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Antipodal convex hypersurfaces

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ABSTRACT

Motivated by a conjecture of Steinhaus, we consider the mapping F, associating to each point x of a convex hypersurface the set of all points at maximal intrinsic distance from x. We first provide two large classes of hypersurfaces with the mapping F single-valued and involutive. Afterwards we show that a convex body is smooth and has constant width if its double has the above properties of F, and we prove a partial converse to this result. Additional conditions are given, to characterize the (doubly covered) balls.

1. INTRODUCTION

A *convex hypersurface* is the boundary of a *convex body* (i.e., compact convex set with interior points) in the Euclidean space $\mathbb{R}^{d+1}$, or a doubly covered convex body in $\mathbb{R}^d$; in the last case we call it *degenerate*. The *intrinsic metric* ρ of a convex hypersurface S is defined, for any two points x, y in S, as the length $\rho(x, y)$ of a *segment* (i.e., shortest path on the hypersurface) from x to y. For any point $x \in S$ let ρ_x denote the *distance function from* x, given by $\rho_x(y) = \rho(x, y)$ for all $y \in S$,

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and let F_x be the set of all *farthest points from* x (i.e., global maxima of ρ_x). For simplicity, we shall not distinguish between the set $F_x = \{y\}$ and the point y, and we shall write $F_x = y$, when the case occurs.

Our paper concerns the following question of H. Steinhaus (see Section A35, (ii) in [3]): *is the sphere characterized by the property that the mapping* F*, associating to any point* x *on the convex surface* S *the set* F_x *of farthest points from* x*, is single-valued,* 1–1 *and symmetrical?*

It is well known that our mapping F is upper semicontinuous. We call F *injective* if $F_x \cap F_y = \emptyset$ for any pair of distinct points $x, y \in S$, and we call F *surjective* if for every point $y \in S$ there is some point $x \in S$ with $y \in F_x$. When we say that F is bijective, we implicitly state that F is single-valued.

While the employment of the mapping F led to a beautiful topological characterization of the spheres [4], the above – most natural – attempt to use F for a metrical characterization of the spheres was not successful.

A class $\mathcal{R}$ of centrally-symmetric surfaces of revolution was provided by the third author in [10], to negatively answer Steinhaus' question. He also asked for alternative descriptions of the set $\mathcal{H}$, of all convex (hyper)surfaces for which F is a single-valued involution ($F \circ F = \mathrm{id}_S$). We shall call the elements of $\mathcal{H}$, referring to the sphere, *antipodal convex (hyper)surfaces*.

Put $\mathcal{I} = \{S \text{ convex} : \mathrm{rad}(S) = \mathrm{diam}(S)\}$, where $\mathrm{rad}(S)$ is the *intrinsic radius* of S, $\mathrm{rad}(S) = \min_{x \in S} \rho(x, F_x)$.

All surfaces in the class $\mathcal{I}$ are antipodal, because $\mathcal{R} \subset \mathcal{I} \subset \mathcal{H}$ [13]. We show, with our Theorem 2, that the inclusion $\mathcal{R} \subset \mathcal{I}$ is strict. Other examples are provided in [6], by showing that all right circular cylinders of small height also belong to $\mathcal{I} \setminus \mathcal{R}$.

The second author [8] proved that no tetrahedron is antipodal, leaving open the existence problem for antipodal polyhedral convex surfaces.

Yet all these considerations were done for surfaces in $\mathbb{R}^3$. The aim of this work is to study antipodal convex hypersurfaces in $\mathbb{R}^{d+1}$, for any integer $d \geqslant 2$.

In the first part of the paper, Theorem 1 generalizes the set $\mathcal{R}$ to higher dimensions (see Section 3 for the precise definition), while Theorem 2 provides a new class of examples, by considering the union of two caps of d-dimensional semispheres of (not necessarily) different radii. We prove that all these hypersurfaces belong to $\mathcal{I} \cap \mathcal{H}$.

In the last part we restrict our study to the set $\mathcal{D}$ of all degenerate convex hypersurfaces. We show in Theorem 3 that every hypersurface in $\mathcal{H} \cap \mathcal{D}$ is the double of a smooth convex body of constant width. Theorem 4 provides a partial converse to Theorem 3; roughly speaking, it states that every double of a C^2-differentiable convex body of constant width has a neighbourhood of its ridge, the restriction of F to which has the properties of Steinhaus.

Theorems 5–7 partly confirm Steinhaus' guess, by proving that it is possible to characterize *the balls* (eventhough not the spheres) by the use of the mapping F. More precisely, a degenerate convex hypersurface D is a doubly covered ball if D is centrally symmetric and the corresponding mapping F is an involutive bijection (Theorem 5), respectively if D is two-dimensional and F is an isometry

(Theorem 6), and the only degenerate convex hypersurfaces in $\mathcal{I}$ are the doubly covered balls (Theorem 7).

We refer to the survey article [11] for general properties of, and references on, farthest point sets on convex surfaces.

The *diameter* of a convex hypersurface S is $\operatorname{diam} S = \sup_{x \in S} \rho_x(F_x)$.

The *width* $w(u)$ of the convex body $K \subset \mathbb{R}^d$ (or of its boundary) in the direction $u \in S^{d-1}$ is the distance between the two supporting hyperplanes of K orthogonal to u. (Here, S^{d-1} denotes the unit sphere in $\mathbb{R}^d$.) K is said to be of *constant width* if $w(u) = w$ for all $u \in S^{d-1}$. See [2] for a survey on this topic.

According to [9], a convex body K is called *smooth* if all its boundary points are regular. The double D of K is called *smooth* if K is smooth, and *of differentiable class C^r* if $\operatorname{bd} K$, the boundary of K, is so.

For $a, b, c \in \mathbb{R}^{d+1}$, the notations abc, $[ab]$, $[abc]$ and $\|a-b\|$ stand for the 2-plane spanned by the points a, b, and c, the line-segment determined by a and b, the Euclidean triangle determined by a, b and c, and the length of $[ab]$, respectively. We denote by $(e_1, \ldots, e_{d+1})$ the canonical basis of $\mathbb{R}^{d+1}$, and the jth component of $\omega \in \mathbb{R}^{d+1}$ by ω^j.

The length of the curve Γ is denoted by $l(\Gamma)$.

2 HYPERSURFACES OF REVOLUTION

The goal of this section is to give some more notations and an auxiliary result for later use.

