# Applications of thermodynamic formalism in complex dynamics on $\mathbb{P}^2$

Eugen Mihailescu

**Abstract**

In this paper, we review several notions from thermodynamic formalism, like topological pressure and entropy and show how they can be employed, in order to obtain information about stable and unstable sets of holomorphic endomorphisms of $\mathbb{P}^2$ with Axiom A.

In particular, we will consider the non-wandering set of such a mapping and its "saddle" part $S_1$, i.e the subset of points with both stable and unstable directions. Under a derivative condition, the stable manifolds of points in $S_1$ will have a very "thin" intersection with $S_1$, from the point of view of Hausdorff dimension. While for diffeomorphisms there is in fact an equality between $HD(W^s_\epsilon(x) \cap S_1)$ and the unique zero of $P(t \cdot \phi^s)$ (Verjovsky-Wu [VW]) in the case of endomorphisms this will not be true anymore; counterexamples in this direction will be provided. We also prove that the unstable manifolds of an endomorphism depend Hölder continuously on the corresponding prehistory of their base point and employ this in the end to give an estimate of the Hausdorff dimension of the global unstable set of $S_1$. This set could be à priori very large, since, unlike in the case of Hénon maps, there is an uncountable collection of local unstable manifolds passing through each point of $S_1$.

**Key words:** thermodynamic formalism, holomorphic endomorphisms, topological pressure, Hausdorff dimension.



## 1 Notation, definitions

The notions of entropy and pressure have appeared initially in thermodynamics. They give a measure of the chaotic behaviour of a system in time, hence they are natural candidates to a study from the point of view of dynamical systems.

In particular, they found applications to complex dynamics, for example Ruelle [Ru] and Verjovsky-Wu [VW].

Our goal here is to generalize some of the results of Verjovsky-Wu ([VW]) from the case of diffeomorphisms to the case of holomorphic endomorphisms of $\mathbb{P}^2$ and to study the differences that appear in the presence of non-invertibility.

The applications of thermodynamic formalism to complex dynamics that constitute the main goal of this paper realize a phenomenologic extension of some very powerful physical ideas.

Let $(X, d)$ be a compact metric space and $f\colon\ X \to X$ a continuous map. Let $n$ a positive integer and $\varepsilon > 0$, and define the metric $d_n(x, y) := \max\{d(f^i x, f^i y),\ i = \overline{0, n-1}\}$.

**Definition 1.** A subset $E \subset X$ is called an $(n, \varepsilon)$-*separated subset* w.r.t. $X, d, f$, if $x, y \in E$, and $x \neq y$, implies $d_n(x, y) \geq \varepsilon$.

**Definition 2.** The *topological pressure* of $f$ is the map $P\colon C(X, \mathbb{R}) \longrightarrow \overline{\mathbb{R}}$ defined on the space of continuous functions as follows:

$$P_f(\varphi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup \left\{ \sum_{x \in E} \exp\left( \sum_{i=0}^{n-1} \varphi(f^i x) \right), E \subset X, (n, \varepsilon)\text{-separated set} \right\}.$$

$f$ is called an *expansive homeomorphism* of $X$ of constant $\delta > 0$, if $(\forall) x, y \in X$ s.t. $d(f^m x, f^m y) < \delta$, $(\forall) m \in \mathbb{Z}$, it follows that $x = y$. It can be (easily) shown that, if $\delta_0$ is an expansive constant, then

$$P_f(\varphi) = \limsup_{n \to \infty} \frac{1}{n} log \sup \left\{ \sum_{x \in E} \exp\left( \sum_{i=0}^{n-1} \varphi(f^i(x)) \right), E(n, \delta)\text{-separated} \right\} \text{ for any } 0 < \delta < \frac{\delta_0}{4}.$$

The following properties of topological pressure are very useful. Proofs can be found in Walters [Wa].

**Theorem (Variational Principle).** *In the above setting,* $P_f(\varphi) = \sup_\mu \{h_\mu(f) + \int \varphi\ d\mu\}$*, where the supremum is taken over all* $f$*-invariant Borel probability measures* $\mu$ *and* $h_\mu(f)$ *is the measure-theoretic entropy of* $\mu$*.*

We do not define formally the entropy, for a reference see for e.g. Walters [Wa].

Enough to say that there exist two types of entropy, the measure-theoretic one, $h_\mu(f)$, and the metric one, $h(f)$.

The metric entropy,$h(f)$, is equal to $P_f(0)$, and from the Variational Principle, $h(f) = \sup\{h_\mu(f), \mu$ $f$-invariant Borel probability measure$\}$.

**Theorem (Properties of Pressure).** *If* $f\colon X \to X$ *is a continuous transformation and* $\varphi, \psi \in \mathcal{C}(X, \mathbb{R})$*, then:*

*1)* $\varphi \leq \psi \Rightarrow P_f(\varphi) \leq P_f(\psi)$

*2)* $P_f(\cdot)$ *is either finitely valued or constantly* $\infty$

*3)* $P_f$ *is convex*

*4) for a strictly negative function* $\varphi$*, the map* $t \longrightarrow P_f(t\varphi)$ *is strictly decreasing.*

5) *If $f_i\colon\ X_i \to X_i$ is a continuous map of a compact metric space $(X_i, d_i)$, $\overline{i=1,2}$ and $\phi\colon\ X_1 \to\ X_2$ is a homeomorphism, then $P_{f_1}(\varphi) = P_{f_2}(\varphi \circ \phi^{-1})$*

$$\begin{array}{ccccc} X_1 & \xrightarrow{f_1} & X_1 & \xrightarrow{\varphi} & \mathbb{R} \\ \phi\downarrow & & \downarrow\phi & & \\ X_2 & \xrightarrow[f_2]{} & X_2 & & \end{array}$$

Let us remind also the notions of Hausdorff dimension and upper box dimension.

Consider $(X, d)$ a metric space and $\alpha > 0$. The Hausdorff $\alpha$-measure of $X$ is defined by $m_\alpha(X) = \sup\limits_{\varepsilon>0} \inf \sum\limits_{i=1}^{\infty} (\text{diam } A_i)^\alpha$, where inf is taken over all covers of $X$ by a family $(A_i)_{i=1}^{\infty}$ of sets of diam $< \varepsilon$.

*The Hausdorff dim of X, $HD(X) = \inf\{\alpha, m_\alpha(X) = 0\} = \sup\{\alpha, m_\alpha(X) = \infty\}$.*

Now, take $\overline{m}_B(X, \alpha) := \varlimsup\limits_{\varepsilon\to 0} \inf \Big\{ \sum\limits_{U\in G} \varepsilon^\alpha, G$ is at most countable cover of $X$ by open sets of diameter $\varepsilon > 0 \Big\}$ and $\underline{m}_B(X, \alpha) := \varliminf\limits_{\varepsilon\to 0} \inf \Big\{ \sum\limits_{U\in G} \varepsilon^\alpha, G$ at most countable cover of $X$ by open sets of diameter $\varepsilon \Big\}$.

Then the upper box dimension, $\overline{dim}_B X = \inf\{\alpha, \overline{m}_B(X, \alpha) = 0\} = \sup\{\alpha, \overline{m}_B(X, \alpha) = \infty\}$. The lower box dimension is, by definition, $\underline{\dim}_B(X) = \inf\{\alpha, \underline{m}_B(X, \alpha) = 0\} = \sup\{\alpha, \underline{m}_B(X, \alpha) = \infty\}$.

**Properties of dimension** (Hausdorff, upper/lower box):

i) For a set $Z \subset \mathbb{R}^m$, $\underline{\dim}_B Z = \lim_{\overline{\varepsilon\to 0}} \frac{\log N(Z,\varepsilon)}{\log\left(\frac{1}{\varepsilon}\right)}$, and $\overline{\dim}_B Z = \varlimsup\limits_{\varepsilon\to 0} \frac{\log N(Z,\varepsilon)}{\log \frac{1}{\varepsilon}}$, where $N(Z, \varepsilon)$ is the least number of balls of radius $\varepsilon$ needed to cover $Z$.

2) $HD(Z_1) \le HD(Z_2)$ if $Z_1 \subset Z_2$

3) $HD\left(\bigcup\limits_{i\ge 1} Z_i\right) = \sup\limits_{i\ge 1} HD(Z_i)$

4) $HD(Z) \le \underline{\dim}_B Z \le \overline{\dim}_B Z$

5) The Hausdorff dimension and lower and upper box dimensions are invariant with respect to a Lipschitz continuous homeomorphism with a Lipschitz continuous inverse.

