw \cap \mathrm{rd}D$. We have

$$\rho(z, v) \leq \min\{||z - x|| + ||x - v||, ||z - y|| + ||y - v||\} \leq ||x - y|| = \rho(x, y),$$

so the diameter of D is realized by points on $\mathrm{rd}D$.

Let $x, y \in \mathrm{rd}D$ be diametrally opposite points. Notice first that, if we have equality in the above inequalities, then the union G of the two line-segments from x to y, one for each of the two faces of D, is a closed geodesic. But, since D is typical, such a geodesic does not exist, by Theorem 2.

Because $y \in F_x$, the sphere of radius equal to $\rho(x, y)$ centered at x is exterior and tangent to B, so xy is normal to B at y. Similarly, since $x \in F_y$, xy is also normal to B at x. Since $\mathrm{rd}D$ is smooth and xy is a double normal of it, $F_x = y$ and $F_y = x$.

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