$\mathcal{H}$armonic $\mathcal{A}$nalysis & $\mathcal{Q}$uantum $\mathcal{M}$echanics

Dedicated to Arne Jensen's 75th birthday

Periodic pseudodifferential operators; structure of their fibres

Radu Purice

"Simion Stoilow" Institute of Mathematics of the Romanian Academy

- collaboration with Horia Cornean and Bernard Helffer -



Aalborg, May 27 - 29, 2026

1. Introduction
2. A Weyl quantization on the d-dimensional torus
3. Quantization of periodic symbols
   - Preliminaries
   - Bloch-Floquet-Zak decomposition
   - Kernels of the fibre operators

Introduction

This talk is based on the paper:

Horia D. Cornean, Bernard Helffer and Radu Purice:
The fibre operators in the Bloch-Floquet decomposition of periodic magnetic pseudo-differential operators.
Journal of Pseudo- Differential Operators and Applications 17, art. 40 (2026);
https://doi.org/10.1007/s11868- 026-00788-w

The Problem:

• Let us consider a d-dimensional *configuration space* $\mathcal{X} \cong \mathbb{R}^d$ with *the canonical orthonormal basis* $\mathfrak{e}_1, \ldots, \mathfrak{e}_d$.

• We are interested in problems invariant under translations with vectors from a lattice $\Gamma := \bigoplus_{1 \leq j \leq d} \mathbb{Z}\, \mathfrak{e}_j$.

• More precisely we consider a quantum system described on *the Hilbert space* $\mathcal{H} := L^2(\mathcal{X})$ with a Hamiltonian commuting with all the translations $\left\{ \mathcal{T}_\gamma \,\middle|\, \gamma \in \Gamma \right\}$.

• We are interested in Hamiltonians given by pseudodifferential operators, extending the main results of the Bloch-Floquet Theory from the case of periodic differential operators.

• Having in view the following developments, we shall consider: $\mathcal{X}^*$, the linear dual of $\mathcal{X} \cong \mathbb{R}^d$ with the Euclidean structure defined by the following dual basis: $\left\{ \mathfrak{e}_1^*, \ldots, \mathfrak{e}_d^* \right\}$, $< \mathfrak{e}_j^* \,,\, \mathfrak{e}_k > := 2\pi\, \delta_{jk}$.

We like to emphasize some group theoretical aspects.

Let us consider the abelian group $\mathbb{S} := \left\{ \mathfrak{z} \in \mathbb{C} \,\middle|\, |\mathfrak{z}| = 1 \right\}$ with its Haar measure with total measure 1.

Given a locally compact abelian group $\mathcal{G}$:

We define its dual, as *the set of characters*
$\widehat{\mathcal{G}} := \left\{ \chi : \mathcal{G} \to \mathbb{S} \,\middle|\, \text{group homomorphism} \right\}$ with the canonical product and the topology of uniform convergence on compact subsets.

Given a pair of dual locally compact abelian groups $\mathcal{G}, \widehat{\mathcal{G}}$:

We define an associated Weyl system as a pair of unitary representations on some given complex separable Hilbert space $\mathcal{K}$:

$$U_{\mathcal{G}} : \mathcal{G} \to \mathbb{U}(\mathcal{K}), \quad V_{\mathcal{G}} : \widehat{\mathcal{G}} \to \mathbb{U}(\mathcal{K}), \text{ such that: } U_{\mathcal{G}}(g) V_{\mathcal{G}}(\hat{g}) = \hat{g}(g) V_{\mathcal{G}}(\hat{g}) U_{\mathcal{G}}(g)$$
$$\forall (g, \hat{g}) \in \mathcal{G} \times \widehat{\mathcal{G}}.$$

The Fourier transform on abelian locally compact groups.

Given a pair of dual abelian locally compact groups $\left(\mathcal{G}, \widehat{\mathcal{G}}\right)$ we define:

$$\mathcal{F}_{\mathcal{G}} : L^1(\mathcal{G}) \to BC(\widehat{G}), \quad \left(\mathcal{F}_{\mathcal{G}} F\right)(\hat{g}) := \int_{\mathcal{G}} (d_{\mathcal{G}} h)\, \hat{g}(h)\, F(h)$$

- It extends to a unitary operator $\mathcal{F}_{\mathcal{G}} : L^2(\mathcal{G}) \to L^2(\widehat{G})$
- with inverse $\left(\mathcal{F}_{\mathcal{G}}^{-} F\right)(g) := \int_{\widehat{\mathcal{G}}} (d_{\widehat{\mathcal{G}}} \hat{h})\, \overline{\hat{h}(g)}\, F(\hat{h})$

Pseudo-differential calculus associated to a Weyl system.

$\forall \Phi \in L^1\big(\mathcal{G} \times \widehat{\mathcal{G}}\big)$ we define:

- the joint Fourier transform: $\widehat{\Phi}(g, \hat{g}) := \int_{\mathcal{G}\times\widehat{\mathcal{G}}} (d_{\mathcal{G}} h \times d_{\widehat{\mathcal{G}}} \hat{h})\, \overline{\hat{g}(h)}\, \hat{h}(g)\, \Phi(h, \hat{h})$,
- the associated 'left' ΨDO: $\mathfrak{Op}_{\mathcal{G}} \Phi := \int_{\mathcal{G}\times\widehat{\mathcal{G}}} (d_{\mathcal{G}} g \times d_{\widehat{\mathcal{G}}} \hat{g})\, \widehat{\Phi}(g, \hat{g})\, V(\hat{g})\, U(g^{-1})$

with the Haar measures $d_{\mathcal{G}} h$ on $\mathcal{G}$ and $d_{\widehat{\mathcal{G}}} \hat{h}$ on $\widehat{\mathcal{G}}$.

One may also define a *'right' ΨDO* by taking the unitaries of the Weyl system in reversed order.

We identify the linear dual $\mathcal{X}^*$ with its dual as abelian group $\widehat{\mathcal{X}}$ through the bijection: $\mathcal{X}^* \ni \xi \mapsto \chi_\xi \in \widehat{\mathcal{X}}, \quad \chi_\xi(x) := e^{-i<\xi, x>}$.

The lattice of symmetries $\Gamma \subset \mathcal{X}$

puts into evidence the closed subgroup $\mathbb{Z}^d \hookrightarrow \mathbb{R}^d$
and the associated compact quotient group $\mathbb{R}^d/\mathbb{Z}^d \cong \mathbb{S}^d$.

We obtain two short exact sequences of abelian groups:

$\Gamma \hookrightarrow \mathcal{X} \rightarrowtail \mathcal{T}$ and $\Gamma_* \hookrightarrow \mathcal{X}^* \rightarrowtail \mathcal{T}_*$ with $\Gamma \cong \mathbb{Z}^d \cong \Gamma_*$ and $\mathcal{T} \cong \mathbb{S}^d \cong \mathcal{T}_*$.
and the associated pairs of dual groups $\left(\mathcal{X}, \mathcal{X}^*\right)$, $\left(\Gamma, \mathcal{T}_*\right)$ and $\left(\mathcal{T}, \Gamma_*\right)$.

The pseudodifferetial operators on $\mathcal{X}$ are affiliated to the algebra generated by the Weyl system associated to the pair $\left(\mathcal{X}, \mathcal{X}^*\right)$.
The $\mathbb{R}$-linear space structure on $\mathcal{X} \cong \mathbb{R}^d$ allows also for what we call a symmetric qunatization for the Weyl system associated to the pair $\left(\mathcal{X}, \mathcal{X}^*\right)$, generating *the Weyl calculus*.