Let us introduce now the notions from higher dimensional complex dynamics that will be needed.

Consider $f\colon\ \mathbb{P}^2 \to \mathbb{P}^2$ a holomorphic map on the complex 2-dimensional projective space; let $f$ be of degree $d$. This means that, in homogeneous coordinates, $f = [P : Q : R]$, where $P, Q, R$ are homogeneous polynomials of degree $d$ with no common zeros other than the origin. We denote the space of holomorphic maps of degree $d$ by $\mathcal{H}_d$.

**Definition 3.** The *non-wandering set* of $f$, denoted by $\Omega_f$, is the set of points $x \in \mathbb{P}^2$ s.t $x$ has no neighbourhood $V$ s.t $f^n V \cap V = \emptyset, \forall n \ge 1$.

A class of holomorphic maps of $\mathbb{P}^2$ whose dynamics has been studied by Fornaess-Sibony are the $s$-hyperbolic maps, [FS1].

First, let $\widehat{\Omega} = \{\hat{x} = (x_k)_{k\le 0}, f(x_{k-1}) = x_k, \text{ and } x_k \in \Omega, \ k \le 0\}$ the hyperbolic cover of $\Omega$. $\hat{f}\colon \ \widehat{\Omega} \to \widehat{\Omega}$ will denote the shift map, $\hat{f}(x_k)_k = (x_{k+1})_k$. The tangent bundle of $\widehat{\Omega}$ is the set $\{(\hat{x}, \xi), \xi \in T_{x_0}\mathbb{P}^2, \text{ if } \hat{x} = (x_k)_{k\le 0}\}$.

**Definition 4.** We say that $f$ is *hyperbolic on* $\Omega$ if $\exists$ a continuous splitting of the tangent bundle of $\widehat{\Omega}, T_{\widehat{\Omega}} = T^s \oplus T^u$, and constants $K > 1$, $C > 0$ s.t. if $\hat{x} \in \widehat{\Omega}$ and $T_{x_0}\mathbb{P}^2 = T^s_{x_0} + T^u_{\hat{x}}$ is a splitting of $T_{x_0}\mathbb{P}^2$, then $|Df^n(x_0)\cdot v| \le C\cdot K^{-n}\cdot |v|$, for $v \in T^s_{x_0}$, and $|Df^n(x_0)\cdot w| \ge C\cdot K^n\cdot |w|$, for $w \in T^u_{\hat{x}}$.

The stable/unstable bundles are also assumed invariant, i.e. $Df(T^s_{x_0}) \subset T^s_{fx_0}$ and $Df(T^u_{\hat{x}}) \subset T^u_{\hat{f}\hat{x}}$.

After a change of metric on $\mathbb{P}^2$, one can assume that $C = 1$. If $x_0 \in \Omega$ and $\varepsilon > 0$, define the local stable manifold by $W^s_\varepsilon(x_0) := \{x \in \mathbb{P}^2, \ d(f^k x, f^k x_0) < \varepsilon, \ k \ge 0\}$, and the local unstable manifold of a prehistory $\hat{x}$, by $W^u_\varepsilon(\hat{x}) := \{y \in \mathbb{P}^2, \ y$ has a prehistory $(y_k)_{k\le 0}$ with $y_0 = y$ s.t. $d(y_k, x_k) < \varepsilon, k \le 0\}$.

These manifolds are tangent at $x_0$ to $T^s_{x_0}$, resp. $T^u_{\hat{x}}$. Let us notice also that, for a holomorphic endomorphism of $\mathbb{P}^2$ the local unstable manifolds are not uniquely defined by a point $x_0 \in \mathbb{P}^2$. Therefore , à priori, through any point $x_0 \in S_1$, may pass an uncountable collection of unstable manifolds.

The global stable set at $x$ is defined as $W^s(x) := \{y \in \mathbb{P}^2, d(f^k x, f^k y) \to 0, \text{ if } k \to \infty\}$.

The global unstable set is $W^u(\hat{x}) := \{y \in \mathbb{P}^2, \ y$ has a prehistory $(y_k)_{k\le 0}$ with $y_o = y$ s.t $d(y_k, x_k) \to 0$, when $k \to -\infty\}$.

**Definition 5.** $f$ is said to satisfy *Axiom A* if it is hyperbolic on its non-wandering set $\Omega$ and the periodic points of $f$ are dense in $\Omega$.

**Definition 6.** $\Omega$ has *local product structure* if $\exists \varepsilon > 0$, so that $(\forall) x \in \Omega, \ \hat{y} \in \widehat{\Omega}, \ W^s_\varepsilon(x) \cap W^u_\varepsilon(\hat{y})$ consists of at most one point, and this point belongs to $\Omega$.

**Theorem (Spectral Decomposition).** *If $f\colon \ \mathbb{P}^2 \to \mathbb{P}^2$ is an endomorphism with Axiom A, then $\Omega$ has local product structure. Also $\Omega$ has a partition into disjoint, closed, invariant sets $\Omega_i$ s.t. $f|_{\Omega_i}$ is topologically transitive, $(\forall) i$ (i.e. $f|_{\Omega_i}$ has a point with dense forward orbit in $\Omega_i$). The above decomposition is unique and $\Omega_i$ are called basic sets.*

**Note:** Let us make the observation that in reality all the results in this paper work for an endomorphism which is hyperbolic and with Axiom A only on its saddle part, i.e on the part where there exist both stable and unstable directions.

In the following, denote with $S_i$ the set of points of $\Omega$ where $\dim T^u = i, i = 0, 1, 2$.

**Definition 7.** $f \in \mathcal{H}_d$ is *s-hyperbolic* if

a) $f$ has Axiom A

b) $f^{-1}(S_2) = S_2$

c) If $C_f$ denotes the critical set of $f$, then $C_f \cap S_1 = \emptyset$

d) $\exists U$ neighborhood of $S_1$ s.t. $f^{-1}(S_1) \cap U = S_1$.

The conditions in Definition 7 are the same as those in Fornaess-Sibony [FS1], with the exception of c), where [FS1] require an arbitrary analytical variety outside $S_1$; so our version of s-hyperbolic map is a little more restrictive.

# 2 Main Results

In this section, we will collect the main theorems of this paper and discuss some of their consequences.

**Theorem 1.** *Consider $\Lambda$ a basic set of the set $S_1$ for an Axiom A, holomophic endomorphism of $\mathbb{P}^2$ with $C_f \cap S_1 = \emptyset$. For any $x \in \Lambda$, $HD(W^s_\varepsilon(x) \cap \Lambda) \leq t^s$, where $t^s$ is the unique real number $t$ such that $P(t \cdot \phi^s) = 0$, with the stable function $\phi^s(y) := log\|Df|_{E^s_y}\|$, $y \in \Lambda$, and the norm $||Df|_{E^s_y}||$ is taken as the norm of a linear application between complex vector spaces. (the size $\varepsilon$ of the stable manifold can be chosen independently of $x$)*

Such a $t^s$ will always exist since $P(0) = h_{top}(f) > 0$ and $P(t \cdot \phi^s) < 0$ for $t$ large enough. Here we are using the properties of topological pressure from Section 1, especially the fact that the function in $t$ , $t \to P(t \cdot \phi^s)$ is strictly decreasing.

We will show in Corollary 1 that, in a certain situation $P(2\phi^s) < 0$, which would then imply that $0 < t^s < 2$.

*Observation:* In the case of real diffeomorphisms, Rufus Bowen and David Ruelle [Bo2] showed that $P(\phi^s) < 0$ if $\Lambda$ is not a repeller. Their method involves two important Volume Lemmas dealing with the Lebesgue measure for balls in the $d_n$ metric. For complex diffeomorphisms their conclusion would translate as $P(2\phi^s) < 0$. Unfortunately, for the case of endomorphisms the non-invertibility makes this proof not to work. In the case of endomorphisms we might not even have always $P(2\phi^s) \leq 0$. Such an example shall be given in the end.

**Corollary 1.** *If $\phi^s$ is the stable function of $\Lambda$, and $|Df|_{E^s}|(x) < \sqrt{\frac{1}{d}}$, for all $x \in \Lambda$, where $d$ is the degree of $f$, then $P(2\phi^s) < 0$. Therefore $HD(W^s_\varepsilon(x) \cap \Lambda) < 2$.*

The proof for Theorem 1 uses the assumption that $C_f \cap S_1 = \emptyset$.

We now come to the main result which involves the collection of all unstable manifolds of points in the set $S_1$ of an s-hyperbolic holomorphic endomorphism of $\mathbb{P}^2$.

