Non-commutative functions in operator-valued non-commutative probability

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Non-commutative probability spaces

$(\mathcal{A}, \varphi)$, where

$\mathcal{A}$= unital, (C*-)($*$-)algebra

φ = normalized, (positive) $\mathbb{C}$-linear functional on $\mathcal{A}$

1. Classical probability spaces $(\Omega, \mathcal{Q}, P)$
   Take $\mathcal{A} = L^\infty(\Omega, P)$ and $\varphi(a) = \int_\Omega a(\omega)dP(\omega), a \in \mathcal{A}$
2. $(M_n(\mathbb{C}), tr)$
3. Random Matrices: $\mathcal{A} = M_n(\mathbb{C}) \otimes L^\infty(\Omega, P) = M_n(L^\infty(\Omega, P)$,
   $\varphi = \int tr(\cdot)dP$
4. G = group
   $\mathcal{A} = \mathbb{C}G$ - group algebra; $\varphi(\sum_{g \in G} \alpha_g \cdot g) = \alpha_e$
5. $\mathcal{H}$=Hilbert space $\xi \in \mathcal{H}$, $||\xi|| = 1$
   $(\mathcal{L}(\mathcal{H}), \langle \cdot \xi, \xi \rangle)$.

Independence relations ($\mathcal{A}_1, \mathcal{A}_2$=subalgebras of $\mathcal{A}$)

- Classical independence ($<$ tensor products of commutative algebras)

  $\mathcal{A}$=commutative, $\varphi(a_1 a_2) = \varphi(a_1)\varphi(a_2)$ for all $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$

- Free independence ($<$ free products of groups, reps)

$$\varphi(a_1 a_2 \cdots a_n) = 0$$

  whenever $a_i \in \mathfrak{A}_{k(i)}$ with
  $\varphi(a_k) = 0$ and $a_k \in \mathcal{A}_{\varepsilon(k)}$ with $k(i) \neq k(i+1)$ and $\varphi(a_k) = 0$.

- Boolean independence ($<$ direct sum of group algebras with identification of 1)

$$\varphi(a_1 \cdots a_n) = \varphi(a_1) \cdots \varphi(a_n)$$

  whenever $a_j \in \mathfrak{A}_{k(j)}$ with $k(i) \neq k(i+1)$.
  $(\mathcal{H}_j, \pi_j, \xi_j) = GNS(\mathcal{A}_j)$, $\quad (\mathcal{H}, \xi) = \bigoplus_j (\mathcal{H}_j, \xi_j)$

$\mathcal{B}$=unital C*-algebra
$\mathcal{B} \subset \mathcal{A}$= inclusion of unital (C)$*$-algebras
$\phi : \mathcal{A} \longrightarrow \mathcal{B}$ positive conditional expectation
-freeness, boolean independence with amalgamation over $\mathcal{B}$.

More general setting: $\mathcal{B} \subset \mathcal{D}$ inclusion of unital C*-algebras
$\theta : \mathcal{A} \longrightarrow \mathcal{D}$ unital $\mathcal{B}$-bimodule map
complete positivity condition:
$\left[\mu(a_j(\mathcal{X})^* a_i(\mathcal{X}))\right]_{i,j=1}^{n} \geq 0$ in $M_n(\mathcal{D})$, $\{a_j\}_{j=1}^{n} \in \mathcal{A}$

-boolean independence, "freeness"

$\mathcal{A} \ni X = X^* \longleftrightarrow \mu_X =$ (compactly supported) measure on $\mathbb{R}$

$$\varphi(X^n) = \int t^n d\mu_X(t)$$

Cauchy transform: $G_{\mu_X}(z) = \int \frac{1}{z-t} d\mu_X(t)$

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Limit theorems:

- Central Limit Theorems:
  - classical independence - Gaussian Distribution,
  - freeness - Wiegner Law (D-V. Voiculescu)
  - boolean independence - Bernoulli law (R. Speicher)
  - monotone independence - Arcsine law (N. Muraki)
- most general limits $<$*infinitesimal arrays* – infinitely divisible distributions