Brief recall of the Hörmander-Weyl quantization

The Weyl system for $(\mathcal{X}, \mathcal{X}^*)$and its Schrödinger representation

$\mathcal{H} := L^2(\mathcal{X}) \ni f.$ $\quad U_{\mathcal{X}} : \mathcal{X} \to \mathbb{U}(\mathcal{H}), \quad \big(U_{\mathcal{X}}(z)f\big)(x) := f(x-z)$

$\quad V_{\mathcal{X}} : \mathcal{X}^* \to \mathbb{U}(\mathcal{H}), \quad \big(V_{\mathcal{X}}(\zeta)f\big)(x) := e^{-i<\zeta,x>} f(x).$

The Weyl unitary map

$$W_{\mathcal{X}} : \mathcal{X} \times \mathcal{X}^* \to \mathbb{U}(\mathcal{H}), \quad W_{\mathcal{X}}(z,\zeta) := e^{(-i/2)<\zeta,z>} V_{\mathcal{X}}(\zeta) U_{\mathcal{X}}(z).$$

The Weyl quantization, $\forall \Phi \in \mathscr{S}(\mathcal{X} \times \mathcal{X}^*)$

$$\mathfrak{Op}_{\mathcal{X}}\Phi := \int_{\mathcal{X}\times\mathcal{X}^*} (dz\, d\zeta) \Big((\mathcal{F}^-_{\mathcal{X}^*} \otimes \mathcal{F}_{\mathcal{X}^*})\Phi \Big)(\zeta, z) \Big(e^{(i/2)<\zeta,z>} \Big) V_{\mathcal{X}}(\zeta) U_{\mathcal{X}}(-z).$$

Explicitely: $\forall f \in \mathcal{H}$, $\forall x \in \mathcal{X}$:

$$\big(\mathfrak{Op}_{\mathcal{X}}(\Phi)f\big)(x) = \int_{\mathcal{X}} dy \Big[\int_{\mathcal{X}^*} d\eta\, e^{i<\eta,(x-y)>} \Phi\big((x+y)/2, \eta\big) \Big] f(y).$$

The Hörmander classes of symbols

For any $p \in \mathbb{R}$ we define:

$$S_1^p(\mathcal{X}\times\mathcal{X}^*) := \left\{F \in C^\infty(\mathcal{X}\times\mathcal{X}^*) \,\middle|\, \nu_{n,m}(F) < \infty \ \forall (n,m) \in \mathbb{N}^2\right\}$$

where: $\nu_{n,m}(F) := \max\limits_{a\in\mathbb{N}^d,|a|\le n} \max\limits_{b\in\mathbb{N}^d,|b|\le m} \sup\limits_{(x,\xi)\in\Xi} <\xi>^{-p+m} \left|\left(\partial_x^a\partial_\xi^b F\right)(x,\xi)\right|.$

- $F \in S_1^p(\mathcal{X}\times\mathcal{X}^*)$ with $p>0$ such that $\exists (C,R)\in\mathbb{R}_+^2$ with
$$\left|F(x,\xi)\right| \ge C^{-1} <\xi>^p \ \forall (x,\xi) \in \left\{\mathcal{X}\times\mathcal{X}^*, |\xi|\ge R\right\}$$
is called *elliptic*.

- For any $s\in\mathbb{R}$ we introduce the symbol $\mathfrak{m}_s(x,\xi) =<\xi>^s$, elliptic for $s>0$.

Theorem A

If $F \in S_1^p(\Xi)$ with $p>0$ is positive and elliptic, then $\mathfrak{Op}(F)$ is essentially self-adjoint on $\mathscr{S}(\mathcal{X})$ and the domain of its closure is:

$$\mathcal{D}\left(\overline{\mathfrak{Op}_{\mathcal{X}}(F)}\right) = \left\{f\in L^2(\mathcal{X}),\ \mathfrak{Op}_{\mathcal{X}}(\mathfrak{m}_p)f \in L^2(\mathcal{X})\right\} =: \mathscr{H}^p(\mathcal{X}).$$

The distribution kernel

The *Kernel Theorem* implies that

For any $F \in \mathscr{S}'(\mathcal{X} \times \mathcal{X}^*)$ there exists $\mathfrak{K}[F] \in \mathscr{S}'(\mathcal{X} \times \mathcal{X})$ such that:

$$\left\langle \mathfrak{K}[F], \overline{\phi} \otimes \psi \right\rangle_{\mathscr{S}(\mathcal{X}\times\mathcal{X})} = \left(\phi, \mathfrak{Op}_{\mathcal{X}}(F)\psi \right)_{L^2(\mathcal{X})}, \quad \forall (\phi, \psi) \in [\mathscr{S}(\mathcal{X})]^2.$$

Applying Lemma 1.3.5 and Proposition 1.3.6 from [*W. Amrein, A-M. Boutet-de-Monvel, V. Georgescu "C_0-Groups, Commutator Methods and Spectral Theory of N-Body Hamiltonians (1996)*] we obtain:

Theorem B.

For $F \in S^p_1(\Xi)$ with $p \in \mathbb{R}$, the operator $\mathfrak{Op}_{\mathcal{X}}(F)$ leaves $\mathscr{S}(\mathcal{X})$ invariant and its distribution kernel $\mathfrak{K}[F] \in \mathscr{S}'(\mathcal{X} \times \mathcal{X})$ has its singular support contained in $\Delta_{\mathcal{X}} := \{(x, x) \in \mathcal{X} \times \mathcal{X},\ x \in \mathcal{X}\}$ and rapid off-diagonal decay. Moreover, for $p < 0$ the kernel has integrable modulus as function of the variable $x - y \in \mathcal{X}$ orthogonal to $\Delta_{\mathcal{X}}$.

A Weyl quantization on the d-dimensional torus

The d-dimensional torus

- For the group isomorphism $\mathbb{R}/\mathbb{Z} \cong \mathbb{S}$
  we choose the following explicit projection function:
  $$\mathbb{R} \ni t \mapsto \mathfrak{p}_1(t) := e^{2\pi i t} \in \mathbb{S}.$$
- Restricting to the interval $\mathcal{I}_1 := [-1/2, 1/2)$ we may define
  a right-inverse $\mathfrak{s}_1 : \mathbb{S} \to \mathcal{I}_1$ such that $\mathfrak{p}_1 \circ \mathfrak{s}_1 = \mathbb{I}\mathrm{d}_{\mathbb{S}}$.
- Using the chosen orthonormal basis on $\mathcal{X}$ and $\mathcal{X}^*$ and taking direct products we define the following explicit projections and *right inverses for them*:
  - $\mathcal{X}/\Gamma = \mathscr{T} \cong \mathbb{S}^d, \quad \mathcal{X}^*/\Gamma_* = \mathscr{T}_* \cong \mathbb{S}^d,$
  - $\mathcal{X} \ni x \mapsto \mathfrak{p}(x) := \left(e^{i<\mathfrak{e}_1^*, x>}, \ldots, e^{i<\mathfrak{e}_d^*, x>}\right) \in \mathscr{T},$
    $\mathfrak{s} : \mathscr{T} \to \bigoplus_{1 \le j \le d} \mathcal{I}\mathfrak{e}_j =: \mathcal{E},$
  - $\mathcal{X}^* \ni \xi \mapsto \mathfrak{p}_*(\xi) := \left(e^{i<\xi, \mathfrak{e}_1>}, \ldots, e^{i<\xi, \mathfrak{e}_d>}\right) \in \mathscr{T}_*,$
    $\mathfrak{s}_* : \mathscr{T}_* \to \bigoplus_{1 \le j \le d} \mathcal{I}\mathfrak{e}_j^* =: \mathcal{B}.$

- On $\mathbb{S}^d$ we consider the Haar measure $d\mathfrak{z}$ giving total mass 1 to the group and the corresponding measures on $d\mathfrak{z}$ on $\mathscr{T}$ and $d\mathfrak{z}^*$ on $\mathscr{T}_*$. We notice that this coincides with the measure induced by the Lebesgue measure on $\mathbb{R}^{2d}$ through the canonical inclusion $\mathbb{S}^d \subset \mathbb{R}^{2d}$.

- The group $\mathbb{S}^d$ is an abelian Lie group having the Lie algebra $\mathfrak{sd} \cong \mathbb{R}^d$. Thus, each vector $v = \sum_{1\leq j\leq d} v_j \mathfrak{e}_j \in \mathbb{R}^d$ generates

an invariant vector field $< v, \mathring{\partial} >= \sum_{1\leq j\leq d} v_j \mathring{\partial}_j$ on $\mathbb{S}^d$

- Identifying $\mathcal{X} \cong \mathbb{R}^d \cong \mathfrak{sd}$ we may easily proove that:
$$\forall \mathring{\phi} \in C^\infty(\mathbb{S}^d): \quad \big(\mathring{\partial}_j \mathring{\phi}\big) \circ \mathfrak{p} = \partial_j\big(\mathring{\phi} \circ \mathfrak{p}\big), \ \forall j \in \{1, \ldots, d\}.$$

- Recalling the discrete Fourier transform: $L^1(\mathbb{S}^d) \ni \mathring{\phi} \mapsto \mathcal{F}_{\mathbb{S}^d} \mathring{\phi} \in \ell^\infty(\mathbb{Z}^d)$
$$\big(\mathcal{F}_{\mathscr{T}_*} \mathring{\phi}\big)_\gamma := \int_{\mathscr{T}_*} e^{i<\mathfrak{s}_*(\mathfrak{z}^*), \gamma>} \mathring{\phi}(\mathfrak{z}^*), \ \forall \gamma \in \Gamma,$$
$$\big(\mathcal{F}_{\mathscr{T}} \mathring{\phi}\big)_{\gamma^*} := \int_{\mathscr{T}} e^{-i<\gamma^*, \mathfrak{s}(\mathfrak{z})>} \mathring{\phi}(\mathfrak{z}), \ \forall \gamma^* \in \Gamma_*$$

- We can define the Sobolev spaces on the torus:
$$\mathscr{H}^p(\mathbb{S}^d) := \Big\{ \mathring{f} \in L^2(\mathbb{S}^d) \Big| \{< \cdot >^p [\mathcal{F}_{\mathbb{S}^d} \mathring{f}] \in \ell^2(\mathbb{Z}^d) \Big\}.$$

Considering the smooth real manifold structure on $\mathbb{S}^d$ one may use the general theory of Hörmander for pseudo-differential operators on real smooth manifolds, in order to obtain a 'local calculus' that has a nice covariant behaviour under changes of coordinates.