In general, through each point $x \in S_1$, there will pass an infinite, uncountable collection of unstable manifolds. It could happen a priori that the union of all global unstable sets might have non-empty interior.

Fornaess-Sibony [FS1] have introduced the set $K^- := \mathbb{P}^2 \setminus U^-$, where $U^- := \{z_0,\ \mathrm{dist}(f^{-n}(z), S_2) \to 0$, uniformly in a neighbourhood of $z_0\}$.

$K^-$ is the analogue of the set of points with bounded backward iterates from the case of Hénon maps. In the Henon diffeomorphisms case, under the hyperbolicity hypothesis, the interior of the set $K^-$ is either empty or it is the union of basins of finitely many repelling orbits ([BS]).

If $f$ is $s$-hyperbolic (see Definition 7), then Fornaess-Sibony [FS1] have shown that $K^- = W^u(\widehat{S}_1) \cup S_0$, and $S_0$ is only a finite set of attracting periodic points [FS]. Here $W^u(\hat{S}_1)$ is the union of all global unstable sets of prehistories of points from $S_1$.

So, whatever is said about $W^u(\widehat{S}_1)$ works in general for $K^-$ also.

The problem in studying $W^u(\widehat{S}_1)$ is that the global unstable sets, and even the local unstable manifolds do not form a lamination; if $\hat{x} \neq \hat{y}$, $W^u_{\hat{x}}$ could still intersect $W^u_{\hat{y}}$.

The only hope is to lift these sets to the hyperbolic cover, where they will be "separated" and then to study the Hausdorff dimension there. We will need a theorem relating the local unstable manifolds with their base points.

First, on the *hyperbolic cover* of $S_1$, denoted $\widehat{S}_1$, we have a metric $\hat{d}(\hat{x}, \hat{y}) = \sum_{i \leq 0} \frac{d(x_i, y_i)}{K^{|i|}}$, where K$>1$, is a constant. This metric depends on the constant K.

**Theorem 2.** *Consider a metric on $\hat{S}_1$, given by $\hat{d}(\hat{x}, \hat{y}) = \sum_{i \leq 0} \frac{d(x_i, y_i)}{K^{|i|}}$, with $K > 1$ fixed. The local unstable manifold at $\hat{x} \in \widehat{S}_1$ can be represented as the image through the exponential map of the graph of a function in $T_{\hat{x}}\mathbb{P}^2$, that depends in a $\theta$-Hölder way on $\hat{x}$, where $\theta$ is any constant satisfying:*

$$\sup_{\hat{x} \in \widehat{S}_1} \|Df|_{E^s_{x_0}}\| \cdot \|Df|_{E^u_{\hat{x}}}\|^{-1} \cdot K^{\theta} < 1$$

*If $\theta = 1$, then the local unstable manifolds depend Lipschitz continuously on their base points.*

Using this Theorem and also a theorem similar to Theorem 1, but for stable sets in $\widehat{S}_1$, one can prove:

**Theorem 3.** *Let $f$ be an $s$-hyperbolic holomorphic mapping of $\mathbb{P}^2$ of degree $d$, with the hyperbolic splitting of the tangent bundle, $E^s_{x_0} \oplus E^u_{\hat{x}}$, such that*

$$(*) \qquad \sup_{\hat{y} \in \widehat{S}_1} \|Df|_{E^s_y}\| \cdot \|Df|_{E^u_{\hat{y}}}\|^{-1} < \inf_{x \in S_1} \|Df|_{E^s_x}\|^{\theta}.$$

*Then both $HD(\hat{W}^u(\hat{S}_1))$ and $HD(K^-)$ are no greater than $2 + \frac{t^s}{\theta}$, where $t^s$ is the only $t$ so that $P(t \cdot \phi^s) = 0$.*

*In particular, if $\sup_{x \in S_1} |Df|_{E^s_x}| < \sqrt{\frac{1}{d}}$ and $\sup_{\hat{y} \in \widehat{S}_1} \|Df|_{E^s_{y_0}}\| \cdot \|Df|_{E^u_{\hat{y}}}\|^{-1} < \inf_{x \in S_1} \|Df|_{E^s_x}\|$, then $HD(K^-) < 4$, so $K^-$ has empty interior.*

Notice also that the condition $(\star)$ in the theorem, when applied to an s-hyperbolic map $f$ without cycles among the basic sets of its $S_1$, is open, i.e. is satisfied by all maps $g$ in a small neighbourhood of $f$ [Mi]. Therefore the estimate on the Hausdorff dimension in the theorem holds for an open set of holomorphic endomorphisms without cycles; one also obtains that the interior of $K^-$ is empty for the maps in that small open set.

The proof of the last theorem uses dimension estimates for $V_\delta^-(\hat{x})$ where $V_\delta^-(\hat{x})$ is the stable set in $\mathbb{P}^2$, estimates similar to those in Theorem 1.

In proving these estimates, however, one has to have an appropiate condition on the constant $K$ appearing in the definition of the metric $\hat{d}$. This, together with the condition for $K$ needed in Theorem 2, will be simultaneously satisfied when the derivative condition in Theorem 3 holds.

We end this section with one last remark. Bedford and Smillie proved in [BS] that if a Hénon diffeomorphism $g$ satisfies $|\det Dg| < 1$, then $K^-$ has empty interior.

The condition on derivatives in our Theorem 3 can be viewed as similar to their condition (although not extending it). However in our case the methods of proof are completely different from theirs.

# 3 Proofs

In the sequel all the norms of derivatives are taken as norms for linear maps between complex vector spaces.

*Proof of Theorem 1.* The proof of this theorem is similar to that of Theorem 1 from Verjovsky-Wu [VW].

For the sake of completeness and for future reference we will sketch the main ideas of the proof. First of all, if $d_{\hat{x}}$ denotes the Riemannian metric induced on the local stable manifold $W^s_\varepsilon(x)$, then there exists a constant $A_x > 0$ s.t.

$$d(y_1, y_2) \leq d_x(y_1, y_2) \leq A_x \cdot d(y_1, y_2), (\forall) \quad y_1, y_2 \in W^s_\varepsilon(x).$$

The stable manifold theorem for an Axiom A map shows that the local stable manifold $W^s_\varepsilon(x)$ depends continuously on $x$, hence the constant $A_x$ can be chosen to depend continuously on $x \in S_1$, and $S_1$ is compact. So there exists $A > 0$ s.t. $A > A_x$, $(\forall) x \in S_1$. Therefore the metrics induced on $W^s_\varepsilon(x)$ are uniformly equivalent to the Fubini metric between points on $W^s_\varepsilon(x)$.

In order to continue the proof, we will need a *Laminated Distortion Lemma*:

**Lemma 1.** *Given $x \in \Lambda$, denote by $x_{-n}$ an element of $f^{-n}(x) \cap \Lambda$. Then, there exists $M > 0$ and $\varepsilon > 0$, both independent of $x$ and $n$ s.t. for any $z \in W^s_{x_{-n}}(\varepsilon)$,*

$$e^{-M} \leq \frac{\|D_z f^n|_{E^s_z}\|}{\|D_{x_{-n}} f^n|_{E^s_{x_{-n}}}\|} \leq e^M.$$

*Proof of Lemma*

Similar as in [VW] Lemma 4. In this Lemma the fact that the map is holomorphic is essential for the writing of the norm of a higher derivative as a product of norms.

As a consequence of the Lemma, $W^s_x(A^{-1}e^{-M}\varepsilon \cdot \|D_{x_{-n}}f^n|_{E^s_{x_{-n}}}\|) \subset f^n(W^s_{x_{-n}}(\varepsilon)) \subset W^s_x(A^2 \cdot e^M \cdot \varepsilon \cdot \|D_{x_{-n}}f^n|_{E^s_{x_{-n}}}\|)$. Hence $A^{-1} \cdot e^{-M} \cdot \varepsilon \cdot \|D_{x_{-n}}f^n|_{E^s_{x_{-n}}}\| \leq \text{diam } f^n(W^s_{x_{-n}}(\varepsilon)) \leq A^2 e^M \cdot \varepsilon \|D_{x_{-n}}f^n|_{E^s_{x_{-n}}}\|$.

Let us continue with the proof of Theorem 1.

The idea now is to take a set $E$ in $f^{-n}W^s_\varepsilon(x) \cap \Lambda$, which is $(n+1, \frac{\varepsilon}{2})$-separated and which has maximal cardinality. Hence $E$ should also be $(n+1, \frac{\varepsilon}{2})$-spanning, i.e. for any $\xi \in f^{-n}W^s_\varepsilon(x) \cap \Lambda$, $\exists y \in E$ s.t. $d_{n+1}(\xi, y) < \frac{\varepsilon}{2}$.

