Spectra of C* algebras, $\mathbb{R}$-actions on $\mathcal{O}_2$, and Extension

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TOC

Conventions and Notations

- Spaces $P, X, Y, \cdots$ are (at least) T_0 and are second countable,
  algebras $A, B, \ldots$ are separable, ...

- ... except corona spaces $\beta(P) \setminus P$, multiplier algebras $\mathcal{M}(B)$, and ideals of corona algebras $\mathrm{Q}(B) := \mathcal{M}(B)/B$, ...

- We use the natural isomorphisms

  $$\mathcal{I}(A) \cong \mathbb{O}(\mathrm{Prim}(A)) \cong \mathcal{F}(\mathrm{Prim}(A))^{op}.$$

- $\mathbb{Q} := [0, 1]^\infty$ denotes the Hilbert cube (with its coordinate-wise order).

- A T_0 space X is called "**sober**" (or "point-complete") if each prime closed subset F of X is a the closure $\overline{\{x\}} = F$ of a singleton $\{x\}$.

Spectra of amenable algebras (1)

Characterization of $\mathrm{Prim}(A)$ for amenable A (H.Harnisch, E.K., M.Rørdam):

Theorem 1. *A sober space X is homeomorphic to a primitive ideal space of an amenable C^*-algebra A, if and only if,*
there is a Polish l.c. space P and a continuous map $\pi\colon P \to X$ such that
*$\pi^{-1}\colon \mathbb{O}(X) \to \mathbb{O}(P)$ is injective (=: π is **pseudo-epimorphic**),*
and
$(\bigcap_n \pi^{-1}(U_n))^\circ = \pi^{-1}((\bigcap_n U_n)^\circ)$ for each sequence $U_1, U_2, \ldots \in \mathbb{O}(X)$ (=: π is pseudo-open).

The algebra $A \otimes \mathcal{O}_2 \otimes \mathbb{K}$ is uniquely determined by X up to (unitarily homotopic) isomorphisms.

Spectra of amenable algebras (2)

Notice: A continuous epimorphism $\pi\colon P \to X$ is not necessarily *pseudo-open*.

We call a map $\Psi\colon \mathbb{O}(X) \to \mathbb{O}(Y)$ "*lower semi-continuous*" if $(\bigcap_n \Psi(U_n))^\circ = \Psi((\bigcap_n U_n)^\circ)$ for each sequence $U_1, U_2, \ldots \in \mathbb{O}(X)$.

(Thus, π is pseudo-open, if and only if, $\Psi := \pi^{-1}$ is lower semi-continuous.)

If one works with *closed sets*, then one has to replace intersections by unions and interiors by closures.

Exotic G-actions (1)

Theorem 2. [N.Ch. Phillips, E.K.] *Suppose that a locally compact amenable group* G *acts topologically free and minimal on* $\mathrm{Prim}(A)$ *of some amenable* C^**-algebra* A*, by* $\alpha\colon G \to \mathrm{Homeo}(\mathrm{Prim}(A))$*.*
Then there exists a continuous group-action $\beta\colon G \to \mathrm{Aut}(B)$ *on the* C^**-algebra* $B := A \otimes \mathcal{O}_2 \otimes \mathbb{K}$ *that implements* α*, and has crossed product* $B \rtimes_\beta G \cong \mathcal{O}_2 \otimes \mathbb{K}$*.*

Definition 3. [N.Ch. Phillips compactification]
Let $\Xi(P)$ *denote the prime* T_0 *space* $P \cup \{\infty\}$ *with topology given by the system of open subsets*

$$\mathbb{O}(\Xi(P)) = \{\emptyset, \Xi(P) \setminus C\,;\ C \subset P,\ \textit{compact in}\, P\,\}\,.$$

Exotic G-actions (2)

Theorem 4. *There exists an amenable C^*-algebra A with $\mathrm{Prim}(A) \cong \Xi(P)$.*

If we apply the above theorems to $\Xi(G)$, we get:

Corollary 5. *Every non-compact amenable l.c. group G has a* co-action *$\widehat{\beta}$ on $\mathcal{O}_2 \otimes \mathbb{K}$ such that $B := (\mathcal{O}_2 \otimes \mathbb{K}) \rtimes \widehat{G}$ is prime and the (dual) action β of G on B is minimal and toplogically free.*

If $G := \mathbb{R} = \widehat{G}$, there is also an action of $\mathbb{R} = \widehat{\mathbb{R}}$ on $\mathcal{O}_2$ itself with this property.