On the other hand, the abelian Lie group structure of $\mathbb{S}^d$ allows for a 'global theory' of pseudo-differential operators based on the general Fourier transform theory on abelian locally compact groups.

In fact, Theorem 4.4 in:
W. McLean, *Local and global description of periodic pseudodifferential operators*. Math. Nachr. 150 (1991)
proves the equivalence of the two procedures and gives a precise correspondence between the two types of symbols:
'local symbols' defined on charts for the Hörmander theory and 'global symbols' defined on the product $\mathbb{S}^d \times \mathbb{Z}^d$ with $\mathbb{Z}^d = \widehat{\mathbb{S}^d}$.
We shall use this 'global theory' for pseudo-differential operators on the d-dimensional torus, choosing the approach based on Weyl systems and proposing a symmetric representation for the Weyl system.

The Weyl system of the torus

The Weyl system for $(\mathscr{T}, \Gamma_*)$ and its Schrödinger representation

$$\mathcal{K} := L^2(\mathscr{T}) \ni \mathring{f}. \qquad U_{\mathscr{T}} : \mathscr{T} \to \mathbb{U}(\mathcal{K}), \quad \left(U_{\mathscr{T}}(\mathfrak{z})\mathring{f}\right)(\mathfrak{z}') := \mathring{f}([\mathfrak{z}']^{-1}\mathfrak{z})$$
$$V_{\mathscr{T}} : \Gamma_* \to \mathbb{U}(\mathcal{K}), \quad \left(V_{\mathscr{T}}(\gamma^*)\mathring{f}\right)(\mathfrak{z}) := e^{-i<\gamma^*, \mathfrak{s}(\mathfrak{z})>}\mathring{f}(\mathfrak{z}).$$

We would like to define, as in the case of $\left(\mathcal{X}, \mathcal{X}^*\right)$, a 'symmetric unitary map' mixing the two unitay groups of the Weyl system:

$$" \boldsymbol{W}(\mathfrak{z}, \gamma^*) := e^{(-i/2)<\gamma^*, \mathfrak{s}(\mathfrak{z})>} \boldsymbol{V}_{\mathscr{T}}(\gamma^*)\boldsymbol{U}_{\mathscr{T}}(\mathfrak{z}) = [\sqrt{\mathfrak{z}}]^{-\gamma^*} \boldsymbol{V}_{\mathscr{T}}(\gamma^*)\boldsymbol{U}_{\mathscr{T}}(\mathfrak{z})"$$

The factor $\sqrt{\mathfrak{z}}$ is no longer continuous on $\mathscr{T}$!

In order to recover the smoothness of the square root function we propose to use its Riemann surface, defined as the $\mathbb{Z}_2$-principal bundle over $\mathbb{S}$

$$\widetilde{\mathbb{S}}_2 := \left\{(\mathfrak{z}, \mathfrak{z}^2) \in [\mathbb{C} \setminus \{0\}]^2, \ \forall \mathfrak{z} \in \mathbb{S}\right\}, \quad \mathfrak{r}_2 : \widetilde{\mathbb{S}}_2 \ni (\mathfrak{z}, \mathfrak{z}^2) \mapsto \mathfrak{z}^2 \in \mathbb{S}$$

having the fibre $\mathfrak{r}_2^{-1}(\mathfrak{z}) = \{\pm\sqrt{\mathfrak{z}}\} \cong \mathbb{Z}_2$.

We shall cover $\mathscr{T}$ with a dirct product of d copies of $\widetilde{\mathbb{S}}_2$.

- $[2\mathbb{Z}^d] := \{2\gamma \,|\, \gamma \in \mathbb{Z}^d\}$, $\quad \mathbb{Z}_2 := \{1, q\}$ with $q^2 = 1$,
- $\mathbb{S}_2^d := \mathbb{R}^d/[2\mathbb{Z}^d] \cong [\widetilde{\mathbb{S}}_2]^d$, $\quad \tilde{\mathfrak{p}}_2(\mathfrak{z}) := \mathfrak{z}^2 \equiv (\mathfrak{z}_1^2, \ldots, \mathfrak{z}_d^2)$,
- $\widehat{\mathbb{S}_2^d} \cong [(1/2)\mathbb{Z}^d]$,
- $\mathfrak{s}_2 : \mathbb{S}_2^d \to [-1, 1)^d$ such that $\mathfrak{p}_2 \circ \mathfrak{s}_2 = \mathbb{I}\mathrm{d}_{\mathbb{S}_2^d}$.

$$
\begin{array}{ccccc}
 & & & & \mathbb{Z}_2^d \\
 & & & & \downarrow{\scriptstyle \hookrightarrow} \\
[2\mathbb{Z}^d] & \overset{\hookrightarrow}{\to} & \mathbb{R}^d & \xrightarrow{\mathfrak{p}_2} & \mathbb{S}_2^d \\
{\scriptstyle \hookrightarrow}\downarrow & & \| & & \downarrow{\scriptstyle \tilde{\mathfrak{p}}_2} \\
\mathbb{Z}^d & \xrightarrow{\hookrightarrow} & \mathbb{R}^d & \xrightarrow{\mathfrak{p}} & \mathbb{S}^d \\
{\scriptstyle \mathfrak{p}_2|_{\mathbb{Z}^d}}\downarrow & & & & \\
\mathbb{Z}_2^d & & & &
\end{array}
$$

There exists $\kappa : \mathbb{S}_2^d \to \mathbb{Z}_2^d$ such that $\mathfrak{s}_2(\widetilde{\mathfrak{z}}) = \mathfrak{s}(\widetilde{\mathfrak{z}}^2) + \kappa(\widetilde{\mathfrak{z}})$.

We shall use the following isomorphisms:

$$\mathbb{Z}^d \ni \nu \mapsto \chi_{1,\nu} \in \widehat{\mathbb{S}^d}, \quad \chi_{1,\nu}(\mathfrak{z}) := e^{-i<\nu,\mathfrak{s}(\mathfrak{z})>} = \mathfrak{z}^{-\nu} \equiv (\mathfrak{z}_1^{-\nu}, \dots, \mathfrak{z}_d^{-\nu}),$$

$$\mathbb{Z}^d \ni \nu \mapsto \chi_{2,\nu} \in \widehat{\mathbb{S}_2^d}, \quad \chi_{2,\nu}(\widetilde{\mathfrak{z}}) := e^{-i<(1/2)\nu,\mathfrak{s}_2(\widetilde{\mathfrak{z}})>} = \widetilde{\mathfrak{z}}^{-\nu} \equiv (\widetilde{\mathfrak{z}_1}^{-\nu}, \dots, \widetilde{\mathfrak{z}_d}^{-\nu}),$$

Definition

We call *symmetric Weyl representation of the d-dimensional torus Weyl system*, the following smooth unitary map:

$$\mathbb{S}_2^d \times \Gamma_* \ni (\tilde{\mathfrak{z}}, \gamma^*) \mapsto \widetilde{\mathbb{W}}_{\mathscr{T}}(\tilde{\mathfrak{z}}, \gamma^*) := \chi_{2,-\gamma^*}(\tilde{\mathfrak{z}}) U_{\mathscr{T}}\big(\tilde{\mathfrak{p}}_2(\tilde{\mathfrak{z}})\big) V_{\mathscr{T}}(\gamma^*) \in \mathbb{U}\big(L^2(\mathscr{T})\big).$$

Proposition

The symmetric Weyl representation $\widetilde{\mathbb{W}}_{\mathscr{T}} : \mathbb{S}_2^d \times \Gamma_* \to \mathbb{U}\big(L^2(\mathscr{T})\big)$ satisfies:

$$\widetilde{\mathbb{W}}_{\mathscr{T}}(\tilde{\mathfrak{z}}, \gamma^*)^* = \widetilde{\mathbb{W}}_{\mathscr{T}}(\tilde{\mathfrak{z}}^{-1}, -\gamma^*);$$

$$\widetilde{\mathbb{W}}_{\mathscr{T}}(\tilde{\mathfrak{z}}, \alpha^*)\widetilde{\mathbb{W}}_{\mathscr{T}}(\tilde{\mathfrak{z}'}, \beta^*) = \chi_{2,\alpha^*}(\tilde{\mathfrak{z}'})^{-1}\chi_{2,\beta^*}(\tilde{\mathfrak{z}})\widetilde{\mathbb{W}}_{\mathscr{T}}(\tilde{\mathfrak{z}}\tilde{\mathfrak{z}'}, \alpha^* + \beta^*).$$

The totoidal symmetric quantization

- Given any $\mathbb{Z}^d$-indexed sequence $\{\vec{f}_\gamma\}_{\gamma\in\mathbb{Z}^d}$ we define: $(\mathfrak{d}_j\vec{f})_\gamma := \vec{f}_{\gamma+\varepsilon_j} - \vec{f}_\gamma$.
- On $C^\infty(\mathscr{T})$ we consider the seminorms: $\nu_n(f) := \sup\limits_{\mathfrak{z}\in\mathscr{T}} \max\limits_{|\mu|\leq n} \left|\left(\mathring{\partial}^\mu f\right)(\mathfrak{z})\right|$.