Denote by $W := W^s_\varepsilon(x) \cap \Lambda$.

In this case, $W \subset \bigcup\limits_{y \in E} f^n(W^s_\varepsilon(y))$ as can be easily noticed from the fact that $E$ is $(n+1, \frac{\varepsilon}{2})$-spanning.

Denote $f^n(W^s_\varepsilon(y)$ with $B^n_y$, for $y \in E$. Then using the Laminated Distortion Lemma,

$$\begin{aligned}\sum_{y \in E} (\text{diam } B^n_y)^t &\leq \sum_{y \in E} (A^2 e^M \cdot \varepsilon)^t \cdot \|D_y f^n|_{E^s_y}\|^t = \\ &= C \cdot \sum_{y \in E} \exp(t \cdot \log \|D_y f^n|_{E^s_y}) \leq \\ &\leq C' \cdot \sum_{y \in E} \exp\left(t \cdot \sum_{j=0}^{n} \phi^s(f^j y)\right).\end{aligned}$$

Hence, using the fact that $P(t \cdot \phi^s) < 0$ if $t > t^s$, and the definition of topological pressure using separated sets, (Definition 2)

$$HD(W^s_\varepsilon(x) \cap S_1) \leq t^s, \text{ for some } \varepsilon > 0.$$

□

We now come to the proof of Corollary 1 which is just an application of the Variational Principle.

*Proof of Corollary 1.* If $||Df|_{E^s}|| < \sqrt{\frac{1}{d}}$, then $\int 2\phi^s \, d\mu < -logd$ for any probability measure $\mu$.

But since $h_\mu \leq h_{top}$ and $h_{top}(f|_\Lambda) = logd$ (from [DJ]), it follows that $h_\mu + \int 2\phi^s \, d\mu < logd - logd = 0$. Hence from the Variational Principle for topological pressure $P(2\phi^s) < 0$, so Theorem 1 implies $HD(W^s_\varepsilon(x) \cap \Lambda) < 2$.

□

We now arrive to the proof of Theorem 2 which gives the Hölder continuous dependence of the local unstable manifolds $W^u_\varepsilon(\hat{x})$ on the base point $\hat{x}$.

In turn, this implies that the distance from a point $\xi \in W^u_\varepsilon(\hat{x})$ to the manifold $W^u_\varepsilon(\hat{y})$ is $\leq C \cdot \hat{d}(\hat{x}, \hat{y})$, for a uniform constant $C > 0$. This last property will be further employed in the proof of Theorem 3.

The case of $C^2$ diffeomorphisms that are hyperbolic over a compact set has been studied extensively in a paper by Pugh-Shub-Wilkinson [PSW]. However in our case the base space will not have any differential structure, being a space of prehistories.

A careful inspection of their proof shows that the following Hölder Section Theorem is essential; after stating it, we will show how it can be employed for hyperbolic covers and what is changing from the diffeomorphism case.

**Hölder Section Theorem:** *Consider $E = X \times Y$, where $X$ is a compact metric space and $Y$ is a closed, bounded subset of a Banach space, and $\pi\colon\ E \to X$ the canonical projection. Assume we have a bundle map $F$, s.t. the diagram*

$$\begin{array}{ccc} E & \xrightarrow{F} & E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ X & \xrightarrow{h} & X \end{array}$$

*is commutative, where $h$ is a homeomorphism. We will make several assumptions on $F$:*

*1) $F$ is assumed to contract fibers of $E$ in the sense that for all $x \in X$, $\exists 0 < k_x < 1$ s.t.*

$$d(F(x,y), F(x,y')) \le k_x d(y,y'), (\forall) y, y' \in Y.$$

*2) There exists a constant $L \ge 1$ s.t. $(\forall) x, x' \in X$ and all $y \in Y$,*

$$|F(x,y) - F(x',y)| \le L \cdot d(x,x')^\theta.$$

*3) $\exists \delta > 0$ s.t. if $(\forall) x \in X$,*

$$\inf \left\{ \frac{d(hx, hx')}{d(x,x')}; x' \in X, d(x,x') < \delta \right\} =: \omega_x > 0,$$

*then $\sup k_x \cdot \omega_x^{-\theta} < 1$.*

*Under hypotheses 1)–3), there will exist a unique $F$-invariant section $\sigma$ of $X \times Y$ (i.e. $\sigma \circ h(x) = F \circ \sigma(x)$, $(\forall) x \in X$), and $\sigma$ is $\theta$-Hölder, meaning that $\exists H > 0$ s.t. if $\sigma(x) = (x, s(x))$, then*

$$|\sigma(x) - \sigma(x')| \le H \cdot d(x,x')^\theta, (\forall) x, x' \in X.$$

*Proof of Theorem 2.* We will show first how the above Holder Section Theorem must be modified in the case of endomorphisms.

In our case, $X = \hat\Lambda, h = \hat f\colon\ \hat\Lambda \to \hat\Lambda$, and we have initially a bundle over $\hat\Lambda$, $T_{\hat\Lambda} = \{(\hat x, v), v \in T_{x_0}\mathbb{P}^2\}$.

We also have the bundle map $\widehat{Df}\colon\ T_{\hat\Lambda} \to T_{\hat\Lambda}$, $\widehat{Df}(\hat x, v) = (\hat f \hat x, D_{x_0} f(v))$.

In the diffeomorphism case, we have to approximate the given stable/unstable spaces distributions with smooth ones.

In our case we must take Lipschitz continuous approximations $\widetilde{E}^s$ and $\widetilde{E}^u$, considered as bundles over $\hat\Lambda$. Indeed $E^u_{\hat x}$ is given by some linear equations as a subspace of $T_{\hat x}\mathbb{P}^2$ and the coefficients appearing in those equations depend continuously on $\hat{x}$. Now, since the subalgebra of Lipschitz functions, $\mathrm{Lip}(\hat{\Lambda}, \mathbb{R})$, is dense in $C(\hat{\Lambda}, \mathbb{R})$, $E^u_{\hat{x}}$ can be approximated by a subspace $\widetilde{E}^u_{\hat{x}}$ depending Lipschitz continuously on $\hat{x}$.

The idea then is to trivialize $\widetilde{E}^u$, i.e. to add another bundle $H_1$ to it, s.t. $\widetilde{E}^u \oplus H_1 \simeq \hat{\Lambda} \times \mathbb{R}^{m_1}$, for some $m_1$. Also $\exists H_2$ s.t. $\widetilde{E}^s \oplus H_2 \simeq \hat{\Lambda} \times \mathbb{R}^{m_2}$ for some $m_2$. In the notation of [PSW], there exist bundle isomorphisms $S_1\colon\ H_1 \to H_1$, $S_2\colon\ H_2 \to H_2$ covering $\hat{f}$ s.t. $S_{1,2}$ preserve the expanding/contracting character of $Df$ on $E^{u,s}$ respectively.

Now we take $\bar{f}\colon\ T_{\hat{\Lambda}}(\delta) \to T_{\hat{\Lambda}}$, $\hat{f}(\hat{x}, v) = \exp^{-1}_{f x_0} \circ f \circ \exp_p(v)$ and $H = H_1 \oplus H_2$ with $S = S_1 \oplus S_2\colon\ H \to H$.

So we obtain the bundle map $F$ by commutativity in the following diagram:

$$\begin{array}{ccc}
\hat{\Lambda} \times \mathbb{R}^{m_1+m_2} & \xrightarrow{F} & \hat{\Lambda} \times \mathbb{R}^{m_1+m_2} \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle a} \\
T_{\hat{\Lambda}}(\delta) \oplus H & \xrightarrow{\bar{f} \oplus S} & T_{\hat{\Lambda}} \oplus H \\
\downarrow & & \downarrow \\
\hat{\Lambda} & \xrightarrow{\hat{f}} & \hat{\Lambda}
\end{array}$$

where $a\colon\ \hat{\Lambda} \times \mathbb{R}^{m_1+m_2} \to T_{\hat{\Lambda}} \oplus H$ is a Lipschitz continuous trivialization isomorphism.

$F$ is differentiable on each fiber $\{\hat{x}\} \times \mathbb{R}^{m_1+m_2}$ and the derivative of $F$ depends Lipschitz continuously on $\hat{x}$.

