On the Generalized Derivation Induced by Two Projections

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The Theorem of Akhiezer and Glazman

Notation

- $\mathcal{L}(\mathscr{H}, \mathscr{K})$ denotes the Banach space of all bounded linear operators between complex Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$.
- $\mathcal{L}(\mathscr{H}) := \mathcal{L}(\mathscr{H}, \mathscr{H})$.
- $\ker T$, respectively $\operatorname{ran} T$ denote the kernel, respectively the range of $T \in \mathcal{L}(\mathscr{H}, \mathscr{K})$.

Let P and Q be orthogonal projections on $\mathscr{H}$.

Theorem (N.I. Akhiezer - I.M. Glazman, 1993)

$$\|P - Q\| = \max\{\|(1 - P)Q\|, \|P(1 - Q)\|\}.$$

The Theorem of Buckholtz

Let $\mathscr{L}$ and $\mathscr{R}$ be closed subspaces in $\mathscr{H}$. We denote by P, respectively Q the orthogonal projections with ranges $\mathscr{L}$, respectively $\mathscr{R}$.

Theorem (D. Buckholtz, 2000)

The following conditions are equivalent:

(a) $\mathscr{H} = \mathscr{L} \dot{+} \mathscr{R}$.

(b) *There exists a bounded idempotent with range $\mathscr{L}$ and kernel $\mathscr{R}$.*

(c) *$P - Q$ is invertible.*

(d) $\|P + Q - 1\| < 1$.

(e) $\|PQ\|,\ \|(1 - P)(1 - Q)\| < 1$.

Other Related Results

Extensions and Generalizations

We extend and/or generalize results by:

- N.I. Akhiezer and I.M. Glazman
- D. Buckholtz
- S. Maeda
- Z. Boulmaarouf, M. Fernandez Miranda and J.-Ph. Labrouse
- T. Kato
- Y. Kato
- J.J. Koliha and V. Rakočević

Possibility of Applications

Applications

These problems have been discusses in connection to various applications in:

- perturbation theory for linear operators
- probability theory
- Fredholm theory
- complex geometry
- statistics
- wavelet theory
- invariant subspace theory

Definition

Let $P \in \mathcal{L}(\mathscr{H})$ and $Q \in \mathcal{L}(\mathscr{K})$ be orthogonal projections.

Definition

The generalized derivation induced by P and Q is defined as

$$\mathcal{L}(\mathscr{K}, \mathscr{H}) \ni X \mapsto \delta_{P,Q}(X) := PX - XQ \in \mathcal{L}(\mathscr{K}, \mathscr{H}).$$

Simple Formulas

- $[\delta_{P,Q}(X)]^* = -\delta_{Q,P}(X^*)$
- $\delta_{P,Q}(X) = -\delta_{1-P,1-Q}(X)$
- $|\delta_{P,1-Q}(X)|^2 + |\delta_{P,Q}(X)|^2 = |\delta_{0,Q}(X)|^2 + |\delta_{0,1-Q}(X)|^2$
- $|\delta_{P,Q}(X)|^2 = |P\delta_{P,Q}(X)|^2 + |\delta_{P,Q}(X)Q|^2$

A Generalized Akhiezer-Glazman Equality

Let $\{T_i\}_{i=1}^n$ be a finite family of bounded linear operators between $\mathscr{H}$ and $\mathscr{K}$.

Lemma

If $T_i^* T_j = 0_{\mathscr{H}}$ and $T_i T_j^* = 0_{\mathscr{K}}$ for every $i \neq j$ then

$$\left\| \sum_{i=1}^n T_i \right\| = \max_{i=1}^n \| T_i \|.$$

Let $P \in \mathcal{L}(\mathscr{H}),\ Q \in \mathcal{L}(\mathscr{K})$ be two orthogonal projections and $X \in \mathcal{L}(\mathscr{K}, \mathscr{H})$.