The spaces of *symbols*

- $o\left(\mathbb{Z}^d; C^\infty(\mathscr{T})\right) :=$

  $$= \left\{(\vec{f}_\gamma)_{\gamma\in\mathbb{Z}^d} \,\middle|\, \vec{f}_\gamma \in C^\infty(\mathscr{T}),\, \exists p\geq 0,\, w^{(p)}_{m,n}(\vec{f}) < \infty\ \forall (m,n)\in\mathbb{N}^2\right\}$$

  where: $w^{(p)}_{m,n}(\vec{f}) := \sup\limits_{\gamma\in\mathbb{Z}^d} <\gamma>^{-p} \max\limits_{|\alpha|\leq m} \nu_n\left((\mathfrak{d}^\alpha\vec{f})_\gamma\right)$.
- For $p\in\mathbb{R}$ and $\rho = 0, 1$:

  $o^p_\rho\left(\mathbb{Z}^d; C^\infty(\mathscr{T})\right) := \left\{\vec{f}\in o\left(\mathbb{Z}^d; C^\infty(\mathscr{T})\right) \middle| w^{(p,\rho)}_{m,n}(\vec{f}) \leq \infty\ \forall (m,n)\in\mathbb{N}^2\right\}$

  where: $w^{(p,\rho)}_{m,n}(\vec{f}) := \sup\limits_{\gamma\in\mathbb{Z}^d} <\gamma>^{-p+m\rho} \max\limits_{|\alpha|\leq m} \nu_n\left((\mathfrak{d}^\alpha\vec{f})_\gamma\right)$.
- $s\left(\mathbb{Z}^d; C^\infty(\mathscr{T})\right) :=$

  $$= \left\{(\vec{f}_\gamma)_{\gamma\in\mathbb{Z}^d} \,\middle|\, \vec{f}_\gamma \in C^\infty(\mathscr{T}),\, w^{(-n)}_{m,n}(\vec{f}) < \infty\ \forall (N,m,n)\in\mathbb{N}^3\right\}.$$

The *The 'symmetric toroidal Fourier transform'*.

Given a sequence $\vec{\Phi} \in o\big(\mathbb{Z}^d; C^\infty(\mathscr{T})\big)$

we shall consider the isomorphism $\mathbb{Z}^d \ni \nu \mapsto \chi_{2,\nu} \in (1/2)\Gamma_* = \widehat{\mathbb{S}_2^d}$

and the Fourier transform $\mathcal{F}^w_{\mathscr{T},\hat{\mathscr{T}}} : o\big(\mathbb{Z}^d; C^\infty(\mathscr{T})\big) \to \ell^1\big(\Gamma_*; \mathscr{D}'(\mathbb{S}_2^d)\big)$

$$\big(\mathcal{F}^w_{\mathscr{T},\hat{\mathscr{T}}}\,\vec{\Phi}\big)_{\alpha_*}(\tilde{\mathfrak{z}}) = 2^{-d}\int_{\mathscr{T}} d\mathfrak{z}'\,\chi_{1,-\alpha^*}(\mathfrak{z}')\sum_{\gamma^*\in\mathbb{Z}^d}\chi_{2,\gamma^*}(\tilde{\mathfrak{z}})\,\vec{\Phi}_{\gamma^*}(\mathfrak{z}')$$

The symmetric toroidal quantization

$$\mathfrak{Op}^w_{\mathscr{T}}(\vec{\Phi}) := 2^{-d}\sum_{\alpha^*\in\Gamma_*}\int_{\mathbb{S}_2^d} d\tilde{\mathfrak{z}}\,\Big(\big(\mathcal{F}^w_{\mathscr{T},\hat{\mathscr{T}}}\,\vec{\Phi}\big)_{\alpha^*}(\tilde{\mathfrak{z}})\Big)\,\widetilde{\mathbb{W}}_{\mathscr{T}}(\tilde{\mathfrak{z}},\alpha^*)\,.$$

The distribution kernel of the quantized operator

$$2^{-d}\sum_{\gamma^*\in\Gamma_*} e^{i<(1/2)\gamma^*,\mathfrak{s}(\mathfrak{z})-\mathfrak{s}(\mathfrak{z}')>}\Big[\sum_{\kappa\in\mathbb{Z}_2^d} e^{i<(1/2)\gamma^*,\kappa>}\,\vec{\Phi}_{\gamma^*}\Big(\mathfrak{p}\big[(1/2)\big(\mathfrak{s}(\mathfrak{z}')+\mathfrak{s}(\mathfrak{z})+\kappa\big)\big)\Big]$$

Quantization of periodic symbols

Preliminaries

Periodic Hörmander type symbols

For any $p \in \mathbb{R}$ let:

$$S^p_1(\mathcal{X}\times\mathcal{X}^*)_\Gamma := \left\{F \in S^p_1(\mathcal{X}\times\mathcal{X}^*)\,\middle|\, F(x+\gamma,\xi) = F(x,\xi)\ \forall \gamma\in\Gamma\right\}$$

- Denoting by $\mathcal{T}_\gamma$ the translation with $\gamma \in \Gamma$ on distributions on $\mathcal{X}$:

$$F \in S^p_1(\mathcal{X}\times\mathcal{X}^*)_\Gamma \implies (\mathcal{T}_\gamma\otimes\mathcal{T}_\gamma)\mathfrak{K}[F] = \mathfrak{K}[F],\ \ \forall\gamma\in\Gamma.$$

Periodic magnetic fields Given a regular magnetic field $B = dA$.

Replacing the usual translations: $\big(U_{\mathcal{X}}(z)f\big)(x) := f(x-z)$

with magnetic translations: $\big(U^A_{\mathcal{X}}(z)f\big)(x) := \exp\Big(-i\int\limits_{[x,x-z]} A\Big) f(x-z)$

we define a *'magnetic Weyl quantization'*:

$$\mathfrak{Op}^A_{\mathcal{X}}\Phi := \int\limits_{\mathcal{X}\times\mathcal{X}^*} (dz\,d\zeta)\,\big((\mathcal{F}^-_{\mathcal{X}^*}\otimes\mathcal{F}_{\mathcal{X}^*})\Phi\big)(\zeta,z)\big(e^{(i/2)<\zeta,z>}\big)V_{\mathcal{X}}(\zeta)U^A_{\mathcal{X}}(-z)$$

that allows for a complete *magnetic pseudo-differential calculus*.

Periodic magnetic fields

Suppose that the magnetic field B has smooth Γ-periodic components and suppose also that it satisfies:

$$\int_0^1 ds \int_0^1 ds' \, \langle B(\gamma + s\mathfrak{e}_j + s'\mathfrak{e}_k), \mathfrak{e}_j \wedge \mathfrak{e}_k \rangle = 0, \quad \forall \gamma \in \Gamma, \ \forall (j,k). \qquad (*)$$

- **Proposition** The above condition is necessary and sufficient in order to exist a smooth, Γ-periodic 1-form A such that $B = dA$.

Theorem

Suppose given a symbol $F \in S_1^p(\mathcal{X} \times \mathcal{X}^*)$ for some $p \in \mathbb{R}$ and a regular Γ-periodic magnetic field $B = dA$ satisfying condition (*) above with A smooth and Γ-periodic. Then $\mathfrak{Op}_{\mathcal{X}}^A(F)$ also has a "standard" Weyl symbol $F_W^A \in S_1^p(\mathcal{X} \times \mathcal{X}^*)$ such that $\mathfrak{Op}_{\mathcal{X}}^A(F) = \mathfrak{Op}_{\mathcal{X}}(F_W^A)$.
Moreover we have that:

$$F_W^A(x,\xi) - F\big(x, \xi - A(x)\big) \in S_1^{p-2}(\Xi). \qquad (3.1)$$

The distribution kernel

Given $F \in S^p_1(\mathcal{X} \times \mathcal{X}^*)_\Gamma$, we have that:

$$\mathfrak{K}[F] = \big(\mathbb{I}\mathrm{d}_{\mathcal{X}} \otimes \mathcal{F}_{\mathcal{X}^*}\big)\big(F \circ \Upsilon\big), \quad \Upsilon(x,y) := \big((x+y)/2, x-y\big)$$

and thus $(\mathcal{T}_\gamma \otimes \mathcal{T}_\gamma)\mathfrak{K}[F] = \mathfrak{K}[F], \ \forall \gamma \in \Gamma$.

