PROPERTIES OF THE DRAZIN SPECTRA FOR BANACH SPACE OPERATORS AND BANACH ALGEBRA ELEMENTS

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Abstract. Given a Banach Algebra A and $a \in A$, several relationships among the Drazin spectrum of a and the ascent, the descent and the Drazin spectra of the multiplication operators L_a and R_a will be presented; the Banach space operator case will be also examined. In addition, a characterization of the spectrum of a in terms of the Drazin spectrum and the poles of the resolvent of a will be considered. Furthermore, several basic properties of the Drazin spectrum in Banach algebras will be studied.

1. Introduction

The main objective of the present talk is to present several results concerning basic properties of the Drazin spectra of Banach algebra elements and Banach space operators studied in [3, 4]. The ascent and the descent spectra will be also considered.

2. The Drazin spectrum

From now on, X will denote a Banach space and $L(X)$ the Banach algebra of all operators defined on and with values in X. In addition, if $T \in L(X)$, then $N(T)$ and $R(T)$ will stand for the null space and the range of T respectively.

Recall that the *descent* and the *ascent* of $T \in L(X)$ are $d(T) = \inf\{n \geq 0\colon R(T^n) = R(T^{n+1})\}$ and $a(T) = \inf\{n \geq 0\colon N(T^n) = N(T^{n+1})\}$ respectively, where if some of the above sets is empty, its infimum is then defined as ∞.

On the other hand, A will denote a unital Banach algebra and e will stand for the unit element of A. If $a \in A$, then $L_a\colon A \to A$ and $R_a\colon A \to A$ will denote the maps defined by left and right multiplication respectively, namely $L_a(x) = ax$ and $R_a(x) = xa$, where $x \in A$.

Next follow the key notions of the present talk, see [6]. Given a Banach algebra A, an element $a \in A$ is said to be *Drazin invertible*, if there exists a necessarily unique $b \in A$ and some $m \in \mathbb{N}$ such that

$$a^m ba = a^m, \qquad bab = b, \qquad ab = ba.$$

If the Drazin inverse of a exists, then it will be denoted by a^D. In addition, the *index* of a, which will be denoted by *index*(a), is the least non-negative integer m for which the above equations hold. Let $\mathcal{DR}(A)$ be the set of all $a \in A$ such that a is Drazin invertible. According to [2], $\mathcal{DR}(A)$ is a *regularity* in the sense of [7, 8]. For the next definition see [2].

Definition 1. *Let A be a unital Banach algebra. The Drazin spectrum of an element $a \in A$ is the set*

$$\sigma_{\mathcal{DR}}(a) = \{\lambda \in \mathbb{C}\colon a - \lambda \notin \mathcal{DR}(A)\}.$$

Naturally $\sigma_{\mathcal{DR}}(a) \subseteq \sigma(a)$, the spectrum of a. In addition, the Drazin spectrum of a Banach algebra element satisfies the spectral mapping theorem for analytic functions defined on a neighborhood of the usual spectrum which are non-constant on each component of its domain of definition. Moreover, $\sigma_{\mathcal{DR}}(a)$ is a closed subset of $\mathbb{C}$, see [7, 8, 2].

When $A = L(X)$, X a Banach space, the left and the right Drazin spectra of an operator have been introduced. Before recalling these notions, consider the sets

$$\mathcal{LD}(X) = \{T \in L(X) \colon a(T) \text{ is finite and } R(T^{a(T)+1}) \text{ is closed}\},$$
$$\mathcal{RD}(X) = \{T \in L(X) \colon d(T) \text{ is finite and } R(T^{d(T)}) \text{ is closed}\}.$$