Infinite divisibility wrt classical (additive) convolution: back to Kolmogorov, Hincin, P. Levy
Levy-Hincin formula:
$\mathcal{F}\nu(t) = exp\left[i\gamma t + \int (e^{itx} - 1 - itx)\frac{x^2+1}{x^2} d\sigma(x)\right]$

Infinite divisibility wrt free additive convolution: H. Bercovici, D-V. Voiculescu ('92, '93), H. Bercovici, V. Pata ('95, '99)

Free Levy-Hincin formula:

$\Phi_\nu(z) = \gamma + \int \frac{1+tz}{z-t} d\sigma(t)$ for $\Phi_\nu(z) = \left(\frac{1}{G_\mu(z)}\right)^{\langle -1 \rangle} - z$
or, equivalently
$\frac{1}{z} R_\nu(z) = \alpha + \int_{\mathbb{R}} \frac{z}{1-tz} d\rho(t)$ where $G_{\mu_X} \circ \left[\frac{1}{z} R_X(z) + \frac{1}{z}\right] = z$ 0

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Bercovici - Pata bijection

Op-valued setting:
"Distributions":

- $\Sigma_{\mathcal{B}} = \{\nu : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{B}$: positive conditional expectation$\}$
- $\Sigma_{\mathcal{B}:\mathcal{D}} = \{\mu : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{D}$: unital $\mathcal{B}$-bimodule maps, c.p. condition$\}$.

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$\Sigma^0_{\mathcal{B}}, \Sigma^0_{\mathcal{B}:\mathcal{D}}$ -realizable as op-valued distributions of self-adjoint elements in C*-algebras: there exists some $M > 0$ such that, for all $b_1, \dots, b_n \in M_m(\mathcal{B})$ and $\mathcal{X}_m = \mathrm{Id}_{M_m(\mathbb{C})} \otimes \mathcal{X}$

$$||(\mathrm{Id}_{M_n(\mathbb{C})} \otimes \mu)(\mathcal{X}_m b_1 \mathcal{X}_m b_2 \cdots \mathcal{X}_m b_n \mathcal{X}_m)|| < M^{n+1} ||b_1|| \cdots ||b_n||. \quad (1)$$

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-Central Limit Theorems, but not much literature about infinite divisibility (R. Speicher, Mem. AMS '98)

- No Nevalinna-Pick integral representation result for this setting

Non-commutative functions

- J. L. Taylor (Adv. in Math. '72)
- D-V. Voiculescu (Asterisque '95, also 2008, '09) ($\sim$fully matricial functions)
- V. Vinnikov, D.S. Kaliuzhnyi-Vebovetskyi ('09)

Good analogue of the Cauchy transform, Taylor expansions, differential calculus

$\mathcal{V}$= vector space over $\mathbb{C}$;

- the *non-commutative space over* $\mathcal{V}$: $\mathcal{V}_{\text{nc}} = \coprod_{n=1}^{\infty} M_n(\mathcal{V})$
- *noncommutative sets*: $\Omega \subseteq \mathcal{V}_{\text{nc}}$ such that $X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}$ for all $X \in \Omega_n$, $Y \in \Omega_m$, where $\Omega_n = \Omega \cap \mathcal{V}^{n \times n}$
- *upper admissible sets*: $\Omega \subseteq \mathcal{V}_{\text{nc}}$ such that for all $X \in \Omega_n$, $Y \in \Omega_m$ and all $Z \in \mathcal{V}^{n \times m}$, there exists $\lambda \in \mathbb{C}$, $\lambda \neq 0$, with

$$\begin{bmatrix} X & \lambda Z \\ 0 & Y \end{bmatrix} \in \Omega_{n+m}.$$

Examples of upper-admissible sets:

- $\Omega = \mathrm{Nilp}\,\mathcal{V} =$ the set of nilpotent matrices over $\mathcal{V}$
- If $\mathcal{V}$ is a Banach space and Ω is open in the sense that $\Omega_n \subseteq \mathcal{V}^{n\times n}$ is open for all n, then Ω is upper admissible.
- $\mathcal{B}$ = unital C*-algebra, $\mathcal{A}$=$*$-algebra containing $\mathcal{B}$;
  $X \in \mathcal{A}$, $I_n \otimes X \in M_n(A)$
  $\rho_n(X : \mathcal{B}) = \{\beta \in M_n(\mathcal{B}) : i_n \otimes X - \beta$ invertible$\}$

  $\rho(X; \mathcal{B}) = \coprod_{n=1}^{\infty} \rho_n(X : \mathcal{B})$ is upper admissible
- Noncommutative half-planes $\mathbb{H}^+(\mathcal{A}_{\mathrm{nc}}) = \{a \in \mathcal{A}_{\mathrm{nc}} : \Im a > 0\}$ over $\mathcal{A}$.

$\Omega \subseteq \mathcal{V}_{\mathrm{nc}}$ = non-commutative (upper admissible) set

noncommutative function: $f : \Omega \to \mathcal{W}_{\mathrm{nc}}$ with

- $f(\Omega_n) \subseteq M_n(\mathcal{W})$
- f respects direct sums: $f(X \oplus Y) = f(X) \oplus f(Y)$ for all $X \in \Omega_n$, $Y \in \Omega_m$.
- f respects similarities: $f(TXT^{-1}) = Tf(X)T^{-1}$ for all $X \in \Omega_n$ and $T \in \mathrm{GL}_n(\mathbb{C})$ such that $TXT^{-1} \in \Omega_n$.

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It turns out that these two conditions are equivalent to a single one: f respects intertwinings, namely if $XS = SY$ then $f(X)S = Sf(Y)$, where $X \in \Omega_n$, $Y \in \Omega_m$, and $S \in M_{n\times m}\mathbb{C}$.

Examples:

- *the full $\mathcal{B}$-resolvent* of X: on each $\rho(I_n \otimes X; M_n(\mathcal{B}))$ consider the analytic function $\beta \mapsto (I_n \otimes X - \beta)^{-1}$
- *the generalized Cauchy transform* of X: $\mathcal{G}_X = (G_X^{(n)})_n$, where

$$\mathbb{H}^+(M_n(\mathcal{B})) \ni b \mapsto G_X^{(n)}(b) = \phi_n[(b - X \otimes \mathrm{Id}_n)^{-1}] \in \mathbb{H}^-(M_n(\mathcal{D}))$$

  for $\phi : \mathcal{A} \longrightarrow \mathcal{D}$ cp map
- $\mathcal{V}$ and $\mathcal{W}$ are operator spaces

  $f_k : \mathcal{V}^{\otimes k} \to \mathcal{W}$ = linear mappings such that $\limsup_{k\to\infty} \sqrt[k]{\|f_k\|_{\mathrm{cb}}} \leq \frac{1}{\rho}$

$$f(X) = \sum_{k=0}^{\infty} f_k(X^k) \text{ is a non-commutative function.}$$

- $\phi : \mathcal{A} \longrightarrow \mathcal{D}$ unital cp $\mathcal{B}$-bimodule map, $X = X^* \in \mathcal{A}$

  $\tilde{\phi}((1 - Xb)^{-1}) = M_X = (M_{n,X})_n$ is a non-commutative function, where

$$M_{n,X}(b) = \sum_{k=0}^{\infty} (I_n \otimes \phi)([(I_n \otimes X)b]^k), \;\; b \in M_n(\mathcal{B})$$