Using Theorem B above, we know that:
$\mathfrak{K}[F] \in \mathscr{S}'(\mathcal{X} \times \mathcal{X})$ has singular support in
$\Delta_{\mathcal{X}} := \{(x,x) \in \mathcal{X} \times \mathcal{X}, \ x \in \mathcal{X}\}$ and rapid off-diagonal decay.
Moreover, for $p < 0$ the singularity on $\Delta_{\mathcal{X}}$ is integrable in $x - y \in \mathcal{X}$ (orthogonal to $\Delta_{\mathcal{X}}$).

Regularization of the distribution kernel.

$$\mathfrak{K}[F]_N(x,y) := \int_{\mathcal{X}^*} d\eta\, e^{i<\eta, x-y>}\, \theta_N(\eta)\, F\big((x+y)/2, \eta\big) \equiv \int_{\mathcal{X}^*} d\eta\, e^{i<\eta, x-y>}\, F^{(N)}\big((x+y)/2, \eta\big)$$

$$\theta \in C_0^\infty\big(\mathcal{X}^*; [0,1]\big), \ |\xi| \leq 1/2 \Rightarrow \theta(\xi) = 1, \ \mathrm{supp}\theta \subset \{|\xi| \leq 1\}$$

$$\theta_N(\xi) := \theta(N^{-1}\xi), \quad \forall N \in \mathbb{N} \setminus \{0\}\}.$$

$F^{(N)} \in S^{-\infty}(\mathcal{X} \times \mathcal{X}^*)$.

Bloch-Floquet-Zak decomposition

Decomposition induced by the inclusion $\mathbb{Z}^d \subset \mathbb{R}^d$

Recall the notation: $\mathcal{I} := [-1/2, 1/2), \quad \mathcal{E} := \sum_{1 \leq j \leq d} \mathcal{I} e_j.$

- We shall use the decomposition: $\mathcal{X} \ni x \mapsto (\iota(x), \hat{x}) \in \Gamma \times \mathcal{E}$
  $\iota(x) := \big(\lfloor x_1 \rfloor, \ldots, \lfloor x_d \rfloor\big), \quad \lfloor t \rfloor := \max\big\{k \in \mathbb{Z} \,\big|\, k \leq t\big\}, \ \forall\, t \in \mathbb{R}.$
- and the associated unitary decomposition:
  $$L^2(\mathcal{X}) \ni f \mapsto \mathcal{U}_\Gamma f \in \ell^2(\Gamma) \otimes L^2(\mathcal{E}) \cong \ell^2\big(\Gamma; L^2(\mathcal{E})\big),$$
  $$\big(\mathcal{U}_\Gamma f\big)(\gamma, \hat{x}) := f(\gamma + \hat{x}).$$

The discrete Fourier transform: $\mathcal{F}_\Gamma : \ell^1(\Gamma) \to C(\mathcal{T}_*), \ \big(\mathcal{F}_\Gamma \vec{f}\big)(\mathfrak{z}^*) := \sum_{\gamma \in \Gamma} \chi_{\mathfrak{z}^*}(\gamma) \vec{f}_\gamma$

allows to define:

$$\mathring{\mathcal{U}}_{BF} := \big(\mathcal{F}_\Gamma \otimes \mathbb{I}\mathrm{d}_{\mathcal{E}}\big) \circ \mathcal{U}_\Gamma : L^2(\mathcal{X}) \to L^2(\mathcal{T}_*) \otimes L^2(\mathcal{E}).$$

We conclude that for $F \in S^p_1(\mathcal{X} \times \mathcal{X}^*)_\Gamma$, the operator $\mathfrak{Op}_{\mathcal{X}}(F)$ commutes with the multiplication operators with $\chi_{\mathfrak{z}^*}$ for any $\mathfrak{z}^* \in \mathcal{T}_*$ and thus with all the multiplication operators with functions in $C(\mathcal{T}_*)$.

A standard argument applied to the above conclusion implies that $\mathring{\mathcal{U}}_{BF}\mathfrak{Op}(F)\mathring{\mathcal{U}}_{BF}^{-1}$ decomposes as a measurable operator valued function:

$$\mathscr{T}_* \ni \mathfrak{z}^* \mapsto \widehat{\mathfrak{Op}(F)}_{\mathfrak{z}^*} \text{ acting on } \mathring{\mathcal{U}}_{BF}\big[\mathscr{S}(\mathcal{X})\big].$$

In order to get a more tractable form for the transformed representation of the group $\mathcal{X} \cong \mathbb{R}^d$ one uses the 'extended version of the $\mathring{\mathcal{U}}_{BF}$, the Bloch-Floquet transform:

- $\big(\mathcal{U}_{BF}\phi\big)(\mathfrak{z}^*, x) = \sum\limits_{\gamma\in\Gamma} e^{-i<\mathfrak{s}_*(\mathfrak{z}^*),\gamma>}\phi(x+\gamma), \quad \forall \phi \in \mathscr{S}(\mathcal{X});$
- $\big(\mathcal{U}_{BF}\phi\big)(\mathfrak{z}^*) \in \Big\{ f \in L^2_{\mathrm{loc}}\big(\mathcal{X}; L^2(\mathcal{E})\big) \,\Big|\, f(x+\gamma) = e^{i<\mathfrak{s}_*(\mathfrak{z}^*),\gamma>} f(x) \ \forall \gamma \in \Gamma \Big\} =: \mathscr{F}_{\mathfrak{z}^*}$
- **Lemma:** The extension of $\mathfrak{Op}(F)$ to $\mathscr{S}'(\mathcal{X})$ by duality leaves each $\mathscr{F}_{\mathfrak{z}^*}$ invariant.
- $\big(\mathcal{U}_{BF}\mathfrak{Op}_{\mathcal{X}}(F)\phi\big)(\mathfrak{z}^*) = \sum\limits_{\gamma\in\Gamma} \chi_{\mathfrak{z}^*}(\gamma) U_{\mathcal{X}}(\gamma)\mathfrak{Op}_{\mathcal{X}}(F)\phi = \sum\limits_{\gamma\in\Gamma} \chi_{\mathfrak{z}^*}(\gamma)\mathfrak{Op}_{\mathcal{X}}(F)U_{\mathcal{X}}(\gamma)\phi =$

$$=: \widehat{\mathfrak{Op}(F)}_{\mathfrak{z}^*}\Big(\sum_{\gamma\in\Gamma} \chi_{\mathfrak{z}^*}(\gamma) U_{\mathcal{X}}(\gamma)\phi\Big) = \widehat{\mathfrak{Op}(F)}_{\mathfrak{z}^*}\big(\mathcal{U}_{BF}\phi\big)(\mathfrak{z}^*)$$

$\widehat{\mathfrak{Op}(F)}_{\mathfrak{z}^*}$ is the restriction to $\mathscr{F}_{\mathfrak{z}^*}$ of the extension of $\mathfrak{Op}(F)$ to $\mathscr{S}'(\mathcal{X})$.

Proof of the Lemma

Let $\mathring{f} \in \mathscr{F}_{\mathfrak{z}^*}$ and $F \in S^p_1(\mathcal{X} \times \mathcal{X}^*)$:

$$\big(\mathfrak{Op}_{\mathcal{X}}(F)\mathring{f}\big)(x+\gamma) = \int_{\mathcal{X}} dy \int_{\mathcal{X}^*} d\eta\, e^{i<\eta, x+\gamma-y>}\, F\big((x+\gamma+y)/2, \eta\big)\, \mathring{f}(y)$$

$$= \int_{\mathcal{X}} dz \int_{\mathcal{X}^*} d\eta\, e^{i<\eta, x-z>}\, F\big((x+z)/2, \eta\big)\, \mathring{f}(z+\gamma) =$$

$$= \int_{\mathcal{X}} dz \int_{\mathcal{X}^*} d\eta\, e^{i<\eta, x-z>}\, F\big((x+z)/2, \eta\big)\, e^{-i<\mathfrak{s}_*(\mathfrak{z}^*), \gamma>}\, \mathring{f}(z)$$

$$= e^{i<\mathfrak{s}_*(\mathfrak{z}^*), \gamma>} \big(\mathfrak{Op}_{\mathcal{X}}(F)\mathring{f}\big)(x) \qquad \blacksquare$$

The Bloch-Floquet space

- The scalar product on $\mathscr{F}_{\mathfrak{z}^*}$: $\big(\hat{f}, \hat{g}\big)_{\mathfrak{z}^*} := \int_{\mathcal{E}} dx\, \overline{\hat{f}(x)}\, \hat{g}(x)$ for any $\mathfrak{z}^* \in \mathscr{T}_*$.
- $\mathscr{F}^{\infty}_{\mathfrak{z}^*} := \Big\{\hat{f} \in C^{\infty}(\mathcal{X}), \Big| \hat{f}(x+\gamma) = e^{i<\mathfrak{s}_*(\mathfrak{z}^*), \gamma>}\, \hat{f}(x)\Big\}$
- $\mathscr{F} := \int^{\oplus}_{\mathscr{T}_*} d\mathfrak{z}^*\, \mathscr{F}_{\mathfrak{z}^*}$; $\quad \mathscr{F}^{\infty} := \Big\{\hat{f} \in C^{\infty}(\mathscr{T}_* \times \mathcal{X}) \Big| \hat{f}(\mathfrak{z}^*) \in \mathscr{F}^{\infty}_{\mathfrak{z}^*},\ \forall \mathfrak{z}^* \in \mathscr{T}_*\Big\}$.