Definition 2. *Let X be a Banach space. An operator $T \in L(X)$ will be called left Drazin invertible (respectively right Drazin invertible), if $T \in \mathcal{LD}(X)$ (respectively if $T \in \mathcal{RD}(X)$). Given $T \in L(X)$, the left Drazin spectrum of T (respectively the right Drazin spectrum of T) is defined as usual from $\mathcal{LD}(X)$ (respectively from $\mathcal{RD}(X)$). These spectra will be denoted by $\sigma_{\mathcal{LD}}(T)$ and $\sigma_{\mathcal{RD}}(T)$ respectively.*

In the conditions of Definition 2, $\sigma_{\mathcal{LD}}(T)$ and $\sigma_{\mathcal{RD}}(T)$ satisfy the spectral mapping theorem under the same hypothesis that the Drazin spectrum does. Moreover, $\sigma_{\mathcal{RD}}(T)$ and $\sigma_{\mathcal{LD}}(T)$ are closed subsets of $\mathbb{C}$, see [8, 1].

Let X be a Banach space and consider the sets

$$\mathcal{R}_4^a(X) = \{T \in L(X) \colon d(T) \text{ is finite}\},$$
$$\mathcal{R}_9^a(X) = \{T \in L(X) \colon a(T) \text{ is finite}\}.$$

The descent and the ascent spectrum of $\in L(X)$ can be derived as for the other spectra just considered. They will be denoted by $\sigma_{dsc}(T)$, $T \in L(X)$ and $\sigma_{asc}(T)$ respectively, see [8, 5].

In the following theorem the relationships among the recalled spectra will be presented.

Theorem 3. *Let X be a Banach space and consider $T \in L(X)$. Then*

$$\sigma_{\mathcal{DR}}(T) = \sigma_{\mathcal{LD}}(T) \cup \sigma_{\mathcal{RD}}(T) = \sigma_{asc}(T) \cup \sigma_{dsc}(T).$$

3. The Drazin Spectrum in Banach Algebras

Next the relationships among the Drazin spectra of a Banach algebra element and of the multiplication operators will be considered.

Theorem 4. *Let A be a unital Banach algebra. Then, the following statements are equivalent.*

(*i*) *The element $a \in A$ is Drazin invertible and index$(a) = k$,*

(*ii*) *$L_a \in L(A)$ is Drazin invertible and index$(L_a) = k$,*

(*iii*) *$R_a \in L(A)$ is Drazin invertible and index$(R_a) = k$.*

Moreover, in this case, if b is the Drazin inverse of a, then L_b (respectively R_b) is the Drazin inverse of L_a (respectively R_a).
As a consequence,
(iv) $\sigma_{\mathcal{DR}}(a) = \sigma_{\mathcal{DR}}(L_a) = \sigma_{\mathcal{DR}}(R_a)$.

Theorem 5. *Consider a unital Banach algebra A, and let $a \in A$. Then, the following statements hold.*

(*i*) $\sigma_{asc}(R_a) \subseteq \sigma_{dsc}(L_a), \sigma_{asc}(L_a) \subseteq \sigma_{dsc}(R_a)$,

(*ii*) $\sigma_{\mathcal{DR}}(a) = \sigma_{\mathcal{RD}}(L_a) \cup \sigma_{\mathcal{RD}}(R_a) = \sigma_{dsc}(L_a) \cup \sigma_{dsc}(R_a)$.

4. Drazin Spectra of Banach Space Operators

In the particular case of $A = L(X)$, X a Banach space, the following results were obtained. See [5, 1] where some of the following results were proved.