We describe any hypersurface of revolution S as

$$S = \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}\colon\ (\|x\|, y) \in \Gamma\right\},$$

where Γ is a curve in $\mathbb{R}_{\geqslant 0} \times \mathbb{R}$ running from $(0, a)$ to $(0, b)$, where $a < b$ and all intermediate points of Γ lie in $\mathbb{R}_{>0} \times \mathbb{R}$. Then the *south* and *north pole* pole of S are $\sigma = (0, a) \in \mathbb{R}^d \times \mathbb{R}$ and $\nu = (0, b) \in \mathbb{R}^d \times \mathbb{R}$, respectively.

Each point $p \neq \sigma, \nu$ in the hypersurface of revolution S lies on a unique *meridian* M_p, defined as the component of $S \cap p\sigma\nu \setminus \{\sigma, \nu\}$ containing p. Denote by M_p^- the *opposite meridian* to M_p (or to p) in the plane $p\sigma\nu$, the image of M_p under the antipodal map in the first d coordinates.

Consider now a continuous map $\phi\colon [0, a] \subset \mathbb{R} \to [0, +\infty[$ such that $\phi(s) > 0$ for all $s \in [0, a[$ and $\phi(a) = 0$, and denote by S_ϕ the hypersurface of revolution generated as above by the curve Γ consisting of the union of the graphs of ϕ and $-\phi$. Alternatively, the symmetrical graph G_ϕ of ϕ, $G_\phi \stackrel{\text{def}}{=} \{(s, 0, \ldots, 0, \pm\phi(s))\colon s \in [0, a]\}$, is included in the (x_1, x_{d+1})-plane $P = \{q\colon x_2(q) = \cdots = x_d(q) = 0\} \subset \mathbb{R}^{d+1}$. Denote by Λ the unit $(d-2)$-sphere of the $(x_2, \ldots, x_d)$-space $P^\perp$. When (λ, α) varies in $\Lambda \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, G_ϕ generates the hypersurface S_ϕ given by

$$X(s, \lambda, \alpha) = s \sin\alpha\, e_1 + s \cos\alpha\, \lambda \pm \phi(s)\, e_{d+1}.$$

The points $\sigma = (0, \ldots, 0, -\phi(0))$ and $\nu = (0, \ldots, 0, \phi(0))$ are respectively the south and the north pole of S_ϕ.

We shall repeatedly make use of the following simple result.

Lemma 1. *The diameter* $\operatorname{diam}(S)$ *of any hypersurface of revolution* S *is equal to the length of any meridian of* S*. If the distance between two points* p, q *on the hypersurface of revolution* S *is equal to* $\operatorname{diam}(S)$ *then* $F_p = q$ *and* p, q *lie on opposite meridians. In particular, if* S *is centrally symmetric then the points are symmetric to each other.*

Proof. The diameter of a hypersurface of revolution S is at least equal to the length of one of its meridians, because any meridian is a segment between the poles of S.

Let p, q be two points in S such that $\rho(p,q) = \operatorname{diam}(S)$, M_p the meridian through p, and $z \in M_p^-$ determined by $\rho(q,\nu) = \rho(z,\nu)$. We have

$$
\begin{aligned}
\operatorname{diam}(S) &= \rho(p,q) \\
&\leqslant \min\{\rho(p,\nu)+\rho(\nu,q), \rho(p,\sigma)+\rho(\sigma,q)\} && (1)\\
&= \min\{\rho(p,\nu)+\rho(\nu,z), \rho(p,\sigma)+\rho(\sigma,z)\} \\
&\leqslant \frac{1}{2}\big(l(M_p)+l(M_p^-)\big) = l(M_p) \leqslant \operatorname{diam}(S), && (2)
\end{aligned}
$$

which implies that $\operatorname{diam}(S) = \rho(p,q) = l(M_p) = l(M)$ for any meridian M.

The the equality case in (2) implies

$$\rho(p,\nu)+\rho(\nu,z) = \rho(p,\sigma)+\rho(\sigma,z) = \operatorname{diam}(S).$$

By the equality case in (1), the two components of $(M_p \cup M_q)\setminus\{p,q\}$ are segments.

Let q' be the point symmetric to q with respect to $\nu\sigma p$. Then the two components of $(M_p \cup M_{q'})\setminus\{p,q'\}$ are segments too. Since segments do not bifurcate, $M_q = M_p^-$ and $q = z$. □

3. A CLASS OF ANTIPODAL HYPERSURFACES

In this section we provide a class $\mathcal{R}$ of antipodal hypersurfaces, by generalizing in arbitrary dimension the last theorem in [10].

Let $\phi : [0,a] \to \mathbb{R}$ be a concave nonincreasing function such that

$$0 = \phi(a) < \phi(0) < a.$$

We assume that the function $\psi : [0,a] \to \mathbb{R}$, given by $\psi(s) = s^2 + \phi^2(s)$, is strictly increasing.

Let $\mathcal{R}$ be the set of all hypersurfaces S_ϕ obtained from the above functions ϕ by rotating in $\mathbb{R}^{d+1}$ their symmetrical graphs $G_\phi \subset x_1 o x_{d+1}$ about the axis $\mathbb{R}e_{d+1}$; i.e., for ϕ as above, $s \in [0,a]$, $\lambda \in \Lambda$, and $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, S_ϕ is given by

$$X(s,\lambda,\alpha) = s \sin\alpha\, e_1 + s\cos\alpha\, \lambda \pm \phi(s)\, e_{d+1}.$$

For the proof of Theorem 1 we need to define, for each concave function ϕ as above, a new function $\eta : [-a,a] \to \mathbb{R}$ by $\eta(\pm s) = \phi(s)$. Let $S_\phi^- \subset \mathbb{R}^{d+1}$ be the hypersurface defined by the rotation of the graph of η about the axis $\mathbb{R}e_1$; that is, for $t \in [-a, a]$, $\mu \in \Lambda$ and $\beta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, S_ϕ^- is given by

$$Y(t, \mu, \beta) = t\, e_1 + \phi(|t|) \cos\beta\, \mu + \phi(|t|) \sin\beta\, e_{d+1}.$$

Lemma 2. *The hypersurfaces S_ϕ and S_ϕ^- are convex, S_ϕ^- is included in* convS_ϕ *and $S_\phi^- \cap S_\phi = S_\phi \cap P$.*

Proof. Let C_ϕ denote the set of all $(x, y) \in \mathbb{R}^d \times \mathbb{R}$ such that $|y| - \phi(\|x\|) \leqslant 0$, that is, the basin $f \leqslant 0$ of the function $f : (x, y) \mapsto |y| - \phi(\|x\|)$. Clearly $S_\phi = \mathrm{bd}C_\phi$.

