BRITTLE CRACK PROPAGATION BASED ON AN OPTIMAL ENERGY BALANCE

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Abstract. We propose a model of quasistatic brittle crack propagation based on the assumption that the damage of the body occurs with minimal loss of energy communicated to the body by the rest of the universe.

1. Introduction

We describe the state of a brittle body by a pair $(\mathbf{v}, S)$. $\mathbf{v}$ stands for the deformation of the body relative to the reference configuration Ω and S is the *active* crack, represented as a union of surfaces in the (closure of) Ω. The *damaged region* K is a variable which takes account on the history of fisuration (hence it is not a state variable). For the physical meaning of K and the necessity to introduce K as a supplementary variable different from S, we refer to Buliga [3].

The deformation $\mathbf{v}$ has to be compatible with the *active* crack , i.e. $\mathbf{v}$ is C^1 outside the surface S. A pair $(\mathbf{v}, S)$ is admissible (with respect to the imposed boundary deformation $\mathbf{u}^0$) if $\mathbf{v} = \mathbf{u}^0$ on $\partial\Omega \setminus (S \cup K)$.

Brittle fracture hypothesis. *The potential energy of the body subject to the boundary deformation* $\mathbf{u}^0$ *depends only on the state of the body* $(\mathbf{v}, S)$ *and it has the expression:*

$$E(\mathbf{v}, S; K, \mathbf{u}^0) \ = \ \int_{\Omega\setminus K} w(\nabla \mathbf{v}) \ dx \ + \ F(S; K, \mathbf{u}^0) \quad . \tag{1}$$

The first term of this functional is the free energy associated to the deformation $\mathbf{v}$*; the second term represents the energy needed to produce the active crack* S*, with the boundary deformation* $\mathbf{u}^0$ *and the damaged region* K *as parameters.*

The most simple form of the function F is:

$$F(S; K, \mathbf{u}^0) \ = \ Const. \ \cdot \ Area \ (S \setminus K) \quad ,$$

that is the energy consumed to create the crack S, in the damaged body $\Omega \setminus K$, is proportional, through a material constant, to the area of $S \setminus K$. With this choice

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of F the potential energy becomes a Mumford-Shah functional (see [4]). From this type of functionals new variational energetic models of brittle crack propagation are emerging in the last years (like [4] , Ambrosio & Braides [1] or Buliga [3]).

The brittle fracture model that we propose is based on two hypotheses:

H1. the state $(\mathbf{u}, S)$ of the body in the presence of the damaged region K and imposed boundary deformation $\mathbf{u}^0$ minimizes the potential energy $E(\mathbf{v}, S'; K, \mathbf{u}^0)$,

H2. the evolution of the triplet $(\mathbf{u}_t, S_t, K_t)$ minimizes the difference between the energy communicated by the rest of the universe to the body minus the elastic energy accumulated in the body.

2. Stable initial configurations

A more convenient setting for the space of states is to consider as a state of the brittle body an element $\mathbf{u} \in \mathbf{SBV}(\Omega) \cap L^\infty(\Omega)$ and identify $\mathbf{u}$ with $(\mathbf{u}, \overline{S_{\mathbf{u}}})$. For notations and properties of the space of functions with bounded variations $\mathbf{SBV}$ we refer to Ambrosio, Fusco & Pallara [2].

As long as we restrain to the quasistatic fracture propagation, it is natural to suppose that (at least) the initial state of the body is stable. The notion of stability that we choose to use is the following: *the state* $\mathbf{u}$ *of the body subject to the given boundary displacement* $\mathbf{u}^0$ *is stable if it minimizes the total energy functional (1) in the class of all admissible pairs* $\mathbf{v}$ *close enough to the stable state.* Close enough means close in the $\mathbf{SBV}$ norm

$$\|\mathbf{u}\|_\infty \ = \ \|\mathbf{u}\|_{L^\infty}(1 + 2\mathcal{H}^{n-1}(S_{\mathbf{u}})) + \|\nabla \mathbf{u}\|_{L^1} \quad .$$

We shall suppose that w is a positive C^2 function, such that $w(I_n) = 0$. From the stability assumption we derive by classical arguments that w has to be quasiconvex in any $\nabla \mathbf{u}(x)$, $x \in \Omega \setminus S$.