NC-Selection and Extensions (1)

Proposition 6. *If B is stable and $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ is a lower semi-continuous action of $\mathrm{Prim}(B)$ on A, then there exists a* lower s.c. *action $\mathcal{M}(\Psi)\colon \mathcal{I}(\mathcal{M}(B)) \to \mathcal{I}(A)$ of $\mathrm{Prim}(\mathcal{M}(B))$ on A, that has the following properties (i)–(iii):*

*(i) $\mathcal{M}(\Psi)$ is **monotone** upper semi-continuous. (:= $\sup$'s of* upward directed *families of ideals will be respected).*

(ii) $\mathcal{M}(\Psi)(J_1) = \mathcal{M}(\Psi)(J_1)$ if $J_1 \cap \delta_\infty(\mathcal{M}(B)) = J_2 \cap \delta_\infty(\mathcal{M}(B))$.

(iii) $\mathcal{M}(\Psi)(\mathcal{M}(B, I)) = \Psi(I)$ for all $I \in \mathcal{I}(B)$.

The "extension" $\mathcal{M}(\Psi)$ of Ψ with (i)–(iii) is unique.

For strongly p.i. (not necessarily separable) B and exact A, there is a nuclear *-morphism $h\colon A \to B$

with $\Psi(J) = h^{-1}(h(A) \cap J)$, if and only if, Ψ is lower s.c. and monotone upper s.c. It yields the following theorem.

NC-Selection and Extensions (2)

Theorem 7. [NC-selection] *Suppose that B is stable, $A \otimes \mathcal{O}_2$ contains a regular* exact *C*-algebra $C \subset A \otimes \mathcal{O}_2$, and that $\Psi \colon \mathcal{I}(B) \to \mathcal{I}(A)$ is a lower s.c. action of $\mathrm{Prim}(B)$ on A.*

*Then there is a *-morphism $h \colon A \to \mathcal{M}(B)$ such that $\delta_\infty \circ h$ is unitarily equivalent to h, $\Psi(J) = h^{-1}(h(A) \cap \mathcal{M}(B, J))$ and that*

$$[h]_J \colon A/\Psi(J) \to \mathcal{M}(B/J) \cong \mathcal{M}(B)/\mathcal{M}(B, J)$$

is weakly nuclear for all $J \in \mathcal{I}(B)$.

Here, a subalgebra $C \subset D$ is **regular** if C separates the ideals of D and $C \cap (I + J) = (C \cap I) + (C \cap J)$ for all $I, J \in \mathcal{I}(D)$.

Theorem 7 applies to necessary and sufficient criteria for (ideal-system-) equivariant semi-splitness of extensions.

NC-Selection and Extensions (3)

Let $\epsilon \colon B \to E$ a *-monomorphism onto a closed ideal of E and $\pi \colon E \to A$ an epimorphism such that $\epsilon(B)$ is the kernel of π. We denote by $\gamma \colon A \to \mathrm{Q}(B) = \mathcal{M}(B)/B$ the Busby invariant of the extension

$$0 \to B \xrightarrow{\epsilon} E \xrightarrow{\pi} A \to 0\,.$$

Now we consider general "actions" $\psi_B \colon S \to \mathcal{I}(B)$, $\psi_E \colon S \to \mathcal{I}(E)$, and $\psi_A \colon S \to \mathcal{I}(A)$, of a set S on B, E and A. We require that the extension E is ψ-equivariant:

(a) $\epsilon(\psi_B(s)) = \epsilon(B) \cap \psi_E(s) = \epsilon(B)\psi_E(s)$, and

(b) $\psi_A(s) = \pi(\psi_E(s))$ for all $s \in S$.

i.e., $0 \to \psi_B(s) \to \psi_E(s) \to \psi_A(s) \to 0$ is exact for each $s \in S$.

NC-Selection and Extensions (4)

An action $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ of $\mathrm{Prim}(A)$ on B is **upper semi-continuous** if Ψ preserves sup of families in $\mathcal{I}(A)$, i.e., $\Psi(I + J) = \Psi(I) + \Psi(J)$ and Ψ is monotone upper semi-continuous.

Lemma 8. *The is a unique maximal upper semi-continuous map $\Phi\colon \mathcal{I}(A) \to \mathcal{I}(B)$ with the property that $\Phi(\psi_A(s)) \subset \psi_B(s)$ for all $s \in S$.*

Every upper semi-continuous action Φ has a lower semi-continuous adjoint map $\Psi\colon \mathcal{I}(B) \to \mathcal{I}(A)$ such that (Ψ, Φ) build a Galois connection, i.e., $\Psi(J) \supset I$ iff $J \supset \Phi(I)$. The rule is: The upper adjoint is lower semi-continuous (preserves inf).