The function $y \mapsto |y|$ is convex, and therefore the function $(x, y) \mapsto |y|$ is convex. The function $x \mapsto \|x\|$ is convex and the function $-\phi$ is nondecreasing and convex, hence the composition $x \mapsto -\phi(\|x\|)$ is convex, and thus the function $(x, y) \mapsto -\phi(\|x\|)$ is convex. As the sum of two convex functions is convex, the function f is convex, and because the subgraph of any convex function is a convex set, it follows that the set C_ϕ is convex, and therefore $S_\phi = \mathrm{bd}C_\phi$ is a convex hypersurface. The proof that S_ϕ^- is a convex hypersurface is analogous.

The hypersurface S_ϕ^- is the set of all $(u, v, y) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$ such that $u = \pm\phi^{-1}(\|(v, y)\|)$, or equivalently $\phi(|u|)^2 = \|v\|^2 + y^2$. Because the function ψ is strictly increasing, we have

$$\begin{aligned}\phi\big(\|(u, v)\|\big)^2 + u^2 + \|v\|^2 &= \phi\big(\|(u, v)\|\big)^2 + \|(u, v)\|^2 \\ &\geqslant \phi(|u|)^2 + u^2 = \|v\|^2 + y^2 + u^2,\end{aligned}$$

with equality if and only if $v = 0$. This is equivalent to $\phi(\|(u, v)\|)^2 \geqslant y^2$ with equality if and only if $v = 0$, that is, $(u, v, y) \in C_\phi$ and $(u, v, y) \in S_\phi$ if and only if $v = 0$ if and only if $(u, v, y) \in S_\phi \cap P$. □

The next well-known lemma can be found, for example, in [1], p. 80. Let K be a convex body in $\mathbb{R}^{d+1}$ and x a point in $\mathbb{R}^{d+1} \setminus \mathrm{int}K$. The *metric projection* of x onto K is the unique point in K closest to x.

Lemma 3. *Let K be a convex body in $\mathbb{R}^{d+1}$ and Γ a curve in $\mathbb{R}^{d+1} \setminus$* int$K$*. Then the length of Γ is at least as long as its metric projection onto K.*

The following result extends to arbitrary dimension Theorem 2 in [10]; the proof we present here is not verbatim the same as the proof given in [10], although it contains the same steps.

Theorem 1. $\mathcal{R} \subset \mathcal{I} \cap \mathcal{H}$.

Proof. Consider a hypersurface S_ϕ in $\mathcal{R}$ and a point $p \in S_\phi$.

We first prove that if p lies on the equator, that is, $p^{d+1} = 0$, then $\rho(p, -p) = \mathrm{diam}(S_\phi)$.

Because of the rotational symmetry, we may assume that $p \in P$, that is, $p^j = 0$ for all $2 \leqslant j \leqslant d$. In combination with $p^{d+1} = 0$ this means that p is a pole of S_ϕ^-, and therefore $\rho_{S_\phi^-}(p, -p) = \operatorname{diam}(S_\phi^-) = l(M_p) = \operatorname{diam}(S_\phi)$ according to Lemma 1. On the other hand, if Γ is any segment in S_ϕ from p to $-p$ and Γ^- its metric projection onto $\operatorname{conv} S_\phi^-$, which lands in S_ϕ^-, then

$$\operatorname{diam}(S_\phi) \geqslant \rho(p, -p) = l(\Gamma) \geqslant l(\Gamma^-) \geqslant \rho_{S_\phi^-}(p, -p) = \operatorname{diam}(S_\phi),$$

where the second inequality follows from Lemma 3. Therefore $\rho(p, -p) = \operatorname{diam}(S_\phi)$.

We now drop the assumption that $p^{d+1} = 0$. Let Γ be a segment from p to $-p$. Then Γ intersects the equator in at least one point q. We obtain from the symmetry of S_ϕ that $\rho(q, -p) = \rho(-q, p)$, and consequently

$$\begin{aligned}\operatorname{diam}(S) &\geqslant \rho(p, -p) = l(\Gamma) = \rho(p, q) + \rho(q, -p) \\ &= \rho(q, p) + \rho(p, -q) \geqslant \rho(q, -q) = \operatorname{diam}(S_\phi).\end{aligned}$$

It follows that $\rho(p, -p) = \operatorname{diam}(S_\phi)$, and therefore Lemma 1 yields that $-p$ is the unique point $p' \in S_\phi$ such that $\rho(p, p') = \operatorname{diam}(S_\phi)$. □

4 ANOTHER CLASS OF ANTIPODAL HYPERSURFACES

In this section we provide another class $\mathcal{J}$ of convex hypersurfaces verifying the Steinhaus' conditions mentioned in Introduction. We consider the union of two caps of d-dimensional semispheres of (possibly not) different radii, and prove that it belongs to $\mathcal{I} \cap \mathcal{H}$. Our argument also holds for the limit case where one of the caps is a disk.

In order to prove the main result of this section, we shall use the following fact of elementary analysis.

Lemma 4. *Let Y be a connected topological space, I and interval in $\mathbb{R}$, and for each $y \in Y$, let $f_y : I \to \mathbb{R}$ be such that:*

(i) *For each $y \in Y$ the function f_y is not constant, and either increasing or decreasing on I,*
(ii) *For each $x \in I$ the function $y \mapsto f_y(x)$ is continuous.*

Then either for each $y \in Y$ the function f_y is increasing, or for each $y \in Y$ the function f_y is decreasing.