- $\mathcal{U}_{BF}\big[\mathscr{S}(\mathcal{X})\big] \subset \mathscr{F}^{\infty}$.

Recalling the regularization:
$\mathfrak{K}[F]_N(x,y) = \int_{\mathcal{X}^*} d\eta\, e^{i<\eta, x-y>}\, F^{(N)}\big((x+y)/2, \eta\big), \quad F^{(N)} \in S^{-\infty}(\mathcal{X} \times \mathcal{X}^*)$
we have the statement:

Proposition

Supose given $F \in S^p_1(\Xi)_\Gamma$ and let us fix $\hat{\phi} \in \mathscr{F}^\infty_{\mathfrak{z}^*}$ for some $\mathfrak{z}^* \in \mathscr{T}_*$. Then given any $\epsilon > 0$, for any $\mu \in \mathbb{N}^d$ and any $n \geq (p+d+1)/2$, there exists $C(n,\mu) > 0$ and $N''_\epsilon \in \mathbb{N} \setminus \{0\}\}$ such that for any $N \geq N''_\epsilon$:

$$\sup_{x\in\mathcal{X}} \Big|\Big[\partial_x^\mu \big(\widehat{\mathfrak{Op}}(F)_{\mathfrak{z}^*} - \widehat{\mathfrak{Op}}(F^{(N)})_{\mathfrak{z}^*}\big)\hat{\phi}\Big](x)\Big| \leq$$

$$\leq\; C(n,\mu)\,\epsilon \Big[\sup_{(x,\xi)\in\Xi} \max_{|\nu|\leq|\mu|+p+d+1} \max_{|\nu'|\leq 2n} \Big[\partial_x^\nu \partial_\xi^{\nu'} F\Big](x,\xi)\Big|\Big]\Big[\sup_{x\in\mathcal{X}} \max_{|\nu''|\leq p+d+1} \Big|\Big[\partial_x^{\nu''} \hat{\phi}\Big](x)\Big|\Big].$$

Proof of the Proposition

Let us write:

$$\Big[\big(\widehat{\mathfrak{Op}}(F)_{\mathfrak{X}^*} - \widehat{\mathfrak{Op}}(F^{(N)})_{\mathfrak{X}^*}\big)\hat{\phi}\Big](\hat{x}+\alpha)$$

$$= \int_{\mathcal{X}} dy \Big[\int_{\mathcal{X}^*} d\eta\, e^{i<\eta,\hat{x}-y+\alpha>}\big(1-\theta_N(\eta)\big)F\big((\hat{x}+y+\alpha)/2,\eta\big)\Big]\hat{\phi}(y)$$

$$= e^{-i<\mathfrak{s}_*(\mathfrak{X}^*),\alpha>}\int_{\mathcal{X}} dy \Big[\int_{\mathcal{X}^*} d\eta\, e^{i<\eta,\hat{x}-(y-\alpha)>}\big(1-\theta_N(\eta)\big)F\big((\hat{x}+y-\alpha)/2+\alpha,\eta\big)\Big]\hat{\phi}(y-\alpha)$$

$$= e^{-i<\mathfrak{s}_*(\mathfrak{X}^*),\alpha>}\int_{\mathcal{X}} dy \Big[\int_{\mathcal{X}^*} d\eta\, e^{i<\eta,\hat{x}-y>}\big(1-\theta_N(\eta)\big)F\big((\hat{x}+y)/2,\eta\big)\Big]\hat{\phi}(y)$$

$$= e^{-i<\mathfrak{s}_*(\mathfrak{X}^*),\alpha>}\int_{\mathcal{X}} dy <y>^{-2n}\int_{\mathcal{X}^*} d\eta <\eta>^{-(p+d+1)} e^{i<\eta,\hat{x}-y>}\times$$

$$\times\Big[\big(\mathbb{Id}-\Delta_y\big)^{(p+d+1)}\Big(\big(\mathbb{Id}-\Delta_\eta\big)^n\big(1-\theta_N(\eta)\big)F\big((\hat{x}+y)/2,\eta\big)\Big)\hat{\phi}(y)\Big]$$

and some usual integration by parts using the oscillating factor $e^{i<\eta,\hat{x}-y>}$ allows to obtain the desired estimation taking into account that:

$$\int_{\mathcal{X}^*} d\eta <\eta>^{-(p+d+1)}\big(1-\theta_N(\eta)\big) \underset{N\nearrow\infty}{\longrightarrow} 0, \quad \big(\partial_\eta^\kappa\theta_N\big)(\eta) \underset{N\nearrow\infty}{\sim} N^{-|\kappa|},\ \forall\kappa\in\mathbb{N}^d.$$

■

Corollary

The limit $\widehat{\mathfrak{Op}}(F^{(N)})_{\mathfrak{Z}^*} \underset{N\nearrow\infty}{\longrightarrow} \widehat{\mathfrak{Op}}(F)_{\mathfrak{Z}^*}$ *exists uniformly on bounded subsets of* $\mathscr{F}^\infty_{\mathfrak{Z}^*}$ *for any* $\mathfrak{Z}^*\in\mathscr{T}_*$.

The Bloch-Floquet-Zak transform

Let us fix any $\hat{\phi} \in \mathscr{F}^{\infty}$ and compute:

$$\big(\mathcal{U}_{BF}\mathfrak{Op}_{\mathcal{X}}(F)\mathcal{U}_{BF}^{-1}\hat{\phi}\big)(\mathfrak{z}^*, x) = \int_{\mathcal{X}} dy \int_{\mathcal{X}^*} d\eta\, e^{i<\eta, x-y>} F\big((x+y)/2, \eta\big)\hat{\phi}(\mathfrak{z}^*, y).$$

For $\mathfrak{z}^*_\circ \in \mathscr{T}_*$, let us consider the unitary operator of multiplication with:

$$\mathfrak{f}_{\mathfrak{z}^*_\circ}(\xi, x) := e^{-i<\xi - \mathfrak{s}_*(\mathfrak{z}^*_\circ), x>}$$

and for $\mathfrak{z}^*$ in a neighbourhood of $\mathfrak{z}^*_\circ$ we have:

$$\big(\mathfrak{f}_{\mathfrak{z}^*_\circ} \circ (\mathfrak{s}_* \times \mathbb{Id})\mathcal{U}_{BF}\mathfrak{Op}_{\mathcal{X}}(F)\mathcal{U}_{BF}^{-1}\mathfrak{f}_{\mathfrak{z}^*_\circ} \circ (\mathfrak{s}_* \times \mathbb{Id})\hat{\phi}\big)(\mathfrak{z}^*, x) =$$

$$= \int_{\mathcal{X}} dy \int_{\mathcal{X}^*} d\eta\, e^{-i<\mathfrak{s}_*(\mathfrak{z}^*) - \mathfrak{s}_*(\mathfrak{z}^*_\circ), (x-y)>} e^{i<\eta, (x-y)>} F\big((x+y)/2, \eta\big)\hat{\phi}(\mathfrak{z}^*, y)$$

$$= \big(\widehat{\mathfrak{Op}}\big((\mathbb{Id}_{\mathcal{X}} \otimes \mathcal{T}_{\mathfrak{s}_*(\mathfrak{z}^*) - \mathfrak{s}_*(\mathfrak{z}^*_\circ)})F\big)_{\mathfrak{z}^*_\circ}\hat{\phi}(\mathfrak{z}^*, \cdot)\big)(x) \quad \boxed{\Rightarrow \text{smoothnes around } \mathfrak{z}^*_\circ}$$

with $(\mathbb{Id}_{\mathcal{X}} \times \mathcal{T}_{\hat{\xi}})F$ the translated of F in the second variable with $\hat{\xi} \in \mathcal{B}$.