We shall admit that the function F has the properties:

h1) is sub-additive

$$\forall A, B \subset \Omega \ , \quad F(A \cup B; K, \mathbf{u}^0) \ \leq \ F(A; K, \mathbf{u}^0) \ + \ F(B; K, \mathbf{u}^0) \quad ,$$

h2) for any $A \subset \Omega$, such that $F(A; K, \mathbf{u}^0) < +\infty$, for any $x \in \Omega$ and $r > 0$, let us denote by $o(x, r)$ the homothety of center x and coefficient r; we have then

$$F(o(x,r)(A) \cap \Omega; K, \mathbf{u}^0) \ \leq \ r^{n-1} F(A; K, \mathbf{u}^0) \quad .$$

Let us denote by $(\mathbf{u}, \Sigma)$ a stable equilibrium state of the body. There are three straightforward consequences of this assumption.

Consider first the class of all admissible pairs $(\mathbf{v}, S)$ such that $S = \Sigma$. We have then for any $\mathbf{v}$, $\mathbf{v} = \mathbf{u}^0$ on $\partial\Omega \setminus (K \cup \Sigma)$, $\mathbf{v} \in C^1(\Omega \setminus (K \cup \Sigma))$,

$$\int_\Omega w(\nabla \mathbf{u}) \, \mathrm{d}x \ \leq \ \int_\Omega w(\nabla \mathbf{v}) \, \mathrm{d}x \quad . \tag{2}$$

The (first order Piola-Kirchoff) stress tensor is defined by:

$$\sigma(\mathbf{u}) \ = \ \frac{\partial w(\mathbf{F})}{\partial \mathbf{F}}(\nabla \mathbf{u}) \ .$$

The variational inequation for $\mathbf{u}$ (2) implies that in the sense of distributions we have:

$$div \ \sigma(\mathbf{u}) \ = \ 0$$

and that on the crack $\Sigma \setminus K$ we have

$$\sigma(\mathbf{u})^+ \mathbf{n} \ = \ \sigma(\mathbf{u})^- \mathbf{n} \ = 0 \ ,$$

where the signs $+$ and $-$ denotes the lateral limits of $\sigma(\mathbf{u})$ with respect to the field of normals $\mathbf{n}$.

A second consequence of the stability assumption is obtained by making smooth variations of the pair $(\mathbf{u}, \Sigma)$, which preserve K.

The set of all diffeomorphisms ϕ of Ω which are equal to the identity map near $\partial\Omega$ is denoted by $\mathcal{D}$; it is obviously non void because it contains at least the identity map id_Ω. Remark also that it is a group with respect to function composition, denoted in this paper by a dot "." .

If $(\mathbf{v}, S)$ is an admissible pair and $\phi \in \mathcal{D}$, such that $\phi(K) = K$, then $(\mathbf{v}.\phi^{-1}, \phi(S))$ is admissible too. Indeed, it is true that $\mathbf{v}.\phi^{-1} \in C^1(\Omega \setminus \phi(S))$, and on $\partial\Omega$ we have $\mathbf{v}(\phi^{-1}(x)) \ = \ \mathbf{v}(x)$. The stability assumption then implies:

$$E(\mathbf{u}, \Sigma; K, \mathbf{u}^0) \ \leq \ E(\mathbf{u}.\phi^{-1}, \phi(\Sigma); K, \mathbf{u}^0) \quad \forall \phi \in \mathcal{D} \ . \tag{3}$$

We may use (3) in order to derive a first variation inequality. Recall that for any C^∞ vector field η on Ω there is an unique associated flow, $\phi : I \times \Omega \to \Omega, \phi_0 = id_\Omega$, where $I \subset R$ is an open neighborhood of $0 \in R$. The vector field $\eta = 0$ generates the constant flow $\phi_t \ = \ id_\Omega$; if η has compact support in Ω then the associated flow $t \mapsto \phi_t$ is a curve in $\mathcal{D}$. In order to have K preserved we impose that $\eta = 0$ on K.

For simplicity, in the sequel we shall make the change of notations: Ω will mean $\Omega \setminus K$ and Σ will mean $\Sigma \setminus K$.

For any smooth vector field η (3) implies then:

$$\frac{d}{dt}_{|t=0} \ F(\phi_t(\Sigma); K, \mathbf{u}^0) \ = \ - \ \frac{d}{dt}_{|t=0} \ \int_\Omega w(\nabla(\mathbf{u}.\phi_t^{-1})) \, \mathrm{d}x \ . \tag{4}$$

Let us compute the right term of (4). For any vector field η with *supp* $\eta \subset\subset \Omega$, let us define, for any $x \in \Sigma$:

$$\lambda(x) \ = \ \eta(x) \cdot \mathbf{n}(x) \ \ , \ \ \eta^T(x) \ = \ \eta(x) - \lambda)(x)\mathbf{n}(x) \quad ,$$

where $\mathbf{n}$ is a fixed field of normals over Σ.