Theorem

$$\|PX - XQ\| = \max\{\|(1-P)XQ\|, \|PX(1-Q)\|\}.$$

Remarks and Consequences 1

Remarks

- $$\|PX - XQ\| \leq \|X\|.$$
- If $X \in \mathcal{L}(\mathscr{H})$ is selfadjoint and $Q = 1 - P$ then
$$\|PX - XP\| = \|(1 - P)XP\|;$$
in particular,
$$\|PQ - QP\| = \|(1 - P)QP\| = \|(1 - Q)PQ\|,$$
where P and Q are two orthogonal projections on $\mathscr{H}$ (S. Maeda, 1990).

Remarks and Consequences 2

Remarks

- $$\|PX - 2PXQ + XQ\| = \|PX - XQ\|;$$
  in particular, if $\mathscr{K} = \mathscr{H}$, $X = 1_{\mathscr{H}}$ and $PQ = QP$ then $P = Q$ or $\|P - Q\| = 1$ (S. Maeda, 1976).
- Let $V \in \mathcal{L}(\mathscr{H}, \mathscr{K})$ be a partial isometry, $P = V^*V$, $Q = VV^*$ and let $X \in \mathcal{L}(\mathscr{K}, \mathscr{H})$. Then
  $$\|PX - XQ\| \leq \max\{\|PX - XV\|, \|VX - XQ\|\};$$
  in particular, when $\mathscr{K} = \mathscr{H}$ and $X = 1_{\mathscr{H}}$ we actually have that
  $$\|P - Q\| \leq \|P - V\| = \|V - Q\|$$
  (S. Maeda, 1990).

Remarks and Consequences 3

Remarks

- Let M be an idempotent and P an orthogonal projection, both acting on the Hilbert space $\mathscr{H}$. Then

$$\|MP - PM\| \leq \frac{\|M\| + \sqrt{\|M\|^2 - 1}}{2};$$

  in particular, if M is an orthogonal projection Q then $\|PQ - QP\| \leq \frac{1}{2}$ (S. Maeda, 1990).
- Let $M \in \mathcal{L}(\mathscr{H})$ be an idempotent and $P \in \mathcal{L}(\mathscr{H})$ the range projection of M. In the special case $Q = 1 - P$ and $X = M^*M$ we deduce that $\|M - P\| = \|M - M^*\|$ (Z. Boulmaarouf, M. Fernandez Miranda and J.-Ph. Labrouse, 1997).

A Generalized Kato Inequality

Let

- $M \in \mathcal{L}(\mathscr{H})$ and $N \in \mathcal{L}(\mathscr{K})$ be two idempotents,
- P and Q the range projections of M and, respectively, N,
- $X \in \mathcal{L}(\mathscr{K}, \mathscr{H})$.

Theorem

$$\|PX - XQ\| \leq \max\{\|MX - XN\|, \|M^*X - XN^*\|\}.$$

Remarks and Consequences

Remarks

- For the case $\mathscr{K} = \mathscr{H}$ and $X = 1_{\mathscr{H}}$ we obtain the following inequality of T. Kato (1995):

$$\|P - Q\| \leq \|M - N\|.$$

- For the case $\mathscr{K} = \mathscr{H}$ and $P = Q$, if M is an idempotent on $\mathscr{H}$, P is the range projection of M and $X \in \mathcal{L}(\mathscr{H})$ then

$$\|PX - XP\| \leq \max\{\|MX - XM\|, \|M^* X - XM^*\|\};$$

in particular, if X is selfadjoint then

$$\|PX - XP\| \leq \|MX - XM\|.$$

A Generalized Maeda Characterization

Let P be a selfadjoint projection on $\mathscr{H}$ and $A \in \mathcal{L}(\mathscr{K}, \mathscr{H}) \setminus \{0\}$.

Lemma

The following conditions are equivalent:

(a) $\operatorname{ran}(PA) = \operatorname{ran} P$.