Hence $D_{\hat{x}}F\colon\ \{\hat{x}\} \times \mathbb{R}^{m_1+m_2} \to \{\hat{f}\hat{x}\} \times \mathbb{R}^{m_1+m_2}$ exists and $|D_{\hat{x}}F(v) - D_{\hat{y}}F(v)| \le \mathrm{Lip}(DF) \cdot \hat{d}(\hat{x}, \hat{y}) \cdot |v|$.

Consider now the Banach space $G = \{g \in C(\mathbb{R}^{m_1}(\delta), \mathbb{R}^{m_2}), \sup \frac{|g(x)|}{|x|} < \infty\}$, equipped with the norm $\|g\|_* = \sup \frac{|g(x)|}{|x|}$; it contains $Y = \{g \in G, \mathrm{Lip}\ g \le 1\}$, which is closed and bounded.

We will thus obtain a bundle map

$$\begin{array}{ccc}
\hat{\Lambda} \times Y & \xrightarrow{F_{\#}} & \hat{\Lambda} \times Y \\
\downarrow & & \downarrow \\
\hat{\Lambda} & \xrightarrow{\hat{f}} & \hat{\Lambda}
\end{array}$$

with $F_{\#}(\hat{x}, g) = (\hat{f}\hat{x}, g_{\hat{x}})$, where graph $g_{\hat{x}} = F_{\hat{x}}(\text{graph } g) \cap (\mathbb{R}^{m_1}(\delta) \times \mathbb{R}^{m_2})$. $F_{\#}$ is called the graph transform. Now we can apply the Hölder Section Theorem with the fiber constant $k_{\hat{x}} = \|Df|_{E^s_{x_0}}\| \cdot \|Df|_{E^u_{\hat{x}}}\|^{-1}$ and the base constant $\omega_{\hat{x}}$.

It remains to see what is the value of $\omega_{\hat{x}}$ in our case.

First, $\hat{d}(\hat{f}\hat{x}, \hat{f}\hat{y}) = d(fx, fy) + \frac{d(x_0, y_0)}{K} + \frac{d(x-1, y-1)}{K^2} + \cdots$ and $\hat{d}(\hat{x}, \hat{y}) = d(x_0, y_0) + \frac{d(x-1, y-1)}{K} + \cdots$, so

$$\frac{\hat{d}(\hat{f}\hat{x}, \hat{f}\hat{y})}{\hat{d}(\hat{x}, \hat{y})} \ge \frac{1}{K}, \quad (\forall)\hat{x} \ne \hat{y}.$$

Now, fix $\hat{x}$ and take $\hat{y}$ s.t. $y_{-N} = x_{-N}, \dots, y_0 = x_0$ and $y_{-N-1} \neq x_{-N-1}$. In this case $\hat{d}(\hat{x}, \hat{y}) = \frac{d(x_{-N-1}, y_{-N-1})}{K^{N+1}} + \cdots$ and if $N$ is chosen large enough we can make $d(\hat{x}, \hat{y})$ as small as we like, for example $\hat{d}(\hat{x}, \hat{y}) < \delta$.

So $\hat{d}(\hat{f}\hat{x}, \hat{f}\hat{y}) = \frac{d(x_{-N-1}, y_{-N-1})}{K^{N+2}} + \cdots = \frac{1}{K}\hat{d}(\hat{x}, \hat{y})$, which implies $\omega_{\hat{x}} = \frac{1}{K}$.

This is an important difference from the case of diffeomorphisms where $\omega_x = \|Df|_{E^s_x}\|$ ([PSW]). Now, using the Lipschitz continuity of $DF$, we can apply the Hölder Section Theorem and get that the unique $F_{\#}$-invariant section $\sigma$ of $\hat{\Lambda} \times Y$ is $\theta$-Hölder, where $\theta$ satisfies

$$\sup_{\hat{x} \in \hat{\Lambda}} \|Df|_{E^s_x}\| \cdot \|Df|_{E^u_{\hat{x}}}\|^{-1} \cdot \left(\frac{1}{K}\right)^{-\theta} < 1.$$

The restriction of this $F_{\#}$-invariant section $\sigma$ to $\widetilde{E}^u_{\hat{x}}(\delta)$ gives exactly the map $\psi_{\hat{x}} \colon \widetilde{E}^u_{\hat{x}}(\delta) \to \widetilde{E}^s_{\hat{x}}$ whose graph is $\exp_x^{-1}(W^u_\delta(\hat{x}))$, and which, consequently depends $\theta$-Hölder on $\hat{x}$. $\square$

**Corollary 2.** *In the setting of Theorem 2, there exists $\delta, \delta' > 0$ and $C > 0$ s.t. if $\xi \in W^u_\delta(\hat{x})$ and $\hat{d}(\hat{x}, \hat{y}) < \delta'$, then*

$$d(\xi, W^u_\delta(\hat{y})) < C \cdot \hat{d}(\hat{x}, \hat{y})^\theta,$$

*where* $\sup_{\hat{x} \in \hat{\Lambda}} \|Df|_{E^s_x}\| \cdot \|Df|_{E^u_{\hat{x}}}\|^{-1} \cdot K^\theta < 1.$

*Proof.* The proof takes into consideration the previous approximation with Lipschitz bundles $\widetilde{E}^u \oplus \widetilde{E}^s$ and the fact that $W^u_\delta(\hat{x}) = \exp_{x_0}(\text{graph } \psi_{\hat{x}})$, where $\psi_{\hat{x}} \colon \widetilde{E}^u_{\hat{x}} \to E^s_{x_0}$ is the restriction of a map which depends $\theta$-Hölder on $\hat{x}$. $\square$

This Corollary will be employed in the proof of Theorem 3.

One can now consider the case of Axiom A holomorphic maps on $\mathbb{P}^2$ and try to "separate" the unstable manifolds thru a given $x \in \mathbb{P}^2$ by taking complete prehistories of their points. We are thus again led to consider hyperbolic covers and properties of the Hausdorff dimension calculated with respect to the metric $\hat{d}$.

*Proof of Theorem 3.* Since $K^- = W^u(\widehat{S}_1) \cup S$ and $S_0$ is finite, it is enough to prove the estimate for $W^u(\widehat{S}_1)$. We will divide the proof into several preliminary steps.

**Step 1.**

The idea is now to show that $HD(W^u(\hat{S}_1)) \leq HD(W^u_\delta(\hat{S}_1), \delta > 0$.

Firstly let us notice that $W^u(\widehat{S}_1) = \bigcup_{\hat{x} \in \widehat{S}_1} W^u(\hat{x})$, and each global unstable set $W^u(\hat{x}) = \bigcup_{n \geq 0} f^n(W^u_\delta(\hat{f}^{-n}\hat{x}))$, for some $\delta > 0$ independent of $\hat{x} \in \widehat{S}_1$.

Therefore $\bigcup_{\hat{x} \in \widehat{S}_1} W^u(\hat{x}) = \bigcup_{\substack{n \geq 0 \\ \hat{x} \in \widehat{S}_1}} f^n(W^u_\delta(\hat{f}^{-n}\hat{x})) = \bigcup_{n \geq 0} f^n(W^u_\delta(\widehat{S}_1))$, where $W^u_\delta(\widehat{S}_1) := \bigcup_{\hat{x} \in \widehat{S}_1} W^u_\delta(\hat{x})$.

Now $HD\left(\bigcup_{n \geq 0} f^n(W^u_\delta(\widehat{S}_1)\right) = \sup_{n \geq 0} HD(f^n(W^u_\delta(\widehat{S}_1)))$.

But $f^n$ is a Lipschitz function, hence

$$HD(f^n W^u_\delta(\widehat{S}_1)) \le HD(W^u_\delta(\widehat{S}_1)).$$

So, $HD(K^-) \le HD(W^u_\delta(\widehat{S}_1))$.

**Step 2.**

Define now the set $V^+_\delta(\hat{x}) := \{\hat{y} \in \widehat{\mathbb{P}}^2 \text{ s.t. } \hat{d}(\hat{f}^{-k}\hat{y}, \hat{f}^{-k}\hat{x}) < \delta, (\forall)k \ge 0\}$,

and $V^-_\delta(\hat{x}) := \{\hat{y} \in \widehat{\mathbb{P}}^2 \text{ s.t. } \hat{d}(\hat{f}^{k}\hat{y}, \hat{f}^{k}\hat{x}) < \delta, (\forall)k \ge 0\}$, for $\hat{x} \in \hat{S}_1$; these are analogues of the unstable/stable manifolds of $f$.

We will show that $V^+_\delta(\hat{x})$ is bi-Lipschitz homeomorphic to $W^u_\delta(\hat{x})$ so it is basically a complex disk and its dimension is equal to 2.