Proof. Let Y^+ and Y^- denote the set of all $y \in Y$ such that f_y is increasing and f_y is decreasing, respectively. It follows from (i) that $Y^+ \cap Y^- = \emptyset$ and $Y = Y^+ \cup Y^-$. Let $y \in Y^+$. Because f_y is not constant, there exist $a, b \in I$ such that $a < b$ and $f_y(a) < f_y(b)$. It follows from (ii) that there exists a neighbourhood Z of y in Y such that for every $z \in Z$ we have $f_z(a) < f_z(b)$, which implies that $f_z \in Y^+$. This proves that Y^+ is an open subset of Y. Similarly we obtain that Y^- is an open subset of Y, and because Y is the disjoint union of Y^+ and Y^- and Y is connected, it follows that $Y = Y^+$ or $Y = Y^-$. $\square$

Let L_{d,R_1,R_2} be the union of two d-dimensional spherical caps of respective radii $R_1 \geqslant 1$ and $R_2 \geqslant 1$, whose common boundary is a unit $(d-1)$-dimensional sphere; i.e., L_{d,R_1,R_2} is the boundary of the intersection of two balls of respective radii R_1 and R_2, with centers at distance $\sqrt{R_1^2-1}+\sqrt{R_2^2-1}$. Denote by $\mathcal{J}$ the set of all hypersurfaces L_{d,R_1,R_2}.

Theorem 2. $\mathcal{J} \subset \mathcal{I} \cap \mathcal{H}$.

Proof. The following argument will be an exercise in trigonometrical geometry.

Denote by V the space spanned by e_{d+1}, by H the hyperplane orthogonal to V and by Λ the unit $(d-1)$-sphere in H.

For $i = 1, 2$, let B_i be a ball of radius R_i in $\mathbb{R}^{d+1}$ whose center ω_i belongs to V, and let S_i be the boundary of B_i. Assume that $1 \leqslant R_1 \leqslant R_2$ and $S_1 \cap S_2 = \Lambda$. Assume, moreover, that $\omega_1^{d+1} < 0$ and $\omega_2^{d+1} > 0$, and put $C_i = \{u \in S_i\colon (-1)^i u^{d+1} < 0\}$. Then the boundary L_{d,R_1R_2} of $B_1 \cap B_2$ is the disjoint union $C_1 \cup \Lambda \cup C_2$. Let $\Theta_i = \angle \mathbf{0}\omega_i\lambda$, where $\lambda \in \Lambda$. It follows from elementary calculus that

$$
\begin{aligned}
&\Theta_i = \sin^{-1}\frac{1}{R_i}\\
(3)\qquad &\omega_i = \bigl(0,\dots,0,(-1)^i h_i\bigr)\\
&h_i = \sqrt{R_i^2-1} = \cot\Theta_i .
\end{aligned}
$$

A parametrization of $C_i^* \overset{\text{def}}{=} C_i \setminus V$ is given by

$$
\begin{aligned}
(4)\qquad &\phi_i : \Lambda \times]0,\Theta_i[\to \mathbb{R}^{d+1}\\
&(\lambda,\theta) \mapsto \omega_i + R_i\bigl(\sin\theta\lambda - (-1)^i\cos\theta e_{d+1}\bigr).
\end{aligned}
$$

We call respectively λ and θ the *longitude* and the *colatitude* of $\phi_i(\lambda,\theta)$.

It is quite obvious that a meridian of L_{d,R_1R_2} has the length equal to

$$
\operatorname{diam}(L_{d,R_1R_2}) = R_1\Theta_1 + R_2\Theta_2.
$$

A *meridian path* of L_{d,R_1R_2} is by definition a path in the union of two opposite meridians of L_{d,R_1R_2}.

If $p = \phi_i(\lambda,\theta)$ belongs to $L_{d,R_1R_2} \setminus V$ then there exists a unique meridian M_p through p. Denote by $a(p)$ the point of $M_p^- \overset{\text{def}}{=} M_{\phi_i(-\lambda,\theta)}$ such that $(M_p \cup M_p^-)\backslash\{p, a(p)\}$ consists of two equally long components. We claim that $a(p)$ is the only farthest point from p. To see this, it suffices to prove that the length of any path from p to $a(p)$ is never less than $\operatorname{diam}(L_{d,R_1R_2})$ and to apply Lemma 1. By continuity, it is sufficient to investigate the following two cases.

Case 1. $p \in C_2^* \cup a(C_2^*)$. We can assume, without loss of generality, that $p \in C_2^*$ (otherwise, exchange p and $a(p)$). Note that p and $a(p)$ have opposite longitudes, and the colatitudes θ_1 of $a(p)$ and θ_2 of p satisfy

$$(5) \qquad \theta_1 R_1 = \theta_2 R_2 \stackrel{\text{def}}{=} \Delta.$$

It follows that $a(C_2^*)$ is the set of those points in C_1^* having the colatitude in $]0, \frac{R_2}{R_1}\Theta_1[$.

Notice that any segment Σ from p to $q = a(p)$ intersects $\Lambda \subset \text{cl}C_2 \cap \text{cl}C_1$ at a single point, say λ. Indeed, if $r, r' \in \Lambda$ then the segment between r and r' of the circle on C_2 with center at ω_2 is shorter than the segment between r and r' of the circle on C_1 with center at ω_1, which in turn is shorter than the segment between r and r' of the great circle on Λ. Therefore, if we start at $p \in C_2$ and want to reach $q \in C_1$ along a segment on L_{d,R_1,R_2} then, after reaching a point on Λ via a circle segment on C_2 with center at ω_2, we must immediately leave Λ into C_1 and go to q along a circle segment in C_1 with center at ω_1.

We can assume without loss of generality that

$$\begin{aligned} p &= \phi_2(e_1, \theta_2), \\ q &= \phi_1(-e_1, \theta_1), \\ \lambda &= (\cos\alpha, \sin\alpha, 0, \ldots, 0). \end{aligned}$$

Σ is the union of two arcs of great circles so, by the cosine rule for spherical triangles, its length is the minimum of

$$\begin{aligned} f_{R_1,R_2,\Delta}(\alpha) = {} & R_1 \cos^{-1}(\cos\Theta_1 \cos\theta_1 + \sin\Theta_1 \cos\alpha \sin\theta_1) \\ & + R_2 \cos^{-1}(\cos\Theta_2 \cos\theta_2 - \sin\Theta_2 \cos\alpha \sin\theta_2). \end{aligned}$$

We use (3) to compute the first derivative of $f_{R_1,R_2,\Delta}$ with respect to α,

$$\begin{aligned} f'_{R_1,R_2,\Delta}(\alpha) &= \sin\alpha \Bigg(\frac{\sin\theta_1}{(1 - (\cos\Theta_1 \cos\theta_1 + \sin\Theta_1 \cos\alpha \sin\theta_1)^2)^{1/2}} \\ &\qquad - \frac{\sin\theta_2}{(1 - (\cos\Theta_2 \cos\theta_2 - \sin\Theta_2 \cos\alpha \sin\theta_2)^2)^{1/2}} \Bigg) \\ &\stackrel{\text{def}}{=} \sin\alpha \, g_{R_1,R_2,\Delta}(\cos\alpha). \end{aligned}$$