Let $\mathfrak{i}_*$ be the identity on $\mathscr{T}_*$

The Bloch-Floquet-Zak (BFZ) transform

$$\big(\mathcal{U}_{BFZ}f\big)(\xi, \mathfrak{z}) := \big[\big(\mathfrak{f}_{\mathfrak{i}_*} \circ (\mathbb{Id} \times \mathfrak{s})\big)\mathcal{U}_{BF}f\big](\xi, \mathfrak{z}) = \sum_{\gamma \in \Gamma} e^{-i<\xi, \gamma + \mathfrak{s}(\mathfrak{z})>} f\big(\gamma + \mathfrak{s}(\mathfrak{z})\big).$$

Structure of periodic ΨDO

- Notice that $\mathcal{U}_{BFZ}\mathcal{S}(\mathcal{X}) \subset BC^{\infty}\big(\mathcal{X}^*; C^{\infty}(\mathcal{T})\big)$.
- If we denote by $\mathcal{G} := \mathcal{U}_{BFZ}\big[L^2(\mathcal{X})\big]$, we have the equality:

$$\mathcal{G} = \Big\{\widetilde{f} \in L^2_{\mathtt{loc}}\big(\mathcal{X}^*; L^2(\mathcal{T})\big) \,\Big|\, \widetilde{f}(\xi+\gamma^*) = \chi^{-1}_{\gamma^*}\widehat{f}(\xi) \in L^2(\mathcal{T}),\ \forall \gamma^* \in \Gamma_*\Big\}.$$

Theorem

Let $\mathfrak{E}_{\mathfrak{z}^*}$ be the multiplication with $\mathfrak{f}\big(\mathfrak{s}_*(\mathfrak{z}^*), \mathfrak{s}(\cdot)\big)$ on $C^{\infty}(\mathcal{T})$.
Given $F \in S^p_1(\mathcal{X}\times\mathcal{X}^*)_\Gamma$, there exists a smooth, Γ_*-periodic function:

$$\mathcal{T}_* \ni \mathfrak{z}^* \mapsto \widetilde{\mathfrak{Op}}(F)_{\mathfrak{z}^*} := \mathfrak{E}_{\mathfrak{z}^*}\widehat{\mathfrak{Op}}\big((\mathbb{I}\mathrm{d}_{\mathcal{X}} \times \mathcal{T}_{\mathfrak{s}_*(\mathfrak{z}^*)})F\big)_{\mathfrak{i}_*}\mathfrak{E}^{-1}_{\mathfrak{z}^*} \in \mathcal{L}\big(C^{\infty}(\mathcal{T})\big)$$

such that
the operator $\mathcal{U}_{BFZ}\mathfrak{Op}(F)\mathcal{U}^{-1}_{BFZ}$ acting in $\mathcal{G}$ is decomposable as the operator-valued function on $\mathcal{X}^*$:

$$\mathcal{X}^* \ni \xi \mapsto \chi_{\iota(\xi)}\widetilde{\mathfrak{Op}}(F)_{\mathfrak{p}_*(\xi)}\chi^{-1}_{\iota(\xi)} \in \mathcal{L}\big(C^{\infty}(\mathcal{T})\big).$$

- We also denote by

$$\mathcal{G}^{\infty} := \Big\{\widetilde{f} \in C^{\infty}\big(\mathcal{X}^*; C^{\infty}(\mathcal{T})\big) \,\Big|\, \widetilde{f}(\xi+\gamma^*) = \chi^{-1}_{\gamma^*}\widehat{f}(\xi) \in C^{\infty},\ \forall \gamma^* \in \Gamma_*\Big\}.$$

Using the above results for $\mathfrak{Op}(F)$ and for its resolvent and for $\mathfrak{m}_p$, one obtains the following result.

Theorem

For a positive, elliptic symbol $F \in S_1^p(\mathcal{X} \times \mathcal{X}^*)_\Gamma$, with $p > 0$ and for any $\xi \in \mathcal{X}^*$, the fibre operator $\chi_{\iota(\xi)} \widetilde{\mathfrak{Op}}(F)_{\mathfrak{p}_*(\xi)} \chi^{-1}_{\iota(\xi)}$ is self-adjoint and lower bounded with self-adjointness domain $\mathscr{H}^p(\mathscr{T})$.

Kernels of the fibre operators

Theorem B above implies that for $F \in S^p_1(\mathcal{X} \times \mathcal{X}^*)_\Gamma$ the ΨDO $\mathfrak{Op}(F)$ has a distribution kernel $\mathfrak{K}[F]$ having at most a singular support contained in the diagonal $\Delta_{\mathcal{X}} := \left\{(x,x) \in \mathcal{X} \times \mathcal{X} \,\middle|\, x \in \mathcal{X}\right\}$.

Let us study first the distribution kernel of the fibre operators $\widehat{\mathfrak{Op}}(F)_{\mathfrak{z}^*}$ for any $\mathfrak{z}^* \in \mathcal{T}_*$.
We use the regularization $F^{(N)} \underset{N \nearrow \infty}{\longrightarrow} F$ defined before.

Proposition

Fixing any $\mathfrak{z}^* \in \mathscr{T}_*$, for any $N \in \mathbb{N} \setminus \{0\}$, $\widehat{\mathfrak{Op}}(F^{(N)})_{\mathfrak{z}^*}$ has the following structure as operator on $\mathscr{F}^{\infty}_{\mathfrak{z}^*} \subset \mathscr{E}(\mathcal{X})$:

$$\widehat{\mathfrak{Op}}(F^{(N)})_{\mathfrak{z}^*} = \mathfrak{I}nt_{\mathcal{X}} \mathfrak{K}[F^{(N)}]_{\mathfrak{z}^*},$$

$$\mathfrak{K}[F^{(N)}]_{\mathfrak{z}^*}(\hat{x}, \hat{y}) = \sum_{\beta \in \Gamma} \hat{\chi}_{\mathfrak{z}^*}(\beta)\, \mathfrak{K}[F^{(N)}](\hat{x}, \hat{y} + \beta)\,.$$

Proof.

Let us fix two functions $\hat{\phi}$ and $\hat{\psi}$ in $\mathscr{F}^{\infty}_{\mathfrak{x}^*}$ and compute:

$$\left(\hat{\psi}\,,\, \widehat{\mathfrak{Op}}(F^{(N)})_{\mathfrak{x}^*}\hat{\phi}\right)_{\mathfrak{x}^*} = \lim_{M \nearrow \infty} \int_{\mathcal{E}} d\hat{x}\, \overline{\hat{\psi}(\hat{x})} \int_{\mathcal{X}} dy\, \mathfrak{K}[F^{(N)}](\hat{x}, y)\, \hat{\phi}_M(y)$$

$$= \lim_{M \nearrow \infty} \int_{\mathcal{E}} d\hat{x}\, \overline{\hat{\psi}(\hat{x})} \int_{\mathcal{E}} d\hat{y} \hat{\phi}(\hat{y}) \sum_{\beta \in \Gamma} \mathfrak{K}[F^{(N)}](\hat{x}, \hat{y} + \beta) \hat{\chi}_{\mathfrak{x}^*}(\beta) \sum_{|\gamma| \leq M} \rho(\hat{y} + \beta + \gamma).$$

□

Remark

The Proposition is true for $\widehat{\mathfrak{Op}}(F)$ when $F \in S^p_1(\mathcal{X} \times \mathcal{X}^*)$ with $p < 0$.

The BFZ-transform of a ΨDO

- Let us fix $F \in S_1^p(\mathcal{X} \times \mathcal{X}^*)_\Gamma$ and $\mathring{\phi} \in \mathcal{G}^\infty$ and compute:

$$\big(\mathcal{U}_{BFZ}\mathfrak{Op}_{\mathcal{X}}(F)\mathcal{U}_{BFZ}^{-1}\mathring{\phi}\big)(\xi,\mathfrak{z}) =$$

$$= \sum_{\alpha\in\Gamma} e^{-i<\xi,\alpha+\mathfrak{s}(\mathfrak{z})>}\int_{\mathcal{X}} dy \int_{\mathcal{X}^*} d\eta\, e^{i<\eta,\alpha+\mathfrak{s}(\mathfrak{z})-y>} F\big((\alpha+\mathfrak{s}(\mathfrak{z})+y)/2,\eta\big)\int_{\mathcal{B}} d\theta\, e^{i<\theta,y>}\,\mathring{\phi}\big(\theta,\mathfrak{p}(y)\big)$$

$$= \sum_{\alpha\in\Gamma} e^{-i<\xi,\alpha+\mathfrak{s}(\mathfrak{z})>}\sum_{\beta\in\Gamma}\int_{\mathcal{E}} d\hat{y} \sum_{\beta^*\in\Gamma_*}\int_{\mathcal{X}^*} d\hat{\eta}\, e^{i<\beta^*+\hat{\eta},\alpha+\mathfrak{s}(\mathfrak{z})-\beta-\hat{y}>} F\big((\alpha+\mathfrak{s}(\mathfrak{z})+\beta+\hat{y})/2,\eta\big)\int_{\mathcal{B}} d\theta\, e^{i<\theta,\beta+\hat{y}>}\,\mathring{\phi}\big(\theta,\mathfrak{p}(\hat{y})\big)$$