(b) $\operatorname{ran}(PAA^*P) = \operatorname{ran} P$.

(c) $\left\|P(\|A\|^2 - AA^*)^{1/2}\right\| < \|A\|$.

(d) PAA^*P is invertible in $P\mathcal{L}(\mathscr{H})P$.

(e) $\|A\|^2(1 - P) + PAA^*P$ is invertible.

(f) $\|A\|^2(1 - P) + PAA^*$ is invertible.

Remark

For the case when A is a selfadjoint projection the equivalence $(a) \Leftrightarrow (c)$ is due to S. Maeda, 1977.

Injectivity

Proposition

The following conditions are equivalent:

(a) $PX - XQ$ is one-to-one.

(b) $\overline{\operatorname{ran}[(1-Q)X^*P]} = \ker Q$ and $\overline{\operatorname{ran}[QX^*(1-P)]} = \operatorname{ran} Q$.

(c) $\|(PX + XQ - X)k\|^2 < \|XQk\|^2 + \|X(1-Q)k\|^2$ for every $k \in \mathscr{K}$, $k \neq 0$.

Remarks

- In the special case when $\mathscr{K} = \mathscr{H}$ and $X = 1_{\mathscr{H}}$ the equivalence $(a) \Leftrightarrow (b)$ is due to Z. Takeda and T. Turumaru, 1952, while $(a) \Leftrightarrow (c)$ to S. Maeda, 1977.
- We can exchange the roles of P with Q and of X with X^* to obtain necessary and sufficient conditions to ensure that $PX - XQ$ has dense range.

A Necessary and a Sufficient Condition for Invertibility

Proposition

If $PX - XQ$ is left invertible then
$\operatorname{ran}(QX^*) = \operatorname{ran} Q$, $\operatorname{ran}[(1-Q)X^*] = \ker Q$ and

$$\|PX + XQ - X\| < \max\{\|XQ\|, \|X(1-Q)\|\}.$$

Proposition

If $\operatorname{ran}(QX^*) = \operatorname{ran} Q$, $\operatorname{ran}[(1-Q)X^*] = \ker Q$ and

$$\|PX + XQ - X\| < \min\left\{ \inf_{\substack{k \in \operatorname{ran} Q \\ \|k\|=1}} \|Xk\|, \inf_{\substack{k \in \ker Q \\ \|k\|=1}} \|Xk\| \right\}$$

then $PX - XQ$ is left invertible.

Necessary and Sufficient Conditions 1

Theorem

The following conditions are equivalent:

(a) $PX - XQ$ *is left invertible.*

(b) $\operatorname{ran}[QX^*(1-P)] = \operatorname{ran} Q$ *and* $\operatorname{ran}[(1-Q)X^*P] = \ker Q$.

(c) $\operatorname{ran}[|(1-P)XQ|^2] = \operatorname{ran} Q$ *and* $\operatorname{ran}[|PX(1-Q)|^2] = \ker Q$.

(d) $\left\|Q[\|X\|^2 - X^*(1-P)X]^{1/2}\right\| < \|X\|$ *and* $\left\|(1-Q)(\|X\|^2 - X^*PX)^{1/2}\right\| < \|X\|$.

(e) $|(1-P)XQ|^2$ *is invertible in* $Q\mathcal{L}(\mathscr{K})Q$ *and* $|PX(1-Q)|^2$ *is invertible in* $(1-Q)\mathcal{L}(\mathscr{K})(1-Q)$.