Notice that $g_{R_1,R_2,\Delta}(X)$ vanishes if and only if

$$\begin{aligned} h_{R_1,R_2,\Delta}(X) &\stackrel{\text{def}}{=} \frac{1 - (\cos\Theta_1 \cos\theta_1 + X \sin\Theta_1 \sin\theta_1)^2}{\sin^2\theta_1} \\ &\qquad - \frac{1 - (\cos\Theta_2 \cos\theta_2 - X \sin\Theta_2 \sin\theta_2)^2}{\sin^2\theta_2} \\ &= \sin^2\Theta_1 \cot^2\theta_1 - \sin^2\Theta_2 \cot^2\theta_2 \\ &\qquad - \big(\sin(2\Theta_1)\cot\theta_1 + \sin(2\Theta_2)\cot\theta_2\big) X \\ &\qquad + \big(\sin^2\Theta_2 - \sin^2\Theta_1\big) X^2 \end{aligned}$$

does so, hence $f_{R_1,R_2,\Delta}$ has at most two extrema on $]0,\pi[$. Therefore, if we prove that 0 and π are both minima for $f_{R_1,R_2,\Delta}$ then it follows that Σ is a meridian path. To do this, we shall apply Lemma 4 and show the property for suitable parameters.

We claim that $h_{R_1,R_2,\Delta}(\pm 1) \neq 0$, except for $R_1 = R_2 = 1$. For $s = \pm 1$,

$$\begin{aligned} h_{R_1,R_2,\Delta}(s) &= \sin^2\Theta_1(\cot^2\theta_1 - 1) - \sin^2\Theta_2(\cot^2\theta_2 - 1) \\ &\quad - s(\sin(2\Theta_1)\cot\theta_1 + \sin(2\Theta_2)\cot\theta_2) \\ &= -(\cot\theta_2\sin\Theta_2 + s\cos\Theta_2)^2 + (\cot\theta_1\sin\Theta_1 - s\cos\Theta_1)^2. \end{aligned}$$

Since $0 < \theta_i \leqslant \Theta_i$, $\cot\theta_i \geqslant \cot\Theta_i$ $(i = 1, 2)$. It follows that both parenthesis above are positive and, by (5), $h(s) = 0$ if and only if

$$k(\Delta) \overset{\text{def}}{=} \cot\frac{\Delta}{R_2}\sin\Theta_2 + s\cos\Theta_2 - \cot\frac{\Delta}{R_1}\sin\Theta_1 + s\cos\Theta_1 = 0.$$

Note that, by (3),

$$k'(\Delta) = \frac{1}{R_1^2\sin^2\frac{\Delta}{R_1}} - \frac{1}{R_2^2\sin^2\frac{\Delta}{R_2}}.$$

Therefore, since the function $y \mapsto y\sin\frac{1}{y}$ is increasing on $]0, \frac{2}{\pi}[$, $k'(\Delta)$ is nonnegative and it vanishes if and only if $R_1 = R_2$. Hence

$$\lim_{\Delta\to 0} k(\Delta) \leqslant k(\Delta) \leqslant k(R_2\Theta_2).$$

By the use of the formula $\cot X = \frac{1}{X} + O(X)$, valid for small numbers X, we can compute the above limit

$$\begin{aligned} k(\Delta) &= \left(\frac{R_2}{\Delta} + O(\Delta)\right)\frac{1}{R_2} - \left(\frac{R_1}{\Delta} + O(\Delta)\right)\frac{1}{R_1} + s(\cos\Theta_1 + \cos\Theta_2) \\ &\underset{\Delta\to 0}{\longrightarrow} s\left(\sqrt{1 - R_1^{-2}} + \sqrt{1 - R_2^{-2}}\right). \end{aligned}$$

Now, if $s = 1$ then $k(\Delta)$ is nonnegative, and it vanishes if and only if $R_2 = R_1 = 1$. If $s = -1$, use (5) to get

$$k(R_2\Theta_2) = -2\cos\Theta_1 \leqslant 0,$$

so $k(\Delta)$ is nonpositive and our claim is proved.

Choose an integer m large enough to ensure that $(R_1, R_2, R_1\Theta_1 + R_2\Theta_2)$ and $y_0 \overset{\text{def}}{=} (\sqrt{2}, \sqrt{2}, \frac{\pi}{6\sqrt{2}})$ belong to Y_m, where

$$Y_m = \left\{(\rho_1, \rho_2, \delta)\colon \rho_1 \in \left[1 + \frac{1}{m}, m\right], \rho_2 \in [\rho_1, m], \delta \in \left[\frac{1}{m}, \rho_2\sin^{-1}\frac{1}{\rho_2}\right]\right\}.$$

We want to apply Lemma 4 for $[a, b] = [0, \pi]$ and $Y = Y_m$. Suppose the assumption of Lemma 4 is not verified; then, for each positive integer n, there exist $x_n \in]0, 1/n[$ and a triple $y_n = (\rho_1, \rho_2, \delta) \in Y_m$ such that $f'_{y_n}(x_n) = 0$ (resp. $f'_{y_n}(\pi - x_n) = 0$). Then, by the compactness of Y_m, we can extract a subsequence of $\{y_n\}_n$ converging to $y \in Y_m$. It follows that the function h_y associated to f_y as above verifies $h_y(1) = 0$ (resp. $h_y(-1) = 0$), in contradiction to the above claim.

Therefore, we can apply Lemma 4 to deduce that 0 and π are both minima of f_y for all y in Y_m if and only if they are minima of f_{y_0}. Finally, use (5) to establish this last statement by a direct computation:

$$f''_{y_0}(\pi) = f''_{y_0}(0) = g_{y_0}(0) > 0.$$

Case 2. $p \in C_1^* \setminus \mathrm{cl}\, a(C_2)$. We can assume, without loss of generality, that $p = \phi(e_1, \theta)$ and $a(p) = \phi(-e_1, \theta')$, where θ and θ' are real positive numbers such that

$$R_1(\theta + \theta') = R_1\Theta_1 + R_2\Theta_2.$$

It follows that θ and θ' are not less than $\frac{R_2}{R_1}\Theta_2$. Assume that $\theta \leqslant \theta'$ (otherwise, exchange them).