$$= \sum_{\gamma\in\Gamma}\sum_{\beta\in\Gamma} e^{-i<\xi,\gamma+\beta+\mathfrak{s}(\mathfrak{z})>}\int_{\mathcal{E}} d\hat{y} \sum_{\beta^*\in\Gamma_*}\int_{\mathcal{X}^*} d\hat{\eta}\, e^{i<\beta^*+\hat{\eta},\gamma+\mathfrak{s}(\mathfrak{z})-\hat{y}>} F\big((\gamma+\mathfrak{s}(\mathfrak{z})+\hat{y})/2,\eta\big)\int_{\mathcal{B}} d\theta\, e^{i<\theta,\beta+\hat{y}>}\,\mathring{\phi}\big(\theta,\mathfrak{p}(\hat{y})\big)$$

$$= \sum_{\tilde{\gamma}\in\Gamma}\sum_{\kappa\in\mathbb{Z}_2^d} e^{-i<\xi,2\tilde{\gamma}+\kappa+\mathfrak{s}(\mathfrak{z})>}\int_{\mathcal{E}} d\hat{y} \sum_{\beta^*\in\Gamma_*}\int_{\mathcal{X}^*} d\hat{\eta}\, e^{i<\beta^*+\hat{\eta},2\tilde{\gamma}+\kappa+\mathfrak{s}(\mathfrak{z})-\hat{y}>} F\big((\kappa+\mathfrak{s}(\mathfrak{z})+\hat{y})/2,\beta^*+\hat{\eta}\big)\, e^{i<\hat{\xi},\hat{y}>}\,\mathring{\phi}\big(\hat{\xi},\mathfrak{p}(\hat{y})\big)$$

$$= e^{-i<\iota(\xi),\mathfrak{s}(\mathfrak{z})>}\int_{\mathscr{T}} d\mathfrak{z}' \sum_{\beta^*\in\Gamma_*}\sum_{\kappa^*\in\mathbb{Z}_2^d} e^{i<\beta^*+\kappa^*/2,\mathfrak{s}(\mathfrak{z})-\mathfrak{s}(\mathfrak{z}')>}\sum_{\kappa\in\mathbb{Z}_2^d}\frac{e^{(i/2)<\kappa^*,\kappa>}}{2^d}\, F\big((\kappa+\mathfrak{s}(\mathfrak{z})+\mathfrak{s}(\mathfrak{z}'))/2,\beta^*+\hat{\xi}+(\kappa^*/2)\big)\,\mathring{\phi}\big(\hat{\xi},\mathfrak{z}'\big)$$

$$= \Big[\mathcal{I}nt_{\mathscr{T}}\Big(\chi_{\iota(\xi)}\mathring{\mathcal{K}}\big[(\mathbb{Id}_{\mathcal{X}}\times\mathcal{T}_{\hat{\xi}})F\big]\chi_{\iota(\xi)}^{-1}\Big)\mathring{\phi}(\xi)\Big](\mathfrak{z}) = \Big[\mathcal{I}nt_{\mathscr{T}}\Big(\mathring{\mathcal{K}}\big[(\mathbb{Id}_{\mathcal{X}}\times\mathcal{T}_{\xi})F\big]\Big)\mathring{\phi}(\xi)\Big](\mathfrak{z})$$

Thus the fibre operator above $\xi \in \mathcal{X}^*$ has the distribution kernel $\mathring{\mathcal{K}}\big[(\mathbb{I}\mathrm{d}_{\mathcal{X}} \times \mathcal{T}_\xi)F\big] \in \mathscr{D}'(\mathscr{T} \times \mathscr{T})$:

$$\mathring{\mathcal{K}}\big[(\mathbb{I}\mathrm{d}_{\mathcal{X}} \times \mathcal{T}_\xi)F\big](\mathfrak{z}, \mathfrak{z}') =$$

$$= \sum_{\beta^* \in \Gamma_*} \sum_{\kappa^* \in \mathbb{Z}_2^d} e^{i<\beta^* + (1/2)\kappa^*, \mathfrak{s}(\mathfrak{z}) - \mathfrak{s}(\mathfrak{z}')>} \sum_{\kappa \in \mathbb{Z}_2^d} \frac{e^{(i/2)<\kappa^*, \kappa>}}{2^d} F\big((\kappa + \mathfrak{s}(\mathfrak{z}) + \mathfrak{s}(\mathfrak{z}'))/2, \beta^* + \xi + (\kappa^*/2)\big).$$

Let us compare it with the generic structure of the distribution kernel of a toroidal ΨDO:

$$2^{-d} \sum_{\gamma^* \in \Gamma_*} e^{i<(1/2)\gamma^*, \mathfrak{s}(\mathfrak{z}) - \mathfrak{s}(\mathfrak{z}')>} \Big[\sum_{\kappa \in \mathbb{Z}_2^d} e^{i<(1/2)\gamma^*, \kappa>} \vec{\Phi}_{\gamma^*}\big(\mathfrak{p}[(1/2)(\mathfrak{s}(\mathfrak{z}') + \mathfrak{s}(\mathfrak{z}) + \kappa)]\big)\Big].$$

If we decompose $\gamma^* = 2\beta^* + \kappa^*$ with $\beta^* \in \Gamma^*$ and $\kappa^* \in \mathbb{Z}_2^d$ we obtain the equality:

$$\big(\mathcal{U}_{BFZ}\mathfrak{Op}(F)\mathcal{U}_{BFZ}^{-1}\mathring{\phi}\big)(\xi, \mathfrak{z}) = \big(\mathfrak{Op}^w_{\mathscr{T}}(\vec{\Phi}_{F,\xi})\mathring{\phi}(\xi, \cdot)\big)(\mathfrak{z})$$

with:

$$(\vec{\Phi}_{F,\xi})_{\gamma^*}(\mathfrak{z}) := [(\mathbb{I}\mathrm{d}_{\mathcal{X}} \times \mathcal{T}_\xi)F]\big(\mathfrak{s}(\mathfrak{z}), (1/2)\gamma^*\big).$$

We may view this as a smooth section in the vector bundle associated to the principal bundle $\mathcal{X}^* \rightarrowtail \big[\mathcal{X}^*/[(1/2)\Gamma_*]\big]$ by the representation of the group $(1/2)\Gamma_*$ on $o^p\big((1/2)\Gamma_*; C^\infty(\mathscr{T})\big)$ by translations.

Let us put into evidence two more forms of the fibre operators:

- The infinite matrix form in $\ell^2(\Gamma_*)$ after a discrete Fourier transform:

$$\left[\mathcal{F}_{\mathscr{T}}\left(\mathcal{U}_{BFZ}\mathfrak{Op}(F)\mathcal{U}_{BFZ}^{-1}\right)_{\xi}\mathcal{F}_{\mathscr{T}}^{-1}\right]_{\alpha^*,\beta^*} =$$
$$= \left[(\mathcal{F}_{\mathscr{T}}\otimes \mathbb{Id}_{\mathcal{X}^*})\left(F\circ(\mathfrak{p}\otimes \mathbb{Id}_{\mathcal{X}})\right]\left(\alpha^*-\beta^*,\xi-(\alpha^*+\beta^*)/2\right).\right.$$

- The form involving the Weyl system of the torus:

$$\left(\mathcal{U}_{BFZ}\mathfrak{Op}(F)\mathcal{U}_{BFZ}^{-1}\mathring{\phi}\right)(\xi,\mathfrak{z}) = \left(\mathfrak{Op}^{w}_{\mathscr{T}}(\vec{\Phi}_{F,\xi})\mathring{\phi}(\xi,\cdot)\right)(\mathfrak{z})$$
$$= 2^{-2d}\sum_{\alpha^*\in\Gamma_*}\int_{\mathbb{S}_2^d} d\tilde{\mathfrak{z}}\Big(\Big[(\mathcal{F}_{\mathscr{T}}\otimes\mathcal{F}_{(1/2)\Gamma_*})\left(F_{\xi}\circ(\mathfrak{s}\otimes\mathfrak{R}_{(1/2)\Gamma_*})\right)\Big]\left(\alpha^*,\tilde{\mathfrak{z}}\right)\Big)\times$$
$$\times\Big[\chi_{2,-\alpha^*}(\tilde{\mathfrak{z}})U_{\mathscr{T}}\left(\tilde{\mathfrak{p}}_2(\tilde{\mathfrak{z}})\right)V_{\mathscr{T}}(\alpha^*)\mathring{\phi}(\xi,\cdot)\Big](\mathfrak{z})$$

with $\mathfrak{R}_{(1/2)\Gamma_*}$ the restriction to $(1/2)\Gamma_* \subset \mathcal{X}^*$ of functions on $\mathcal{X}^*$.

Many Happy Years Dear Arne !