The Taylor-Taylor expansion:
If f is a non-commutative function, then
$f(X) = \sum_{k=0}^{\infty} \Delta_R^k f(\underbrace{0, \ldots, 0}_{k+1})(X^k)$

where the multilinear forms $\Delta_R^k f(\underbrace{0, \ldots, 0}_{k+1}) \colon \mathcal{V}^k \longrightarrow \mathcal{W}$ are the values at $(0, \ldots, 0)$ of the kth order noncommutative difference-differential operators applied to f; they can be calculated directly by evaluating f on block upper-triangular matrices!

$$
f\left(\begin{bmatrix} 0 & Z_1 & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 0 & Z_k \\ 0 & \cdots & \cdots & 0 & 0 \end{bmatrix}\right)
$$

$$
= \begin{bmatrix} f(0) & \Delta_R f(0,0)(Z_1) & \cdots & \cdots & \Delta_R^k f(0,\dots,0)(Z_1,\dots,Z_k) \\ 0 & f(0) & \ddots & & \Delta_R^{k-1} f(0,\dots,0)(Z_2,\dots,Z_k) \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & f(0) & \Delta_R f(0,0)(Z_k) \\ 0 & \cdots & \cdots & 0 & f(0) \end{bmatrix}.
$$

Consequence: M_X **indeed** encodes all the moments of X, since the previous results implies that for

$$b = \begin{bmatrix} 0 & b_1 & 0 & \dots & 0 \\ 0 & 0 & b_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & b_N \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \in M_{n+1}(\mathcal{B})$$

we have that

$$(I_{n+1} \otimes \phi)\big([(I_{n+1} \otimes X)b]^n\big) = \begin{bmatrix} 0 & \dots & 0 & \phi(Xb_1 c \dots Xb_n) \\ 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 \end{bmatrix}.$$

At the moment we do not have a analogue of the Nevalinna-Pick integral representation result for the non-commutative functions - In some situations, the results holds true for sure.

For infinitely divisibility results one can still use Hilbert bimodules techniques and combinatorial methods.

Idea: infinite divisibility is equivalent to a construction giving the moments of the linearizing transform for the additive convolution in as the moments of a sum of creation, annihilation and preservation operators

Non-commutative R-transform

$1 \in \mathcal{B} \in \mathcal{A}$, $\mathcal{B}$=C*-algebra, $\mathcal{A}$=$*$-algebra; $\phi : \mathcal{A} \longrightarrow \mathcal{B}$ positive conditional expectation

$\mathcal{A} \in X^* = X \longleftrightarrow \mu_x \in \Sigma_{\mathcal{B}}$,

$$\mu_X(\mathcal{X} b_1 \mathcal{X} b_2 \cdots b_{n-1} \mathcal{X}) = \phi(X b_1 X b_2 \cdots b_{n-1} X)$$

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The non-commutative R-transform of $X \in \mathcal{A}$ (or of $\mu_X \in \Sigma_X$) is given via

$$M_\nu(b) - 1 = R_\nu\left(b M_\nu(b)\right)$$

The equation is always meaningful for $b \in \mathrm{Nilp}(\mathcal{B})$; if $\mu_X \in \Sigma_{\mathcal{B}}^0$, then also for b in a non-commutative ball around 0.
Prop:

1. R_X is a non-commutative function
2. If $X, Y \in \mathcal{A}$ are free w. r. t. ϕ, then $R_{X+Y} = R_X + R_Y$.

Theorem (M.P., '10) $\mu \in \Sigma_{\mathcal{B}}$ is free infinitly divisible if and only if there exist some selfadjoint $\alpha \in \mathcal{B}$ and some completely positive ($\mathbb{C}$-)linear map $\nu : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{B}$ such that

$$R_{\mu}(b) = \left[\alpha \cdot \mathrm{Id} + \widetilde{\nu}\left(b(1 - \mathcal{X}b)^{-1}\right)\right] b.$$

s cf i

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Conditional freeness:

$\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ subalgebras containing $\mathcal{B}$.
$\Phi : \mathcal{A} \longrightarrow \mathcal{D}$ $\mathcal{B}$-bimodule map, $\psi : \mathcal{A} \longrightarrow \mathcal{B}$ conditional expectation
$\mathcal{A}_1, \mathcal{A}_2$ are c-free if:

(1) $\mathcal{A}_1, \mathcal{A}_2$ are free w.r.t. ψ
(2) $\Phi(a_1 \cdots a_n) = \Phi(a_1) \cdots \Phi(a_n)$ whenever $\psi(a_j) = 0$ and $a_j \in \mathcal{A}_{\epsilon(j)}$ with $\epsilon(j) \neq \epsilon(j+1)$

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- F. Boca (JFA, '91) - "amalgamated free product of cp maps"
- R. Speicher, M. Bozejko (Pac. J. Math, '96) - scalar case, "c-freeness", ${}^cR$-transform, limit laws
- K. Dykema, E. Blanchard (Pac. J. Math., '01) - reduced free products and embeddings of free products of von Neumann algebras
- M. P., J-C Wang ('08, to appear in Trans. AMS) - multiplicative properties, c-free S-transform
- M. P. (Comm. Stoch. Anal. '08) - op-valued ${}^cR$-transform

$\Sigma_{\mathcal{B}:\mathcal{D}} = \{\mu : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{D}$ $\mathcal{B}$-bimodule maps satisfying (2)$\}$

$$\left[\mu(f_j^* f_i)\right]_{i,j=1}^{N} \in M_N(\mathcal{D})_+ \text{ for all } N \in \mathbb{N}, f_1, \ldots, f_N \in \mathcal{B}\langle\mathcal{X}\rangle \qquad (2)$$

$\mathcal{A} \ni X = X^* \leftrightarrow (\mu_X, \nu_X) \in \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$

$\mathcal{A}$=C*-algebra, $(\mu_X, \nu_X) \in \Sigma^0_{\mathcal{B}:\mathcal{D}} \times \Sigma^0_{\mathcal{B}}$

where $\mu \in \Sigma^0_{\mathcal{B}:\mathcal{D}}(\Sigma^0_{\mathcal{B}})$ is there exists some $M > 0$ such that, for all $b_1, \ldots, b_n \in M_m(\mathcal{B})$ and $\mathcal{X}_m = \mathrm{Id}_{M_m(\mathbb{C})} \otimes \mathcal{X}$

$$||(\mathrm{Id}_{M_n(\mathbb{C})} \otimes \mu)(\mathcal{X}_m b_1 \mathcal{X}_m b_2 \cdots \mathcal{X}_m b_n \mathcal{X}_m)|| < M^{n+1} ||b_1|| \cdots ||b_n||. \qquad (3)$$

c-freeness induces

$\boxed{\mathrm{c}} : (\Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}) \times (\Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}) \longrightarrow \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$

Op-valued Boolean independence:$\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{A}$ are Boolean independent if

$$\theta(a_1 a_2 \cdots a_n) = \theta(a_1) \cdots \theta(a_n) \text{ whenever } a_i \in \mathcal{A}_{\epsilon(i)} \text{ with } \epsilon(i) \neq \epsilon(i+1).$$

$\uplus : \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}:\mathcal{D}} \longrightarrow \Sigma_{\mathcal{B}:\mathcal{D}}$ is linearized on $\Sigma^0_{\mathcal{B}:\mathcal{D}}$ by $\mu \to b - \mathcal{F}_\mu(b)$,

$$\mathcal{F}_\mu(b) = (\mathcal{G}_\mu(b))^{-1}$$

Theorem(M.P.'10):

1. Any $\mu \in \Sigma_{\mathcal{B}:\mathcal{D}}$ is infinitely divisible with respect to boolean convolution.
2. There exists a selfadjoint $\alpha \in \mathcal{D}$ and a $\mathbb{C}$-linear map $\sigma : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{D}$, satisfying the cp condition such that

$$\mathcal{F}(b) = \alpha \cdot \mathrm{Id} + \widetilde{\sigma}\left([b - \mathcal{X}]^{-1}\right) \tag{4}$$