The estimate in the conclusion of the theorem will follow from adding the dimension of $V^+_\delta(\hat{x})$ to the dimension of $V^-_\delta(\hat{x}) \cap \hat{S}_1$ taking also into consideration that $V^+_\delta(\hat{x})$ depends $\theta$-Holder continuously on $\hat{x}$.

In the sequel we are proving all these facts, starting with the laminated structure of $V^+_\delta(\hat{S}_1)$.

First if $\hat{y} \in V^-_\delta(\hat{x})$, then $d(f^k y, f^k x) \le \hat{d}(\hat{f}^k\hat{y}, \hat{f}^k\hat{x}) < \delta, k \ge 0 \Rightarrow \pi(V^-_\delta(\hat{x})) \subset W^s_\delta(x)$.

Let us show that $V^+_\delta(\hat{x}) \cap V^-_\delta(\hat{y})$ has at most one point.

Indeed, if $\exists \hat{\xi}, \hat{\eta} \in V^+_\delta(\hat{x}) \cap V^-_\delta(\hat{y})$, then $\hat{d}(\hat{f}^k\hat{\xi}, \hat{f}^k\hat{\eta}) < 2\delta,\ k \in \mathbb{Z} \Rightarrow d(\xi_k, \eta_k) < 2\delta\ (\forall)k \in \mathbb{Z} \Rightarrow \hat{\xi} = \hat{\eta}$, since $f$ is expansive.

Also, by a local product structure type of argument it is easy to prove that $V^+_\delta(\hat{x}) \cap V^-_\delta(\hat{y}) = \{\hat{z}\} \in \hat{S}_1$ for $\hat{x}, \hat{y}$ sufficiently close.

Hence, if $\hat{y}$ is close enough to $\hat{x}$, $(V^+_\delta(\hat{y}))_{\hat{y}}$ gives a local lamination of $V^+_\delta(\widehat{S}_1)$, indexed by points in $V^-_\delta(\hat{x}) \cap \widehat{S}_1$, which will satisfy our goal of "separating" the local unstable manifolds.

We will show now that $HD(V^+_\delta(\hat{x})) = 2$, and, also, that $\overline{\dim}_B(V^+_\delta(\hat{x})) = 2$, $(\forall)\hat{x} \in \widehat{S}_1$.

First, it is clear that $W^u_\delta(\hat{x}) \supset \pi(V^+_\delta(\hat{x}))$.

Also, if $y \in W^u_\delta(\hat{x})$, we can find a prehistory $\hat{y}$ s.t. $d(y_{-k}, x_{-k}) < \delta,\ k \ge 0$

$$\hat{d}(\hat{f}^{-k}\hat{y}, \hat{f}^{-k}\hat{x}) = d(y_{-k}, x_{-k}) + \frac{d(y_{-k-1}, x_{-k-1})}{K} + \cdots$$

$< \delta + \frac{\delta}{K} + \cdots = \frac{K}{K-1}\delta = K' \cdot \delta$, where $K' = \frac{K}{K-1}$. So, this means that $\hat{y} \in V^+_{K'\cdot\delta}(\hat{x})$, hence:

$$\pi(V^+_\delta(\hat{x})) \subset W^u_\delta(\hat{x}) \subset \pi(V^+_{K'\cdot\delta}(\hat{x})).$$

Also, if $y \in W^u_\delta(\hat{x})$, there exists some $0 < \lambda < 1$ so that $d(y_{-k}, x_{-k}) < \lambda^k \cdot d(y_0, x_0) \Rightarrow d(\hat{y}, \hat{x}) = d(x_0, y_0) + \frac{d(x_{-1}, y_{-1})}{K} + \cdots < d(x_0, y_0) + \frac{\lambda}{K} d(x_0, y_0) + \frac{\lambda^2}{K^2} d(x_0, y_0) + \cdots = d(x_0, y_0) \cdot K''$, for some $K''$ independent of $\hat{x}, \hat{y}$.

Hence $d(x_0, y_0) \le \hat{d}(\hat{x}, \hat{y}) \le d(x_0, y_0) \cdot K''$, so $\pi\colon\ V^+_\delta(\hat{x}) \to \pi(V^+_\delta(\hat{x}))$ is **bi-Lipschitz**.

Therefore $\overline{\dim}_B(V^+_\delta(\hat{x})) \le \overline{\dim}_B(W^u_\delta(\hat{x})) = 2$, and in fact using the properties of upper box dimension it is seen that $\overline{\dim}_B(V^+_\delta(\hat{x})) = 2$.

Also, let us notice that, since $W^u_\delta(\hat{x})$ is a small complex disk which depends continuously on $\hat{x} \in \widehat{S}_1$, and $\widehat{S}_1$ is compact, the way in which we can cover $W^u_\delta(\hat{x})$ with sets of diameter $\varepsilon$ is uniform in $\hat{x}$.

This means that there exists a positive constant $C > 1$ s.t. for any $\varepsilon > 0$, we can cover any given $W^u_\delta(\hat{x})$, $\hat{x} \in \widehat{S}_1$ with the same number $N_\varepsilon$ of sets of diameter $C \cdot \varepsilon$, and $\lim\limits_{\varepsilon \to 0} \frac{\log N_\varepsilon}{\log \frac{1}{\varepsilon}} = 2$.

**Step 3.**

We now concentrate on the Hausdorff dimension of $V^-_\delta(\hat{x}) \cap \widehat{S}_1$, for $\hat{x} \in \widehat{S}_1$, and show that it is less than $t^s$. First, it is obvious that $\pi(V^-_\delta(\hat{x}) \cap \widehat{S}_1) \subset W^s_\delta(x) \cap S_1$.

Take $E$ to be $(n+1, \delta)$-separated in $\hat{f}^{-n} V^-_\delta(\hat{x}) \cap \widehat{S}_1$ and assume that $E$ has maximal cardinality. Hence, $(\forall)\hat{\xi} \in \hat{f}^{-n} V^-_\delta(\hat{x}) \cap \widehat{S}_1$, $\exists \hat{y} \in E$ with $\hat{d}_{n+1}(\hat{\xi}, \hat{y}) < \delta$, otherwise $E \cup \{\hat{\xi}\}$ would also be separated.

So, if $\hat{z} \in V^-_\delta(\hat{x}) \cap \widehat{S}_1 \Rightarrow \exists \hat{y} \in E$ with $\hat{d}_{n+1}(\hat{f}^{-n}\hat{z}, \hat{y}) < \delta$, hence $\hat{d}(\hat{f}^{-n}\hat{z}, \hat{y}) < \delta$, $\hat{d}(\hat{f}^{-n+1}\hat{z}, \hat{f}\hat{y}) < \delta, \dots, \hat{d}(\hat{z}, \hat{f}^n\hat{y}) < \delta$, and since $\hat{z} \in V^-_\delta(\hat{x})$, $\hat{d}(\hat{f}^k\hat{z}, \hat{f}^{k+n}\hat{y}) < \delta$, $k \geq 0$.

So $\hat{f}^{-n}\hat{z} \in V^-_\delta(\hat{y}) \Rightarrow \hat{z} \in \hat{f}^n V^-_\delta(\hat{y})$ for some $\hat{y} \in E$.

This shows that $V^-_\delta(\hat{x}) \cap \widehat{S}_1 = \bigcup\limits_{\hat{y} \in E} \hat{f}^n(V^-_\delta(\hat{y}))$

What can we say about the diameter of $\hat{f}^n V^-_\delta(\hat{y})$?

If $\hat{\xi}, \hat{\eta} \in V^-_\delta(\hat{y})$, then $\xi, \eta \in W^s_\delta(y)$, and $d(f^k\xi, f^k\eta) \leq C \cdot \|Df^k|_{E^s_y}\|$, $k \geq 0$, where $C$ is independent on $k, \hat{y}, \hat{\xi}, \hat{\eta}$. This is proved by the Distortion Lemma used for Theorem 1.