Proposition 6. *Consider a Banach space X, and let $T \in L(X)$.*

(*i*) *If $dsc(L_T)$ is finite, then $dsc(T)$ is finite. In addition, $dsc(T) \leq dsc(L_T)$.*

(*ii*) *If $dsc(T) = d$ is finite and $N(T^{d+1})$ has a direct complement, then $dsc(L_T)$ is finite. Moreover, $dsc(L_T) = dsc(T)$.*

(*iii*) *Necessary and suffcient for $asc(L_T)$ to be finite, is the fact that $asc(T)$ is finite. Furthermore, in this case $asc(L_T) = asc(T)$.*

(*iv*) *If $R(L_T)$ is closed, then $R(T)$ is closed.*

(*v*) *If $R(T)$ is closed and $N(T)$ has a direct complement, then $R(L_T)$ is closed.*

(*vi*) *If $dsc(R_T)$ is finite, then $asc(T)$ is finite. In addition, $asc(T) \leq dsc(R_T)$.*

(*vii*) *If $asc(T) = a$ is finite and T^{a+1} is a regular operator, then $dsc(R_T)$ is finite. Moreover, $dsc(R_T) = asc(T)$.*

(*viii*) *If $asc(R_T) = a$ is finite and there exists $k \geq a$ such that $R(T^{k+1})$ has a direct complement, then $dsc(T)$ is finite. Furthermore, $dsc(T) = asc(R_T)$.*

(*ix*) *If $dsc(T)$ is finite, then $asc(R_T)$ is finite. What is more, $asc(R_T) \leq dsc(T)$.*

Theorem 7. *Consider a Banach space X, and $T \in L(X)$. Then, the following statements hold.*

$$\begin{array}{ll} (i) & \sigma_{dsc}(T) \subseteq \sigma_{dsc}(L_T),\ \sigma_{asc}(T) = \sigma_{asc}(L_T), \\ (ii) & \sigma_{\mathcal{RD}}(T) \subseteq \sigma_{\mathcal{RD}}(L_T),\quad \sigma_{\mathcal{LD}}(T) \subseteq \sigma_{\mathcal{LD}}(L_T), \\ (iii) & \sigma_{asc}(T) \subseteq \sigma_{dsc}(R_T),\ \sigma_{asc}(R_T) \subseteq \sigma_{dsc}(T). \end{array}$$

Theorem 8. *Consider a Hilbert space H, and let $T \in L(H)$. Then, the following statements hold.*

$$\begin{array}{ll} (i) & \sigma_{dsc}(T) = \sigma_{dsc}(L_T). \\ (ii) & \sigma_{\mathcal{RD}}(T) = \sigma_{\mathcal{RD}}(L_T). \\ (iii) & \sigma_{\mathcal{LD}}(T) = \sigma_{\mathcal{LD}}(L_T). \\ (iv) & \sigma_{\mathcal{LD}}(T) = \sigma_{\mathcal{RD}}(R_T). \\ (v) & \sigma_{\mathcal{RD}}(T) = \sigma_{\mathcal{LD}}(R_T). \end{array}$$

As a result,

$$(vi) \qquad \sigma_{\mathcal{DR}}(T) = \sigma_{\mathcal{LD}}(L_T) \cup \sigma_{\mathcal{LD}}(R_T).$$

Compare Theorem 5 (ii) with Theorem 8 (vi).

5. A Characterization of the Drazin Spectrum

Let $K \subseteq \mathbb{C}$ be a compact set. Then iso K will stand for the set of all isolated points of K and acc $K = K\setminus$ iso K. When A is a unital Banach algebra and $a \in A$, $\Pi(a)$ will denote the sets of *poles* of $a \in A$ and $\mathcal{IES}(a) = \text{iso } \sigma(a) \setminus \Pi(a)$. Recall also that $\rho_{\mathcal{DR}}(a) = \mathbb{C} \setminus \sigma_{\mathcal{DR}}(a)$.