A segment between p and $p' = a(p)$ consists either of a meridian path included in C_1, in which case $\rho(p, p') = \mathrm{diam}(L_{d,R_1R_2})$, or of three arcs of circles: the first one in C_1 from p to some point $u \in \Lambda$, the second one in C_2 from u to some point $u' \in \Lambda$, and the third one in C_1 from u' to p'.

Let $p'' = \phi(-e_1, \theta)$ be the point symmetrical to p with respect to $e_1^\perp$.

We claim that the shortest path from p to p'', among all paths intersecting C_2, is the meridian path. This claim would end the proof, since it directly implies that the meridian paths are segments between p and p'.

To prove the claim, suppose on the contrary that a shortest path Σ joining p to p'' and intersecting C_2 is shorter that $2(R_2\Theta_2 + R_1(\Theta_1 - \theta))$.

Notice that Σ is symmetrical with respect to $e_1^\perp$. Indeed, denote by Σ' the path symmetrical to Σ with respect to $e_1^\perp$, and assume $\Sigma \neq \Sigma'$ (as point sets). Take the first point along Σ, say q, in $\Sigma' \cap e_1^\perp$. Then the arcs of Σ and Σ' from p to q have the same length, and they are segments of our hypersurface L_{d,R_1R_2}. If Σ and Σ' would not have opposite directions at q then the first variation formula (see Theorem 3.5 in [7]) would show the existence (around q) of a path shorter then Σ, which is impossible by our choice. So Σ and Σ' have opposite directions at q, and therefore they coincide as point sets.

We may assume that Σ consists of an arc of circle in C_1 from p to $u = \cos\alpha\, e_1 + \sin\alpha\, e_2$ ($\alpha \in]0, \frac{\pi}{2}[$), an arc of circle in C_2 from u to $u' = -\cos\alpha\, e_1 + \sin\alpha\, e_2$, and an arc of circle in C_1 from u' to p''. Then the length of Σ is given by

$$L(\alpha) = R_2 \cos^{-1}\left(1 - \frac{2\cos^2\alpha}{R_2^2}\right) + 2R_1 \cos^{-1} \frac{\sin\theta\cos\alpha + \cos\theta\sqrt{R_1^2 - 1}}{R_1}.$$

By a straightforward computation, the first derivative of L is

$$L'(\alpha) = 2\sin\alpha\left(\frac{\sin\theta}{\sqrt{1-(\sin\theta\cos\alpha+\cos\theta\sqrt{R_1^2-1})^2/R_1^2}}\right.$$
$$\left. - \frac{1}{\sqrt{1-\cos^2\alpha/R_2^2}}\right)$$

and it vanishes on $]0, \frac{\pi}{2}[$ if and only if $\cos\alpha$ is a solution of the equation

$$(R_2^2 - R_1^2)X^2 + 2\cot\theta\sqrt{R_1^2-1}\,R_2^2 X - \cot^2\theta R_2^2 = 0. \tag{6}$$

Since the product of the solutions of (6) is negative, there exists at most one local extremum for L in $]0, \frac{\pi}{2}[$, say at α_0. Furthermore,

$$L'\left(\frac{\pi}{2}\right) = \frac{\sin\theta}{\sqrt{1-\cos^2\theta(R_1^2-1)/R_1^2}} - 1 < 0,$$

whence α_0 is a point of maximum, and the global minimum of L on $[0, \frac{\pi}{2}]$ is either at 0 or at $\frac{\pi}{2}$. Now we compute

$$\begin{aligned}
&\frac{1}{2R_1}\left(L(0) - L\left(\frac{\pi}{2}\right)\right)\\
&= -\cos^{-1}\frac{\cos\theta\sqrt{R_1^2-1}}{R_1}\\
&\quad + \cos^{-1}\frac{\sin\theta+\cos\theta\sqrt{R_1^2-1}}{R_1} + \frac{R_2}{2R_1}\cos^{-1}\left(1-\frac{2}{R_2^2}\right)\\
&= -\cos^{-1}(\cos\theta\cos\Theta_1)\\
&\quad + \cos^{-1}(\sin\theta\sin\Theta_1 + \cos\theta\cos\Theta_1)\\
&\quad + \frac{\sin\Theta_1}{2\sin\Theta_2}\cos^{-1}\left(1-2\sin^2\Theta_2\right)\\
&= -\cos^{-1}(\cos\theta\cos\Theta_1) + \Theta_1 - \theta + \frac{\sin\Theta_1}{\sin\Theta_2}\Theta_2\\
&\leqslant -\Theta_1 + \Theta_1 - \theta + \frac{R_2}{R_1}\Theta_2 \leqslant 0.
\end{aligned}$$

So, the global minimum of L is at 0, and the shortest path Σ is a meridian path. This completes the proof of Theorem 2. □

5 DEGENERATE ANTIPODAL HYPERSURFACES

The usual definition of a convex hypersurface covers, beside the boundaries of convex bodies, the degenerate case too. Formally, a *d-dimensional degenerate convex hypersurface* D is the union of two isometric copies K and K' of a convex body (also denoted by) $K \subset \mathbb{R}^d$ ($d \geqslant 2$), glued together along their boundary by identifying the points $x \in \mathrm{bd}K$ and $x' \doteq \iota(x) \in \mathrm{bd}K'$, where $\iota : K \to K'$ is the isometry between K and K'. Call K and K' the *faces* of D, and D the *double* of K; the *ridge of* D is $\mathrm{rd}D = K \cap K'$. Thus, D is (seen as) limit in $\mathbb{R}^{d+1}$ of d-dimensional convex hypersurfaces containing $\mathrm{rd}D$.

Let $\mathcal{D}$ denote the space of all degenerate convex hypersurfaces of some fixed dimension.

For any point x in the double D_T of an arbitrary simplex T of dimension at least 4, the set F_x is included in the vertex set of T [5]. Since F is upper semi-continuous and its image is closed, there exists a neighbourhood of D_T in $\mathcal{D}$, all of which hypersurfaces have F properly multivalued. Moreover, the same happens for F on *most* (in the sense of Baire categories) hypersurfaces in $\mathcal{D}$ [12], so $\mathcal{H} \cap \mathcal{D}$ is a small subset of $\mathcal{D}$. We refine this by Theorem 3.

The next auxiliary result, a complete proof of which can be found in [12], follows from the first variation formula.