(f) $\|X\|^2(1-Q) + |(1-P)XQ|^2$ *and* $\|X\|^2 Q + |PX(1-Q)|^2$ *are invertible.*

(g) $\|X\|^2(1-Q) + QX^*(1-P)X$ *and* $\|X\|^2 Q + (1-Q)X^*PX$ *are invertible.*

Necessary and Sufficient Conditions 2

Remarks

- We can exchange the roles of P with Q and of X with X^* in the previous propositions and theorem to obtain necessary and/or sufficient conditions for the right invertibility, respectively invertibility of $PX - XQ$.
- In the special case $\mathscr{K} = \mathscr{H}$ and $X = 1_{\mathscr{H}}$:
  - $(b) \Leftrightarrow (d) \Leftrightarrow (e)$ S. Maeda, 1977
  - $(a) \Leftrightarrow (d)$ D. Buckholtz, 2000
  - $(a) \Leftrightarrow (f) \Leftrightarrow (g)$ J.J. Koliha and V. Rakočević, 2002 (in the setting of rings).

The Kato Condition

Theorem (Y. Kato, 1976)

If $\|P+Q-1\|<1$ *(equivalently,* $P-Q$ *is invertible) then*

$$\|P+Q-1\| = \|PQ\| = \|(1-P)(1-Q)\|.$$

Theorem

If $PX-XQ$ *is left invertible and* $\mathrm{ran}\, P$ *is invariant under* $X|PX-XQ|^{-1}X^*$ *then*

$$\|PX+XQ-X\| = \|PXQ\| = \|(1-P)X(1-Q)\|.$$

Remark

If $PX-XQ$ is invertible then QX^*P and $(1-P)X(1-Q)$ are unitarily equivalent.

$PX - XQ$ invertible and $\operatorname{ran} X$ not closed

If $PX - XQ$ is invertible then operators $PX(1-Q)$, $(1-P)XQ$, PX, $(1-P)X$, XQ and $X(1-Q)$ have closed ranges. However, the invertibility of $PX - XQ$ does not imply that X has closed range:

Example

Let P and Q be two orthogonal projections $\mathcal{L}(\mathscr{H}) \setminus \{0,1\}$ such that $\|P-Q\|, \|P+Q-1\| < 1$, U a unitary operator on $\ker P$ onto $\ker Q$ (B. Sz.-Nagy, 1942) and Z a bounded linear operator on $\ker P$ which does not have closed range. We define

$$X := Q + \frac{1}{2\|Y\|\|(PQ-QP)^{-1}\|} Y,$$

where, for $h \in \mathscr{H}$, $Yh := UZ(h - Ph)$. Then $PX - XP$ is invertible and X does not have closed range.

Invertibility and Operators with Dense Ranges

Proposition

The following conditions are equivalent:

(a) $PX - XQ$ has closed range.

(b) $PX(1-Q)$ and $(1-P)XQ$ have closed ranges.

Theorem

The following conditions are equivalent:

(a) $PX - XQ$ *is invertible.*

(b) $\operatorname{ran}[PX(1-Q)] = \operatorname{ran} P$, $\operatorname{ran}[(1-P)XQ] = \ker P$, $\overline{\operatorname{ran}[QX^*(1-P)]} = \operatorname{ran} Q$ *and* $\overline{\operatorname{ran}[(1-Q)X^*P]} = \ker Q$.

(c) $\overline{\operatorname{ran}[PX(1-Q)]} = \operatorname{ran} P$, $\overline{\operatorname{ran}[(1-P)XQ]} = \ker P$, $\operatorname{ran}[QX^*(1-P)] = \operatorname{ran} Q$ *and* $\operatorname{ran}[(1-Q)X^*P] = \ker Q$.

Lemma

Lemma

(i) If PX has closed range and the sum $\ker(PX) + \ker Q$ is closed and direct then $\mathrm{ran}\,[(1-Q)X^*P] = \ker Q$. The converse is, in general, false.

(ii) The following conditions are equivalent:

- **(a)** $X(1-Q)$ has closed range, the sum $\ker(PX) + \ker Q$ is direct and the sum $\mathrm{ran}\,P + \ker[(1-Q)X^*]$ is closed;
- **(b)** $\mathrm{ran}\,[(1-Q)X^*P] = \ker Q$.

(iii) If PX and $X(1-Q)$ have closed ranges then $\ker(PX) + \ker Q$ is closed if and only if $\mathrm{ran}\,P + \ker[(1-Q)X^*]$ is closed.