- on $\Sigma^0_{\mathcal{B}:\mathcal{D}}$, $\uplus$ is linearized also by $b - \mathcal{F}_\mu(b)$, where

$$\begin{aligned}\mathcal{F}_\mu(b) &= (\mathcal{G}_\mu(b))^{-1} \\ &= B_\mu(b^{-1})b + b\end{aligned}$$

henceforth

$$\mathcal{F}(b) = \alpha \cdot \mathrm{Id} + \widetilde{\sigma}\left((b - \mathcal{X})^{-1}\right)$$

for some selfadjoint $\alpha \in \mathcal{D}$ and some $\mathbb{C}$-linear map $\sigma : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{D}$

- Non-commutative ${}^cR$-transform for $(\mu, \nu) \in \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$

$$(M_\mu(b) - 1) \cdot {}^cR_{\mu,\nu}(bM_\nu(b)) = (M_\mu(b) - 1) \cdot M_\mu(b)$$

  - ${}^cR$ is a non-commutative function
  - if $X, Y \in \mathcal{A}$ are c-free w.r.t. (Φ, ψ), then ${}^cR_{X+Y} = {}^cR_X + {}^cR_Y$

- **Theorem**(M.P., V. Vinnikov '10) A pair $(\mu, \nu) \in \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$ is c-free infinitely divisible iff ν is free infinitely divisible and there exist some selfadjoint $\alpha \in \mathcal{B}$ and some completely positive ($\mathbb{C}$-)linear map $\nu : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{D}$ such that

$${}^cR_{\mu,\nu}(b) = \left[\alpha \cdot \mathrm{Id} + \widetilde{\nu}\left(b(1 - Xb)^{-1}\right)\right] b.$$

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The Non-commutative Boolean-to-Free Bercovici-Pata bijection:

$\mathcal{BP} : \Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}} \to \boxed{\text{c}}$-infinitely divisible elements of $\Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$

$$\begin{tikzcd}
(\mu,\nu) \arrow[rr, "\mathcal{BP}"] \arrow[d] & & (\mu',\nu') \\
(B_\mu, B_\nu) \arrow[dr] & & ({}^cR_{\mu',\nu'}, R_{\nu'}) \arrow[u] \\
& (\alpha_1,\sigma_1,\alpha_2,\sigma_2) \arrow[ur] &
\end{tikzcd}$$

Theorem (Ş. Belinschi, M.P.)
Let $(\mu_n, \nu_n)_n$ be an infinitesimal sequence from $\Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}$ and $\{k_n\}_n$ be an increasing sequence of positive integers. The following properties are equivalent:

(1) $(\mu_n, \nu_n)^{\uplus k_n}$ norm-converges in moments to some $(\mu, \nu) \in \Sigma^0_{\mathcal{B}:\mathcal{D}} \times \Sigma^0_{\mathcal{B}}$

(2) $(\mu_n, \nu_n)^{\boxed{\mathrm{c}}\, k_n}$ norm-converges in moments in $\Sigma^0_{\mathcal{B}:\mathcal{D}} \times \Sigma^0_{\mathcal{B}}$

(3) $(\mu_n, \nu_n)^{\boxed{\mathrm{c}}\, k_n}$ norm-converges in moments to $\mathcal{BP}(\mu, \nu)$ for some $(\mu, \nu) \in \Sigma^0_{\mathcal{B}:\mathcal{D}} \times \Sigma^0_{\mathcal{B}}$