A point $y \in S$ is called *critical* with respect to ρ_x (or to x), if for any direction τ of S at y for there exists a segment from y to x whose direction at y makes an angle $\alpha \leqslant \pi/2$ with τ.

Lemma 5. *If $y \in S$ is a local maximum for ρ_x then it is critical with respect to ρ_x.*

We say that a convex body K has the *property of double normals* if any line normal to K at some boundary point of K is also normal to K at the other intersection point with the boundary of K.

The following result can be found, for example, in [2].

Lemma 6. *A convex body K has constant width if and only if it has the property of double normals.*

The next result provides a link between the intrinsic geometry of convex hypersurfaces and the (extrinsic) geometry of convex bodies. Compared to Theorems 1 and 2, it also shows that the degeneracy is a quite strong restriction.

Theorem 3. *If the double of the convex body $K \subset \mathbb{R}^d$ has the corresponding mapping F an involutive bijection then K is smooth and has constant width.*

Proof. Denote by D the double of K and consider a point z in $S_K = \mathrm{bd}K \subset D$. Then clearly $v = F_z \in S_K$ and there are precisely two segments from v to z, one on each face of D. By Lemma 5, v is a critical point for ρ_z, hence the line zv is normal to S_K at v. Since $z = F_v$, z is a critical point for ρ_v and the line vz is normal to S_K at z, so S_K and consequently K has the property of double normals. Then K has constant width, by Lemma 6.

Assume there exists a supporting cone T_x of S_K, at some point $x \in S_K$, which is not a hyperplane, hence the normal cone N_x of S_K at x is not reduced to a vector. Then there exist two distinct 0-extreme unit vectors v', v'' in N_x, and two sequences of smooth points x'_n, x''_n convergent to x such that the unit normal vectors $n_{x'_n}, n_{x''_n}$ converge to v', v'' respectively (see Theorem 2.2.7 in [9]).

Put $y'_n = F_{x'_n}$, $y''_n = F_{x''_n}$ and $y' = \lim y'_n$, $y'' = \lim y''_n$. The Hausdorff dimension of the set of all singular points of S_K is at most $\dim S_K - 1$ (see [9]), so we may take x'_n, x''_n such that y'_n, y''_n are smooth, too. We get $y' \neq y''$, because the lines $x'_n y'_n$ and $x''_n y''_n$ are normal to S_K at x'_n, y'_n, and respectively x''_n, y''_n, and the limit directions v' and v'' are distinct. On the other hand, $y', y'' \in F_x$ because $y'_n = F_{x'_n}$, $y''_n = F_{x''_n}$, a contradiction to the bijectivity of F which proves the smoothness of D. □

6. DOUBLES OF CONSTANT WIDTH CONVEX BODIES

Degenerate antipodal convex hypersurfaces are smooth and have constant width, as shown by Theorem 3. In this section we prove a partial converse of this fact.

Theorem 4. *Let K be a C^2-differentiable convex body of constant width w, and D the double of K. Then there exists a neighbourhood N of the ridge of D such that for all $x \in N$,* $\max \rho_x = w$. *Consequently, the restriction of F to N is single-valued and involutive.*

Proof. To begin with, notice that the restriction of F to $S = \mathrm{rd}D$ is single-valued and involutive. Indeed, since K is of constant width, its boundary has the property of double normals (Lemma 6), and for any double normal the distance between the points of contact with $\mathrm{bd}K = S$ equals w.

Let $x, x' \in S$ be two mutually antipodal points in D, and Σ one of the two segments between them. We denote by x_t (resp. x'_t) the point of Σ (resp. of $\iota(\Sigma)$) such that $\rho(x, x_t) = t$ (resp. $\rho(x', x_t) = t$). We claim that there exists a positive number ε such that for all $t \in [0, \varepsilon]$, $F_{x_t} = x'_t$; moreover, there are precisely two segments between x_t and x'_t, the union of which is $\Sigma \cup \iota(\Sigma)$. In particular, $\max \rho_{x_t} = \rho(x_t, x'_t) = w$.

Assume on the contrary that for $t \to 0$ there exists a segment from x_t to x'_t whose length is strictly less than w. Such a segment consists of a segment Γ_t from x_t to some point $y_t \in S$ and a segment Γ'_t from y_t to x'_t. Since Σ and $\iota(\Sigma)$ are the only segments between x and x', y_t goes to x (or to x') if t goes to 0.

Let H be the hyperplane normal to Σ through x', and π the orthogonal projection onto H. As a convex hypersurface, S is locally the graph of some function defined on some neighbourhood of $x' \in H$, namely there exists an open set U containing x such that

$$S \cap U = \left\{ z + \big(w - \psi(z)\big)\frac{\overrightarrow{x'x}}{w} \colon z \in \pi(U) \right\}$$

Since S is C^2-differentiable, there exists a positive number A such that

$$0 \leqslant \psi(z) \leqslant A \|x'z\|^2. \tag{7}$$

With $z_t = \pi(y_t)$, the distance between x_t and x'_t is expressed by

$$g(z_t, t) \overset{\text{def}}{=} \sqrt{\|x'z_t\|^2 + (t - \psi(z_t))^2} + \sqrt{\|x'z_t\|^2 + (w - \psi(z_t) - t)^2}.$$

Using the simple fact that $\sqrt{U} + \sqrt{V} \leqslant w$ if and only if $w^2 \geqslant U + V$ and $(U + V - w^2)^2 \geqslant 4UV$, we obtain by a straightforward computation that $g(z_t, t) \leqslant w$ if and only if

$$0 \leqslant -\|x'z_t\|^2 + t(w - t) + \psi(z_t)(w - \psi(z_t)),$$
$$0 \leqslant -\|x'z_t\|^2 w^2 + 4t\psi(z_t)(w - \psi(z_t))(w - t) \overset{\text{def}}{=} h(z_t, t).$$

From (7) we get, for $t < \frac{1}{4A}$,

$$\begin{aligned} h(z_t, t) &\leqslant -\|x'z_t\|^2 w^2 + 4w^2 tA\|x'z_t\|^2 \\ &= \|x'z_t\|^2 w^2(4tA - 1) < 0. \end{aligned}$$

It follows that $g(z_t, t) > w$ for t small enough, which is in contradiction with Lemma 1, and proves the claim.