(iv) If the sums $\ker(PX) + \ker Q$ and $\mathrm{ran}\,P + \ker[(1-Q)X^*]$ are direct then $\mathrm{ran}\,[PX(1-Q)] = \mathrm{ran}\,P$ if and only if $\mathrm{ran}\,[(1-Q)X^*P] = \ker Q$.

Sufficient Conditions for Invertibility

The Condition $(P, Q, X)_1$

PX has closed range and the sum $\ker(PX) + \ker Q$ is closed and direct.

Theorem

Each of the following conditions

(i) $(P, Q, X)_1$ *and* $(1 - P, 1 - Q, X)_1$.

(ii) $(P, Q, X)_1$ *and* $\mathrm{ran}\,[QX^*(1 - P)] = \mathrm{ran}\,Q$.

(iii) $\mathrm{ran}\,[(1 - Q)X^*P] = \ker Q$ *and* $(1 - P, 1 - Q, X)_1$.

implies that $PX - XQ$ is left invertible.

Necessary and Sufficient Conditions for Invertibility 1

The Condition $(P, Q, X)_2$

$X(1-Q)$ has closed range, the sum $\ker(PX) + \ker Q$ is direct and the sum $\operatorname{ran} P + \ker[(1-Q)X^*]$ is closed.

Theorem

The following conditions are equivalent:

(a) *$PX - XQ$ is invertible;*

(b) *$(P, Q, X)_2$ and $(1-P, 1-Q, X)_2$;*

(c) *$(P, Q, X)_2$ and $\operatorname{ran}[QX^*(1-P)] = \operatorname{ran} Q$;*

(d) *$\operatorname{ran}[(1-Q)X^*P] = \ker Q$ and $(1-P, 1-Q, X)_2$.*

Necessary and Sufficient Conditions for Invertibility 2

Theorem

The following conditions are equivalent:

(a) $PX - XQ$ *is invertible.*

(b) $\operatorname{ran}[PX(1-Q)] = \operatorname{ran} P$, $\overline{\operatorname{ran}[(1-P)XQ]} = \ker P$, $\operatorname{ran}[QX^*(1-P)] = \operatorname{ran} Q$ *and* $\overline{\operatorname{ran}[(1-Q)X^*P]} = \ker Q$.

(c) $\overline{\operatorname{ran}[PX(1-Q)]} = \operatorname{ran} P$, $\operatorname{ran}[(1-P)XQ] = \ker P$, $\overline{\operatorname{ran}[QX^*(1-P)]} = \operatorname{ran} Q$ *and* $\operatorname{ran}[(1-Q)X^*P] = \ker Q$.

A Final Example

If $PX - XQ$ is invertible then the sums $\ker(PX) + \ker Q$, $\ker[(1-P)X] + \operatorname{ran} Q$, $\ker P + \ker(QX^*)$ and $\operatorname{ran} P + \ker[(1-Q)X^*]$ are closed and direct. The converse is, in general, false:

Example

Let T be any operator on a Hilbert space $\mathscr{H}_0$ which is one-to-one, selfadjoint, but not invertible. Let $\mathscr{H} := \mathscr{H}_0 \oplus \mathscr{H}_0$ and X be the operator defined on $\mathscr{H}$ by $X(h_0, h_1) := (Th_1, Th_0),\ (h_0, h_1) \in \mathscr{H}$. If P is the orthogonal projection onto the first component of $\mathscr{H}$ then $PX - XP$ is one-to-one, but not invertible. The sums $\ker(PX) + \ker P$, $\ker[(1-P)X] + \operatorname{ran} P$, $\ker P + \ker(PX^*)$ and $\operatorname{ran} P + \ker[(1-P)X^*]$ reduce to the orthogonal decomposition $\ker P \oplus \operatorname{ran} P = \mathscr{H}$; hence they are closed and direct.