Theorem 9. *Let A be a unital Banach algebra and consider $a \in A$. Then, the following statements hold.*

$$\begin{array}{ll} (i) & \Pi(a) = \Pi(L_a) = \Pi(R_a), \\ (ii) & \mathcal{IES}(a) = \mathcal{IES}(L_a) = \mathcal{IES}(R_a). \end{array}$$

In particular, if X is a Banach space and $T \in L(X)$, then

$$\Pi(T) = \Pi(L_T) = \Pi(R_T), \quad \mathcal{IES}(T) = \mathcal{IES}(L_T) = \mathcal{IES}(R_T).$$

Theorem 10. *Let A be a unital Banach algebra and consider $a \in A$. Then, the following statements hold.*

(*i*) $\Pi(a) = \sigma(a) \cap \rho_{\mathcal{DR}}(a), \sigma(a) = \sigma_{\mathcal{DR}}(a) \cup \Pi(a)$,

(*ii*) $\sigma_{\mathcal{DR}}(a) \cap \Pi(a) = \emptyset, \text{iso } \sigma(a) \cap \sigma_{\mathcal{DR}}(a) = \mathcal{IES}(a)$.

(*iii*) $\text{acc } \sigma(a) = \text{acc } \sigma_{\mathcal{DR}}(a), \text{iso } \sigma_{\mathcal{DR}}(a) = \mathcal{IES}(a)$.

(*iv*) $\sigma_{\mathcal{DR}}(a) = \text{acc } \sigma(a) \cup \mathcal{IES}(a)$.

(*v*) *Necessary and sufficient for a to be Drazin invertible is that $\lambda = 0$ is a pole of the resolvent operator of a.*

6. Basic properties of the Drazin Spectrum

Given a unital Banach algebra A and $a \in A$, $\partial\sigma(a)$ will denote the topological boundary of $\sigma(a)$ and $\rho(a) = \mathbb{C} \setminus \sigma(a)$.

Theorem 11. *Let A be a unital Banach algebra and consider $a \in A$. Then, the following statements are equivalent.*

(i) $\sigma_{\mathcal{DR}}(a) = \emptyset$, *(ii)* $\partial\sigma(a) \subseteq \rho_{\mathcal{DR}}(a)$, *(iii) a is algebraic.*

Theorem 12. *Let A be a unital Banach algebra and consider $a \in A$. Then, the following statements are equivalent.*

(i) $\sigma(a)$ is at most countable, (ii) $\sigma_{\mathcal{DR}}(a)$ is at most countable.

Furthermore, in this case

$$\begin{aligned} \sigma_{\mathcal{DR}}(a) &= \sigma_{\mathcal{LD}}(L_a) = \sigma_{\mathcal{RD}}(L_a) = \sigma_{dsc}(L_a) \\ &= \sigma_{\mathcal{LD}}(R_a) = \sigma_{\mathcal{RD}}(R_a) = \sigma_{dsc}(R_a). \end{aligned}$$

Given a Banach algebra A and a and $b \in A$, the identity

$$\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\},$$

is well known. However, in the case of the Drazin spectrum, both spectra coincide.

Theorem 13. *Let A be a unital Banach algebra and consider a and $b \in A$. Then,*

$$\sigma_{\mathcal{DR}}(ab) = \sigma_{\mathcal{DR}}(ba).$$

Remark 14. Let A be a unital Banach algebra. Note that

$$\Pi(ab) \setminus \{0\} = \Pi(ba) \setminus \{0\},$$

where a and b belong to A. Furthermore, this identity can not be improved. Consider for example the Banach space $X = l^2(\mathbb{N})$ and the operator S, $T \in L(X)$ defined by

$$S((x_n)_{n\geq 1}) = (0, x_1, x_2, \ldots, x_n, \ldots), \quad T((x_n)_{n\geq 1}) = (x_2, x_3, \ldots, x_n, \ldots),$$

where $(x_n)_{n\geq 1} \in X$. Then, a straightforward calculation shows that

$$\Pi(ST) = \{0, 1\}, \qquad \Pi(TS) = \{1\}.$$

Theorem 15. *Let A be a unital Banach algebra and consider Ω a connected component of $\rho_{\mathcal{DR}}(a)$, $a \in A$. Then $\Omega \setminus \Pi(a) \subseteq \rho(a)$. Furthermore, Ω is not contained in $\sigma(a)$, unless $\Omega = \emptyset$.*