Put

$$\varepsilon(z) \overset{\text{def}}{=} \max\left\{\varepsilon > 0\colon\ \rho\left(z + \varepsilon\frac{\overrightarrow{zz'}}{w}, F_{z+\varepsilon\frac{\overrightarrow{zz'}}{w}}\right) = w\right\}.$$

Suppose that there exists a sequence z_n tending to z_0 such that $\varepsilon(z_n)$ is tending to zero. The above argument shows that $h(z_n, \varepsilon(z_n)) < 0$ for n large enough, which is impossible. Hence $\min_{z\in S}\varepsilon(z) > 0$ and the proof is complete. □

Remark. In the proof of Theorem 4 we have used the $\mathcal{C}^2$-differentiability only to obtain (7). Since this inequality also holds under the weaker hypothesis that the boundary of K has finite *upper curvatures* at every point (see [1] p.14 for the definition), the statement of Theorem 4 can be accordingly strengthened.

7. THREE CHARACTERIZATIONS OF BALLS

In this section we partly confirm Steinhaus' guess, by proving with our Theorems 5–7 that it is possible to characterize *the balls* (eventhough not the spheres) by the use of the mapping F.

Theorem 1 provides many hypersurfaces in $\mathcal{H} \setminus \mathcal{D}$ with central symmetry, and thus with the mapping F an isometry. Nevertheless, since any centrally symmetric body of constant width is a ball (see, for example, [2]), we directly obtain from Theorem 3 the following result.

Theorem 5. *If the double of the centrally symmetric convex body $K \subset \mathbb{R}^d$ has the corresponding mapping F an involutive bijection then K is a ball.*

Theorem 6. *If the double of the convex body $K \subset \mathbb{R}^2$ has the corresponding mapping F an isometry then K is a ball.*

Proof. This follows simply from Theorem 5 and Theorem 4 in [13], stating that the convex surface S is a centrally symmetric surface in $\mathcal{H}$ if and only if the associated mapping F is an isometry. □

Let S_K denote the *sphere inscribed to* K; i.e., the sphere of maximal radius included in K. We shall repeatedly and implicitly use the following simple fact (see, for example, [14]).

Lemma 7. *Any closed half sphere of S_K contains a point in the set $S_K \cap K$.*

While Theorems 1 and 2 provide many hypersurfaces in $\mathcal{I} \setminus \mathcal{D}$, the next result shows that there are very few degenerate hypersurfaces in $\mathcal{I}$.

Theorem 7. *If the double D of the convex body K satisfies* $\text{rad}(D) = \text{diam}(D)$ *then K is a ball.*

Proof. Suppose $D \in \mathcal{I} \cap \mathcal{D}$ is the double of the convex body $K \subset \mathbb{R}^d$. Then it is easily seen that K has constant width $w = \rho(x, F_x)$, $x \in D$, and also has the property of double normals.

Let o and r denote the centre and the radius of the sphere S_K inscribed to K, respectively. We claim that $F_o = \iota(o)$. If so, then $\rho(o, F_o) = 2r$, hence $w = 2r$ and K is a ball.

To prove the claim, assume there exists a point $\iota(y) \in (F_o \setminus \{\iota(o)\}) \cap \iota(K)$. Denote by S_- and S_+ the closed semispheres of S_K determined by the hyperplane orthogonal at o to the line yo, such that $y \in S_-$. Clearly, $y \in S_- \setminus S_+$.

The point o belongs to $F_{\iota(y)}$, because $\rho(o, F_o) = \rho(\iota(y), F_{\iota(y)})$. By Lemma 5, there is some segment Γ from $\iota(y)$ to o which intersects S_+, say at a. Put $\{b\} = \Gamma \cap \text{rd}D$, hence $\Gamma = [\iota(y)b] \cup [bo]$ and $a \in [bo]$.

Since S_K is inscribed to K, there exists a point $c \in S_- \cap K$. Let H be the hyperplane orthogonal to the line yo through c, and d the point in $H \cap S_- \cap yao$ determined by $[ya] \cap [od] \neq \emptyset$.

Put $\{e\} = [od] \cap [ay]$ and apply twice the triangle inequality, for $[aeo]$ and $[dey]$. We get $\|a - o\| + \|y - d\| \leqslant \|e - o\| + \|a - e\| + \|y - e\| + \|d - e\| = \|d - o\| + \|y - a\|$. Since $\|a - o\| = \|d - o\| = r$, it follows that $\|y - d\| \leqslant \|y - a\|$.

We have $\|\iota(y) - c\| = \|y - c\|$ and therefore, by the previous inequality,

$$\begin{aligned} l(\Gamma) &= \|\iota(y) - b\| + \|b - o\| = \|y - b\| + \|b - o\| \\ &\geqslant \|y - a\| + \|a - o\| \geqslant \|y - d\| + \|d - o\| \\ &= \|y - c\| + \|c - o\| = \|\iota(y) - c\| + \|c - o\|. \end{aligned}$$

The length-minimality of Γ implies that the inequalities above actually are all equalities, hence $\|y - d\| = \|y - a\|$, $\{a, d, c\} \subset S_- \cap S_+$, and any shortest path from $\iota(y)$ to o intersects S_K at a point in $S_K \cap K$.

Let E be the hyperellipsoid of revolution with the foci at o and y, and with the sum of the focal radii equal to $\rho(o, y)$. It follows that E is included in K, because all segments from $\iota(y)$ to o have the same length. Since E is tangent to K at c, its normal line n at c is also normal to K. Of course, n bisects the angle $\angle yco$. But S_K and K are tangent at c, so n is also normal to S_K at c, and thus $n = oc$, impossible.

This completely proves the claim and the theorem. □

8. OPEN QUESTIONS

We conclude with three questions related to our work.

1. The inclusion $\mathcal{I} \subset \mathcal{H}$ holds for convex surfaces in $\mathbb{R}^3$ (see [13]); is it true in arbitrary dimension? An affirmative answer would simplify Theorems 1 and 2, and would give a further motivation for Theorem 7.
2. Theorems 1 and 2, and the fact that all right circular cylinders of small height belong to $\mathcal{I} \cap \mathcal{H}$ [6], suggest the following problem.
   Find all convex (hyper)surfaces of revolution in $\mathcal{I} \cap \mathcal{H}$. Or, at least those whose generating function ϕ (see Section 2) has piecewise constant curvature.
3. Find all smooth hypersurfaces of constant width which belong to $\mathcal{H} \cap \mathcal{D}$ (see Theorems 3, 4 and 7).

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