Applications of Theorem 7 to the adjoint Ψ of Φ in Lemma 8 implies the following necessary and sufficient criterion (ii):

NC-Selection and Extensions (5)

Theorem 9. *Let B, E, A, ϵ, π, γ, $\psi_Y : S \to \mathcal{I}(Y)$ (for $Y \in \{B, E, A\}$) be as above, and let $\Phi : \mathcal{I}(A) \to \mathcal{I}(B)$ the map given in Lemma 8.*

Suppose, in addition, that A is exact *and that B is weakly injective (has the* WEP *of Lance).*

Then the following properties (i) and (ii) of the extension are equivalent:

(i) The extension has an S-equivariant c.p. splitting map, i.e., there is a c.p. map $V : A \to E$ with $\pi \circ V = \mathrm{id}_A$ and $V(\psi_A(s)) \subset \psi_E(s)$ for all $s \in S$.

(ii) The Busby invariant $\gamma : A \to \mathrm{Q}(B)$ is nuclear, and,

$$\pi_B(\mathcal{M}(B, \Phi(J))) \supset \gamma(J) \qquad \forall\ J \in \mathcal{I}(A)$$

Coherent l.q-compact spaces (1)

A subset C of X is **saturated** if $C = \mathrm{Sat}(C)$, where $\mathrm{Sat}(C)$ means the intersection of all $U \in \mathbb{O}(X)$ with $U \supset C$.

Definition 10. *A sober T_0 space X is called "**coherent**" if the intersection $C_1 \cap C_2$ of two* saturated *quasi-compact subsets $C_1, C_2 \subset X$ is again quasi-compact.*

Now we give some partial results concerning the open **Question:**
Is every (second-countable) *coherent* locally quasi-compact sober T_0 space X homeomorphic to the primitive ideal spaces $\mathrm{Prim}(A)$ of some *amenable* C^*-algebra A?

Coherent l.q-compact spaces (2)

Let X a locally quasi-compact sober T_0 space, and $\mathcal{F}(X)$ the lattice of closed subsets $F \subset X$.

Definition 11. *The topological space $\mathcal{F}(X)_{\mathrm{lsc}}$ is the set $\mathcal{F}(X)$ with the T_0 **order topology** that is* generated *by the complements*

$$\mathcal{F}(X) \setminus [\emptyset, F] = \{G \in \mathcal{F}(X)\,;\; G \cap U \neq \emptyset\} =: \mu_U$$

of the intervals $[\emptyset, F]$ for all $F \in \mathcal{F}(X)$ (where $U = X \setminus F$).

*The **Fell-Vietoris topology** on $\mathcal{F}(X)$ is the topology, that is* generated *by the sets μ_U ($U \in \mathbb{O}(X)$) and the sets $\mu_C := \{G \in \mathcal{F}(X)\,;\; G \cap C = \emptyset\}$ for all* quasi-compact *$C \subset X$. (The induced topology on $\mathbb{O}(X) \cong \mathcal{F}(X)^{op}$ is called **Larson** topology.)*

Coherent l.q-compact spaces (3)

The space $\mathcal{F}(X)_{\mathrm{lsc}}$ is a *coherent* second countable locally quasi-compact sober T_0 space, and the space $\mathcal{F}(X)_H$ (with Fell-Vietoris topology of Def. 11) is a *compact Polish* space.

Definition 12. *A map $f\colon X \to [0,\infty)$ is a* ***Dini function*** *if it is lower semi-continuous and $\sup f(\bigcap_n F_n) = \inf_n\{\sup f(F_n)\}$ for every decreasing sequence $F_1 \supset F_2 \supset \cdots$ of closed subsets of X.*

There are several other definitions — e.g. by the generalized Dini Lemma — that are equivalent for all sober spaces.

For sober spaces X one has also that a function $f\colon X \to [0,1]$ is Dini, if and only if, $f\colon X \to [0,1]_{\mathrm{lsc}}$ is continuous and the restriction $f\colon X \setminus f^{-1}(0) \to (0,1]_{\mathrm{lsc}}$ is **proper**.

Coherent l.q-compact spaces (4)

The ordered Hilbert cube $\mathbb{Q}$ is nothing else $\mathcal{F}(Y)$ for $Y := X_0 \uplus X_0 \uplus \cdots$ where $X_0 := (0, 1]_{\mathrm{lsc}}$. The Fell-Vietoris topology is just the ordinary Hausdorff topology on $\mathbb{Q}$.