We work with $fs$-hyperbolic, so $C_f \cap S_1 = \emptyset \Rightarrow \exists \lambda' > 0$ s.t. $0 < \lambda' \leq \|Df|_{E^s_y}\| < \lambda < 1$, $y \in S_1$

$$\|Df^n|_{E^s_y}\| = \|Df^{n-1}|_{E^s_y}\| \cdot \|Df|_{E^s_{f^{n-1}_y}}\|$$

$$\|Df^{n-1}|_{E^s_y}\| = \frac{\|Df^n|_{E^s_y}\|}{\|Df|_{E^s_{f^{n-1}_y}}\|} \leq \frac{\|Df^n|_{E^s_y}\|}{\lambda'}.$$

So $d(f^{n-1}\xi, f^{n-1}\eta) \leq C \cdot \frac{\|Df^n|_{E^s_y}\|}{\lambda'}$, and similarly,

$$d(f^k\xi, f^k\eta) \leq C \cdot \frac{\|Df^n|_{E^s_y}\|}{(\lambda')^{n-k}}, \quad k \geq 0.$$

Thus $\hat{d}(\hat{f}^n\hat{\xi}, \hat{f}^n\hat{\eta}) = d(f^n\xi, f^n\eta) + \frac{d(f^{n-1}\xi, f^{n-1}\eta)}{K} + \dots + \frac{d(\xi,\eta)}{K^n} + \frac{d(\xi_{-1},\eta_{-1})}{K^{n+1}} + \dots \leq C \cdot \|Df^n|_{E^s_y}\| \Big(1 + \frac{1}{K\lambda'} + \dots + \frac{1}{(K\lambda')^n}\Big) + \frac{M}{K^n}$, for some constant $M$.

Assume now the following condition:

$$\frac{1}{K} < \lambda'$$

in this case, also $\frac{1}{K^n} < \lambda'^n < \|Df^n|_{E^s_y}\| \Rightarrow \hat{d}(\hat{f}^n\hat{\xi}, \hat{f}^n\hat{\eta}) \leq C' \cdot \|Df^n|_{E^s_y}\|$ for a constant $C' > 0$

$$\Rightarrow \text{diam } \hat{f}^n V^-_\delta(\hat{y}) \leq C' \cdot \|Df^n|_{E^s_y}\|$$

Consequently for a positive number $\alpha$,

$$\sum_{\hat{y}\in E}(\text{diam } \hat{f}^n V_\delta^-(\hat{y}))^\alpha \le (C')^\alpha \cdot \sum_{y\in E} \|Df^n|_{E^s_y}\|^\alpha =$$

$$= (C')^\alpha \cdot \sum_{\hat{y}\in E} e^{\alpha\cdot \ln \|Df^n|_{E^s_y}\|} = (C')^\alpha \cdot \sum_{\hat{y}\in E} e^{\alpha(\ln \|Df|_{E^s_y}\| + \cdots + \ln \|Df|_{E^s_{f^{n-1}_y}}\|)}$$

$$= (C')^\alpha \cdot \sum_{\hat{y}\in E} \exp[\alpha \cdot (\phi^s(y) + \cdots + \phi^s(f^{n-1}y))].$$

But, if $t^s$ is the unique number for which $P(t^s\phi^s) = 0$, and if $t > t^s$, then $P(t\cdot\phi^s) < 0$. Hence if $t > t^s$, $\overline{\lim_{n\to\infty}} \frac{1}{n} \log \left\{ \sup \sum_{\hat{y}\in F} e^{t(\phi^s(y)+\cdots+\phi^s f^n(y))},\ F(n+1,\delta)\text{-separated for } \hat{f} \right\} \le \beta < 0$ for some $\beta < 0$.

Combining this with the above estimate for $\sum_{\hat{y}\in E}(\text{diam } \hat{f}^n V_\delta^-(\hat{y}))^\alpha$ we find that

$$HD(V_\delta^-(\hat{x}) \cap \widehat{S}_1) \le t^s \tag{1}$$

**Final Step.**

We are now ready to prove the final conclusion. Let us take $\delta > 0$ small so that if $\hat{d}(\hat{y},\hat{x}) < \delta$, then $V_\delta^+(\hat{x}) \cap V_\delta^-(\hat{y})$ consists of a single point. So, close to $\hat{x}, V_\delta^+(\widehat{S}_1)$ is laminated by $V_\delta^+(\hat{y})$, with $\hat{y} \in V_\delta^-(\hat{x}) \cap \widehat{S}_1$.

Let us consider now a covering of $V_\delta^-(\hat{x}) \cap \widehat{S}_1$ with sets $U_i$ of diameter $< \varepsilon$, for $\varepsilon > 0$ small, and take a point $\hat{\xi}_i$ in each $U_i$.

Cover $V_\delta^+(\hat{\xi}_i)$ with open balls of diameter equal to diam $U_i$ so that we get $N_{\text{diam } U_i}$ balls in this covering ($N_\varepsilon$ is the least number of sets of diameter $\varepsilon$ needed to cover some $V_\delta^+(\hat{x})$, for some $\varepsilon > 0$).

According to our observation above, $N_\varepsilon$ does not depend on $\hat{x}$. Now, if $\hat{z} \in U_i$ and $\hat{\xi} \in V_\delta^+(\hat{z})$, it will exist a point $\hat{\eta} \in V_\delta^+(\hat{\xi}_i)$ s.t. $\hat{d}(\hat{\xi},\hat{\eta}) \le C \cdot d(\hat{z},\hat{\xi}_i)^\theta$, if $\sup_{\hat{y}\in\widehat{S}_1} \|Df|_{E^s_{\hat{y}}}\| \cdot \|Df|_{E^u_{\hat{y}}}\|^{-1} < \frac{1}{K^\theta}$, from Theorem 2.

So, we will be able to cover $\bigcup_{\hat{y}\in U_i} V_\delta^+(\hat{y})$ with $N_{\text{diam } U_i}$ sets of diameter $\le C\cdot(\text{diam } U_i)^\theta$. Using the fact that $\overline{\dim}_B V_\delta^+(\hat{x}) = \overline{\lim_{\varepsilon\to 0}} \frac{\log N_\varepsilon}{\log \frac{1}{\varepsilon}} = \underline{\dim}_B V_\delta^+(\hat{x}) = \underline{\lim}_{\varepsilon\to 0} \frac{\log N_\varepsilon}{\log\frac{1}{\varepsilon}} = 2$, we see that corresponding to the covering $(U_i)_{i\in I}$ of $V_\delta^-(\hat{x}) \cap \widehat{S}_1$, we get

$$\sum_{i\in I}(\text{diam } U_i)^{\theta t} \cdot N_{(\text{diam } U_i)^\theta} \le C \cdot \sum_{i\in I}(\text{diam } U_i)^{\theta t} \cdot \frac{1}{(\text{diam } U_i)^{(2+\varepsilon')\theta}},$$

for $\varepsilon' > 0$ arbitrary and if $t > 2 + \frac{t^s}{\theta}$, the above goes to 0 if $\varepsilon \to 0$, because we saw above in (1) that $HD(V_\delta^-(\hat{x}) \cap \widehat{S}_1) \le t^s$ if $\frac{1}{K} < \inf_{y'\in S_1} \|Df|_{E^s_{y'}}\|$.

In conclusion, $HD(V_\delta^+(\widehat{S}_1)) \le 2 + \frac{t^s}{\theta}$, given that the two conditions, $\left.\begin{array}{l} \sup_{\hat{y}\in\widehat{S}_1} \|Df|_{E^s_{\hat{y}}}\| \cdot \|Df|_{E^u_{\hat{y}}}\|^{-1} < \frac{1}{K^\theta} \\ \frac{1}{K} < \inf_{y'\in S_1} \|Df|_{E^s_{y'}}\| \end{array}\right\}$ are satisfied.

This in fact works for some $K > 1$ if and only if

$$(*) \qquad \sup_{\hat{y}\in\widehat{S}_1} \|Df|_{E^s_y}\| \cdot \|Df|_{E^u_{\hat{y}}}\|^{-1} < \inf_{x\in S_1} \|Df|_{E^s_x}\|^{\theta}.$$

In particular in the conditions of Corollary 1, it follows that $t^s < 2$, which implies:

$$HD(K^-) < 2 + \frac{1}{\theta}, \text{ therefore if } \theta = 1$$

in condition $(*)$, we get $HD(K^-) < 4$.

Also, this means that, in fact for any $\theta > 0$ with $\theta > \frac{t^s}{2}$, the interior of $K^-$ is empty.

□

**Examples:**

We will give a few examples of s-hyperbolic maps on $\mathbb{P}^2$ for which the conclusion of Theorem 3 applies.

First let us give a general lemma which guarantees that small perturbations of a hyperbolic map without cycles give small perturbations of the pressure of the corresponding stable functions.