(4) There exist two pairs (γ, σ) and (γ_0, σ_0) determining B_μ and B_ν such that

$$\begin{aligned}
\lim_{n \longrightarrow \infty} (\mu_n)(\mathcal{X}) &= \gamma \\
\lim_{n \longrightarrow \infty} (\nu_n)(\mathcal{X}) &= \gamma_0 \\
\lim_{n \longrightarrow \infty} k_n \cdot (\mu_n \otimes \mathrm{Id}_{M_N(\mathbb{C})})((\mathcal{X} b)^m \mathcal{X}) &= \sigma \otimes \mathrm{Id}_{M_N(\mathbb{C})}(b(\mathcal{X} b)^{m-1}) \\
\lim_{n \longrightarrow \infty} k_n \cdot (\nu_n \otimes \mathrm{Id}_{M_N(\mathbb{C})})((\mathcal{X} b)^m \mathcal{X}) &= \sigma_0 \otimes \mathrm{Id}_{M_N(\mathbb{C})}(b(\mathcal{X} b)^{m-1})
\end{aligned}$$

ht

Proposition (Ş. Belinschi, M.P.)
For any compactly supported distributions $(\mu_1, \nu_1), (\mu_2, \nu_2)$, we have

$$\mathcal{BP}((\mu_1, \nu_1) \boxtimes_c (\mu_2, \nu_2)) = \mathcal{BP}(\mu_1, \nu_1) \boxtimes_c \mathcal{BP}(\mu_2, \nu_2).$$

The c-free convolution of positive probability measures on the positive half-line is not necessarily well-defined, while the Boolean - or c-free - Bercovici-Pata bijection is not well defined for measures on the unit circle, as sums of unitaries are not unitaries. Thus, the above result must be viewed in terms of algebraic distributions.

Idea: μ determines $\rho_\mu : \mathcal{B}\langle\mathcal{X}\rangle \longrightarrow \mathcal{B}$ via the moment-free cumulant recurrence relations. On $\mathcal{B}\langle\mathcal{X}\rangle_0 = \mathcal{B}\langle\mathcal{X}\rangle \setminus \mathcal{B}$ consider the positive pairing $\langle f(\mathcal{X}), g(\mathcal{X})\rangle = \rho_\mu(g(\mathcal{X})^* f(\mathcal{X}))$ and the self-adjoint operator $T : f(\mathcal{X}) \mapsto \mathcal{X} f(\mathcal{X})$.
Then consider the self-adjoint operator V on the full Fock $\mathcal{B}$-bimodule over $\mathcal{B}\langle\mathcal{X}\rangle_0$, which $\mathcal{B}$-valued distribution with respect to the ground $\mathcal{B}$-state coincides with μ, given by

$$V = a_X + a_X^* + \widetilde{T} + \mu(X)\mathrm{Id}.$$

The terms from the Taylor-Taylor development of $\widetilde{R}_\mu$ are the moments of $\widetilde{T}$ with respect to the mapping $\langle \cdot\mathcal{X}, \mathcal{X}\rangle$ and the conclusion follows using the additivity property of $\widetilde{R}$ and some cb-norm inequalities. r

Theorem (Ş. Belinschi, M. P.)
Assume that $\{X_{jk}\}_{j\in\mathbb{N};1\leq k\leq k_j}$ is a triangular array of random variables in $(\mathcal{A}, E_{\mathcal{B}}, \mathcal{B}, \theta, \mathcal{D})$ of elements free (c-free, boolean independent) so that $\{X_{jk} \colon 1 \leq k \leq k_j\}$ have the same distribution with respect to E_B (θ, E_B, θ) for each $j \in \mathbb{N}$ (i.e. rows are identically distributed).
Assume in addition that

$$\limsup_{j\to\infty} \|X_{j1} + \cdots + X_{jk_j}\| \leq M$$

for some $M \geq 0$. If $\lim_{j\to\infty} X_{j1} + X_{j2} + \cdots + X_{jk_j}$ exists in distribution as norm-limit of moments in $\Sigma_{\mathcal{B}}$ $(\Sigma_{\mathcal{B}:\mathcal{D}} \times \Sigma_{\mathcal{B}}, \Sigma_{\mathcal{B}:\mathcal{D}})$, then the limit distribution is free (c-free, boolean) infinitely divisible. bp