If X is locally quasi-compact sober T_0 space, then a dense sequence $g_1, g_2, \ldots$ in the Dini functions g on X with $\sup g(X) = 1$ defines an order isomorphism $\iota \colon \mathcal{F} \to \mathbb{Q}$ onto a $\max$-closed subset $\iota(\mathcal{F})$ of $\mathbb{Q}$ with $\iota(\emptyset) = 0$, $\iota(X) = 1$ (Construction of J.M.G. Fell in case of C*–algebras):

$$\iota(F) := (\sup g_1(F), \sup g_2(F), \ldots) \in \mathbb{Q}\,.$$

The image $\iota(\mathcal{F}(X))$ is closed in $\mathbb{Q}$ (with Hausdorff topology) and ι defines an isomorphism from $\mathcal{F}(X)$ onto $\iota(\mathcal{F}(X))$ with respect to both topologies on $\mathcal{F}(X)$ and $\mathbb{Q}$.

Coherent l.q-compact spaces (5)

In this way, $X \cong \eta(X) \subset \overline{\eta(X)}^{H} \setminus \{0\} \subset \mathcal{F}(X) \subset \mathbb{Q}$, considered as Polish spaces, with $X \ni x \mapsto \eta(x) := \overline{\{\, x \,\}} \in \mathcal{F}(X)$.

Theorem 13. *Let X a second countable locally (quasi-)compact sober T_0 space. Following properties (i)-(v) of X are equivalent:*

*(i) X is **coherent**.*

*(ii) The image $\eta(X) \cong X$ in $\mathcal{F}(X) \setminus \{\emptyset\}$ is **closed** in $\mathcal{F}(X) \setminus \{\emptyset\}$ with respect to the Fell-Vietoris topology on $\mathcal{F}(X)$.*

*(iii) The set $\mathcal{D}(X)$ of Dini functions on X is **convex**.*

*(iv) $\mathcal{D}(X)$ is $\min$-**closed**.*

*(v) $\mathcal{D}(X)$ is **multiplicatively closed**.*

Coherent l.q-compact spaces (6)

Lemma 14. *(I) Each closed subset $F \subset \mathbb{Q}_H$ is a coherent sober subspace F_{lsc} of $\mathbb{Q}_{\mathrm{lsc}}$, and is the intersection of an decreasing sequence F_k of closed subspaces of $\mathbb{Q}_H$ that are continuously order-isomorphic to spaces $G_k \times \mathbb{Q}$ with $G_k \subset [0,1]^{n_k}$ a finite union of n_k-dimensional (small) cubes.*

(II) If $F = \bigcap_k F_k$ for a sequence $F_1 \supset F_2 \supset \cdots$ of closed subsets in $\mathcal{F}(\mathbb{Q}_H)$, and if each $(F_k)_{\mathrm{lsc}}$ is the primitive ideal space of an amenable C^*-algebra, then F_{lsc} is the primitive ideal space of an amenable C^*-algebra.

Coherent l.q-compact spaces (7)

Corollary 15. *If there is a coherent sober l.c. space X that is not homeomorphic to the primitive ideal space of an amenable C^*-algebra, then there is $n \in \mathbb{N}$ and a finite union Y of (Hausdorff-closed) cubes in $[0,1]^n$ such that Y with induced order-topology is not the primitive ideal space of any amenable C^*-algebra.*

Theorem 16. [O.B. Ioffe, E.K.] *If $G \subset [0,1]^n$ is a finite union of cubes, then the space G_{lsc} has a finite decomposition series $U_1 \subset U_2 \subset \cdots \subset U_k$, by open subsets $U_\ell \subset G_{\mathrm{lsc}}$ such that $U_{\ell+1} \setminus U_\ell$ is the primitive ideal space of an amenable C^*-algebra.*

It leads us to the following conjecture (open question).

Coherent l.q-compact spaces (8)

Conjecture 17. *Suppose that X is a locally quasi-compact sober space, and $U \subset X$ is open.*

X is homeomorphic to the primitive ideal space of an amenable C-algebra if U and $X \setminus U$ are homeomorphic to primitive ideal spaces of amenable C*-algebras.*

A proof of this Conjecture would imply that *sober l.q-c. spaces are primitive ideal spaces of amenable C*-algebras — if they have decomposition series by open subsets $\{U_\alpha\}$ with coherent spaces $U_{\alpha+1} \setminus U_\alpha$.*