We have supposed that Σ is a piecewise C^1 surface and that Σ is closed. The edge of the crack Σ is denoted by $d\Sigma$. We shall denote further by $B_r(d\Sigma)$ the tubular neighborhood of radius r of $d\Sigma$ and by $[f] = f^+ - f^-$ the jump of the function f over the surface Σ with respect to the field of normals $\mathbf{n}$.

We use the information previously gained to obtain:

$$- \frac{d}{dt}_{|t=0} \int_\Omega w(\nabla(\mathbf{u}.\phi_t^{-1})) \ \mathrm{d}x \ = \ \int_\Sigma [w(\nabla \mathbf{u})] \lambda \ \mathrm{d} \ \mathcal{H}^{n-1} \ +$$

$$+ \lim_{r \rightarrow 0} \int_{\partial B_r(d\Sigma)} \left\{ [w(\nabla \mathbf{u})]\lambda \ - \ [\sigma(\mathbf{u})_{ij} (\nabla \mathbf{u})_{ik}] \eta_k \mathbf{n}_j \right\} \ \mathrm{d} \ \mathcal{H}^{n-1} \quad . \tag{5}$$

We may think at three kind of variations: *a)* for $\eta = 0$ on Σ; in this case we have $\phi_t(\Sigma) = \Sigma$ for any t, *b)* for $\eta = \lambda \mathbf{n}$ on $\Sigma \setminus K$ (normal variations), $\lambda : \Sigma \rightarrow R$, *c)* for $\eta \cdot \mathbf{n} = 0$ on $\Sigma \setminus K$ (tangent variations).

In the case a) (4) gives no new information. In the case b), if we suppose moreover that $\eta = 0$ on $d\Sigma$ and $F(\Sigma; K, \mathbf{u}^0) = \mathcal{H}^{n-1}(\Sigma \setminus K)$ then we obtain:

$$\int_\Sigma \left\{ [w(\nabla \mathbf{u})] \ + H \right\} \lambda \ \mathrm{d}\mathcal{H}^{n-1} \ = \ 0 \quad ,$$

where $H = -div_s \mathbf{n} = - \ div \ \mathbf{n} \ + \mathbf{n}_{i,j} \mathbf{n}_i \mathbf{n}_j$ is the curvature of the surface Σ.

In the case c) we obtain

$$\frac{d}{dt}_{|t=0} F(\phi_t(\Sigma); K, \mathbf{u}^0) \ = \tag{6}$$

$$\lim_{r \rightarrow 0} \int_{\partial B_r(d\Sigma)} \left\{ [w(\nabla \mathbf{u})]\lambda \ - \ [\sigma(\mathbf{u})_{ij} (\nabla \mathbf{u})_{ik}] \eta_k \mathbf{n}_j \right\} \ \mathrm{d}\mathcal{H}^{n-1} \ .$$

Let us consider the case $n = 2$ and the function

$$F(S; \mathbf{u}^0) \ = \ G \ \mathcal{H}^1(S) \quad ,$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, i.e. the length measure. Let us suppose, for simplicity, that the crack Σ from the stable state $(\mathbf{u}, \Sigma)$ has only one edge, i.e. $d\Sigma = \{x_0\}$. Let us choose a vector field η with compact support in Ω such that η is tangent to Σ. The equality (6) becomes then

$$G \ \eta(x_0) \cdot \tau(x_0) \ = \ \lim_{r \rightarrow 0} \int_{\partial B_r(x_0)} \left\{ [w(\nabla \mathbf{u})] \eta \cdot \mathbf{n} \ - \ [\sigma(\mathbf{u})_{ij} (\nabla \mathbf{u})_{ik}] \eta_k \mathbf{n}_j \right\} \ \mathrm{d}\mathcal{H}^{n-1} \ ,$$

where $\tau(x)$ is the unitary tangent in $x \in \Sigma$ at Σ. If we suppose moreover that the crack Σ is straight near x_0, and the material coordinates are chosen such that near

x_0 we have $\eta(x) = \tau(x) = (1,0)$, then the equality (6) takes the form:

$$G = \lim_{r\to 0} \int_{\partial B_r(x_0)} \{[w(\nabla \mathbf{u})]\mathbf{n}_1 - [\sigma(\mathbf{u})_{ij}(\nabla \mathbf{u})_{i1}]\mathbf{n}_j\} \, \mathrm{d}\mathcal{H}^{n-1} . \tag{7}$$

We recognize in the right term of (7) the integral J of Rice; therefore at the edge of the crack the integral J has to be equal to the constant G, interpreted as the constant of Griffith.