Consider $f$ an s-hyperbolic map and $\Lambda$ one of its basic sets. From stability results for hyperbolic maps without cycles, it follows that if $g$ is close enough to $f$ then $g$ has also hyperbolic structure on a set $\Lambda_g$, and there is a homeomorphism $\Phi : \hat{\Lambda}_f \to \hat{\Lambda}_g$ commuting with $\hat{f}$ and $\hat{g}$. It is necessary to consider the hyperbolic covers because in general there is no conjugating homeomorphism between $\Lambda_f$ and $\Lambda_g$.

Also, the set $\Lambda_g$ will be close to $\Lambda_f$.

**Lemma 2.** *Let $f$ an s-hyperbolic map without cycles. Then if $\phi^s_f(x) := log||Df|_{E^s_x}||, x \in \Lambda_f$ and $\phi^s_g(y) := ||Df|_{E^s_y}||, y \in \Lambda_g$, we have that $P_g(\phi^s_g) \to P_f(\phi^s_f)$, when $g \to f$ in the uniform convergence metric.*

*Proof.* Denote by $\hat{\phi}^s_f(\hat{x}) := \phi^s_f(x), \hat{x} \in \hat{\Lambda}_f, x \in \Lambda$. It is not difficult to prove that

$$P_h(\psi) = P_{\hat{h}}(\psi \circ \pi), \tag{2}$$

for all s-hyperbolic endomorphisms $h$, where $\pi : \hat{\Lambda}_h \to \Lambda_h$ is the canonical projection and $\psi$ is a continuous function in $C(\Lambda_h, \mathbb{R})$ [Wa]; in particular this property will be true for $f$ and $g$ .

If $d(f, g) < \varepsilon$, with $\varepsilon$ small enough, there will exist $\delta > 0$, so that

$$(1 + \delta)\hat{\phi}^s \le \hat{\phi}^s_g \circ \Phi \le (1 - \delta)\hat{\phi}^s < 0 \tag{3}$$

where $\phi^s$denotes $\phi^s_f$ and $\hat{\phi}^s$ denotes $\hat{\phi}^s_f$. Here $\Phi : \hat{\Lambda}_f \to \hat{\Lambda}_g$ is the homeomorphism reffered to above. But pressure is invariated by a homeomorphism, so $P_{\hat{f}}(\hat{\phi}^s_g \circ \Phi) = P_{\hat{g}}(\hat{\phi}^s_g)$, hence it is enough to show that $P_{\hat{f}}(\hat{\phi}^s_g \circ \Phi) \to P_f(\phi^s) := P(\phi^s)$.

Let $\mu$ an ergodic invariant measure on $\Lambda$, then by the Variational Principle (Section 1),

$$(1+\delta)h_\mu + (1+\delta)\int \phi^s \, d\mu \le \delta h_\mu + P((1+\delta)\phi^s),$$

therefore again, by the Variational Principle, $(1+\delta)P(\phi^s) \le \delta h_{top} + P((1+\delta)\phi^s)$.

Similary we prove $P((1-\delta)\phi^s) \le (1-\delta)P(\phi^s) + \delta h_{top}$. This together with (2) and (3) implies that

$$(1+\delta)P(\phi^s) - \delta h_{top} \le P((1+\delta)\phi^s) \le P_{\hat{f}}(\hat{\phi}_g^s \circ \Phi) \le P((1-\delta)\phi^s) \le (1-\delta)P(\phi^s) + \delta h_{top} \quad (4)$$

But $h_{top} < \infty$, so by taking $\delta \to 0$ in inequalities (4) we get that $P_{\hat{f}}(\hat{\phi}_g^s \circ \Phi) \to P(\phi^s)$ when $g \to f$.

□

**Example 1.** Let $f[z:w:t] = [P(z:t) : Q(w:t) : t^2]$, where $P, Q$ are polynomials of degree 2 in one variable. Take these polynomials to be hyperbolic on their Julia sets.

Since any quadratic polynomial is conjugated to a polynomial of the form $z^2 + c$, it will be enough to take examples of this form.

Now let us take $P(z) = z^2+c, c \neq 0$ where $c$ is in the interior of the component of the Mandelbrot set corresponding to attracting fixed points. If $c$ is in that component, then $P$ is hyperbolic on its Julia set ([CG], section 8.1).

Consider also $Q(w) = w^2 + d, d \neq 0$, with $d$ in the same component of the Mandelbrot set as $c$. In order to have the critical set outside $S_1$, $c$ and $d$ must be $\neq 0$. Assume also that

$$|\frac{1 \pm \sqrt{1-4c}}{2}| < \frac{1}{2\sqrt{2}}, |\frac{1 \pm \sqrt{1-4d}}{2}| < \frac{1}{2\sqrt{2}}.$$

This condition will guarantee that $P(2\phi^s) < 0$ because it is clear that for our example $P(2\phi^s) = h_{top} + 2log|Df(z_0)|$, with $z_0$ a fixed point and $h_{top}(P) = h_{top}(Q) = log2$.

Hence $t^s < 2$.

Denote the Julia sets of $P, Q$ by $J_P, J_Q$.

Then the basic sets for $S_1$ are in $t = 1$ , {sinks of P} $\times J_Q$ or $J_P \times$ {sinks of Q}; the basic set in $t = 0$ is the Julia set for $[P(z:0) : Q(z:0)]$. For this functin, all the unstable manifolds $W_\varepsilon^u(\hat{x})$ coincide and are contained in complex lines.

However, if we perturb $f$ to an $f_\varepsilon$, the unstable manifolds will not be in general contained in complex lines and so it is not clear a priori if the estimate from Theorem 3 on the Hausdorf dimension is true. Still, on a basic set of the form $\{(z_0, w), z_0$ fixed attracting point of P and $w \in J_Q\}$, $||Df|_{E^s_{(z_0,w)}}|| = |P'(z_0)|$, hence the condition ($\star$):

$$sup_{(z_0,w)}||Df|_{E^s_{(z_0,w)}}|| \cdot ||Df|E^u_{\widehat{(z_0,w)}}||^{-1} < inf_{(z_0,w)}||Df|_{E^s_{(z_0,w)}}||$$

is satisfied with $\theta = 1$.

Now condition ($\star$) will be satisfied also for $f_\varepsilon$, because $S_1(f_\varepsilon)$ is very close to $S_1(f)$ for $\varepsilon$ small (Stability Theorem). And because $P(2\phi^s_f) < 0$, it follows from Lemma 2 that $P(2\phi^s_{f_\varepsilon}) < 0$ as well, hence $t^s_{f_\varepsilon} < 2$. In conclusion, Theorem 3 gives $HD(K^-(f_\varepsilon)) < 4$. In particular $K^-(f_\varepsilon)$ has empty interior too.

**Example 2.** Lastly, we shall provide an example showing that $P(2 \cdot \phi^s)$ can be positive, unlike in the case of diffeomorphisms.

Take the canonical extension of $f(z, w) = (z^2 + c, w^2 + d)$ to the projective space $\mathbb{P}^2$, with $c \neq 0, d \neq 0$. Denote $P(z) = z^2 + c, Q(w) = w^2 + d$ and choose $d \neq 0$ so that $Q$ is hyperbolic.

Consider $\Lambda = \{z_0\} \times J_Q$, a basic set for $f$, with $z_0$ an attractive fixed point for $P$.

Then $h_{top}(f|_\Lambda) = h_{top}(Q)$ =log 2. If the fixed point is $z_0 = \frac{1-\sqrt{1-4c}}{2}$, then the multiplier at this point will be $1 - \sqrt{1-4c}$ and we assume $|1 - \sqrt{1-4c}| < 1$. In this case $P$ is hyperbolic [CG].

Now one can take an ergodic invariant measure $\mu$ on $\Lambda$ of entropy close to log 2 due to the fact that $h_{top}(f|_\Lambda)$ =log 2 (there is in fact an invariant measure of maximal entropy log 2, but we do not need this here).

Obviously $\int 2\phi^s d\mu = log|1 - \sqrt{1-4c}|^2$, so $P(2\phi^s) \geq h_\mu$+log$|1 - \sqrt{1-4c}|^2$. However one can choose $c$ so that $|1 - \sqrt{1-4c}| = \frac{4}{5} < 1$, giving $2 \cdot |1 - \sqrt{1-4c}|^2 = \frac{32}{25} > 1$, hence $P(2\phi^s) > 0$.

Finally let us make the observation that $P(2\phi^s) > 0$ means that $t^s > 2$; however in our example HD$(W^s_\varepsilon(x) \cap \Lambda) = 0$, so in Theorem 1 one does not have equality in general.

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Eugen Mihailescu's email address: Eugen.Mihailescu@math.tamu.edu

Address: Department of Mathematics, Texas A&M University
College Station, TX 77843-3368, USA