Theorem 16. *Let A be a $\mathbb{C}^*$-algebra. Then, the following statements hold.*

$$
\begin{array}{llll}
(i) & \sigma_{dsc}(L_{a^*}) = \overline{\sigma_{dsc}(R_a)}, & & \sigma_{dsc}(R_{a^*}) = \overline{\sigma_{dsc}(L_a)}. \\
(ii) & \sigma_{\mathcal{RD}}(L_{a^*}) = \overline{\sigma_{\mathcal{RD}}(R_a)}, & & \sigma_{\mathcal{RD}}(R_{a^*}) = \overline{\sigma_{\mathcal{RD}}(L_a)}. \\
(iii) & \sigma_{asc}(L_{a^*}) = \overline{\sigma_{asc}(R_a)}, & & \sigma_{asc}(R_{a^*}) = \overline{\sigma_{asc}(L_a)}. \\
(iv) & \sigma_{\mathcal{LD}}(L_{a^*}) = \overline{\sigma_{\mathcal{LD}}(R_a)}, & & \sigma_{\mathcal{LD}}(R_{a^*}) = \overline{\sigma_{\mathcal{LD}}(L_a)}.
\end{array}
$$

Furthermore, when a is a hermitian element of A, all the spectra considered in statements (i)-(iv) are contained in the real line, and

$$
\begin{array}{ll}
(v) & \sigma_{asc}(L_a) = \sigma_{asc}(R_a),\ \sigma_{\mathcal{LD}}(L_a) = \sigma_{\mathcal{LD}}(R_a), \\
(vi) & \sigma_{\mathcal{DR}}(a) = \sigma_{dsc}(L_a) = \sigma_{dsc}(R_a) = \sigma_{\mathcal{RD}}(L_a) = \sigma_{\mathcal{RD}}(R_a).
\end{array}
$$

Theorem 17. *Let H be a Hilbert space. Then, the following statement hold.*

$$
\begin{array}{ll}
(i) & \sigma_{dsc}(R_T) = \overline{\sigma_{dsc}(L_{T^*})} = \overline{\sigma_{dsc}(T^*)}, \quad \sigma_{dsc}(R_{T^*}) = \overline{\sigma_{dsc}(L_T)} = \overline{\sigma_{dsc}(T)}. \\
(ii) & \sigma_{asc}(R_T) = \overline{\sigma_{asc}(L_{T^*})} = \overline{\sigma_{asc}(T^*)}, \quad \sigma_{asc}(R_{T^*}) = \overline{\sigma_{asc}(L_T)} = \overline{\sigma_{asc}(T)}.
\end{array}
$$

Furthermore, when $T = T^*$, $\sigma_{dsc}(R_T)$ and $\sigma_{asc}(R_T)$ are subsets of the real line, and

$$
\begin{array}{ll}
(iii) & \sigma_{asc}(R_T) = \sigma_{asc}(L_T) = \sigma_{asc}(T), \\
(iv) & \sigma_{\mathcal{DR}}(T) = \sigma_{dsc}(R_T) = \sigma_{dsc}(L_T) = \sigma_{dsc}(T).
\end{array}
$$

Theorem 18. *Let X be a Banach space and consider $T \in L(X)$.*
(a) The following statements are equivalent.

(i) T is meromorphic, *(ii) $\sigma_{\mathcal{DR}}(T) \subseteq \{0\}$,*

(iii) $L_T \in L(L(X))$ is meromorphic, *(iv) $R_T \in L(L(X))$ is meromorphic.*

(b) Let S and $T \in L(X)$. Then, necessary and sufficient for ST to be meromorphic is the fact that TS is meromorphic.
(c) Let $F \in L(X)$ and suppose that there exists a positive integer n such that F^n has finite dimensional range and F commutes with T. Then, if T is meromorphic, $T + F$ is meromorphic.

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