The equality (7) tells us that at the edge of the crack prescribed by the energetic model the Griffith criterion is fulfilled. It is to be remarked that the equality (7) is based on the fact that the edge of the crack exists, i.e. that Σ, as a surface with boundary, has a part of it's boundary inside the body; otherwise the equality (6), the source of (7), becomes simply the trivial relation $0 = 0$. Notice that we have supposed that the damaged region K is void; if $S \subset K$ then again we obtain from (7) a trivial equality.

Let us go back to the older notations and suppose that K is itself a reunion of surfaces and that

$$F(S;K,\mathbf{u}^0) = F(S \cup K; K, \mathbf{u}^0) \ . \tag{8}$$

Consider a vector field η with associated flow ϕ_t such that $K \subset \phi(K)$. If we replace in the previous consideration Σ by $\Sigma \cup K$, we find that (6) becomes an inequality:

$$\frac{d}{dt}_{|t=0} F(\phi_t(\Sigma \cup K); K, \mathbf{u}^0) \ \geq \tag{9}$$

$$\lim_{r\to 0} \int_{\partial B_r(d\Sigma \cup K)} \{[w(\nabla \mathbf{u})]\lambda - [\sigma(\mathbf{u})_{ij}(\nabla \mathbf{u})_{ik}]\eta_k \mathbf{n}_j\} \, \mathrm{d}\mathcal{H}^{n-1} .$$

The interpretation of this fact (evident in the 2-dimensional case) is the following: if K is a crack set and the state $(\mathbf{u}, K)$ is stable then at the "edges" of K the generalized integral of Rice is at most equal to the first variation of the energy F.

The third consequence of the stability assumption is particularly important for our goals. This consequence is obtained by comparison of the energy of the equilibrium state $(\mathbf{u}, \Sigma)$ with other particular admissible pairs.

Let $x_0 \in \Omega$ be a fixed point and $r > 0$. We construct the following admissible pair $(\mathbf{v}_r, S_r)$:

$$\mathbf{v}_r(x) = \begin{cases} \mathbf{u}(x) & \text{if } x \in \Omega \setminus B(x_0, r) \\ x + \mathbf{u}(x_0) & \text{if } x \in \Omega \cap B(x_0, r) \end{cases} ,$$

We have then:

$$S_r = (\Sigma \setminus B(x_0, r)) \cup \partial B(x_0, r) \ .$$

We have the inequality $E(\mathbf{u}, K; \mathbf{u}^0) \leq E(\mathbf{v}_r, S_r; \mathbf{u}^0)$, for any $r > 0$. We use the properties of F to deduce that there is a positive number C such that for any $x_0 \in \Omega$

and $r > 0$ we have :

$$\int_{\Omega \cap B(x_0,r)} w(\nabla \mathbf{u}) \, \mathrm{d}x \ \leq \ C \, r^{n-1} \quad . \tag{10}$$

This inequality lead us to the following energy concentration property for $\mathbf{u}$:

$$\limsup_{r \to 0} \frac{\int_{\Omega \cap B(x_0,r)} w(\nabla \mathbf{u}) \, \mathrm{d}x}{r^{n-1}} \ \leq \ C \ < +\infty \tag{11}$$

For any $A \subset \Omega$ and any $r > 0$ we denote by $B(A,r)$ the set

$$B(A,r) \ = \ \cup \{ B(x,r) \mid x \in A \} \cap \Omega \quad .$$

Let us consider the functions, defined for any closed set $M \subset \Omega$:

$$CM(\mathbf{u}, \Sigma)(B) \ = \ \limsup_{r \to 0} \frac{\int_{B(d\Sigma \cap B, r) \cap \Omega} w(\nabla \mathbf{u}) \, \mathrm{d}x}{r} \quad ,$$

$$CF(\Sigma; K, \mathbf{u}^0)(B) \ = \ \limsup_{r \to 0} \frac{F(\partial B(dK \cap B, r); \mathbf{u}^0)}{r} \quad .$$

These functions are sub-auditive and have the $n-1$ density finite. With their aid we can write, for a stable state $(\mathbf{u}, \Sigma)$, the inequality, for any closed set B:

$$CM(\mathbf{u}, \Sigma)(B) \ \leq \ CF(\Sigma; K, \mathbf{u}^0)(B) \tag{12}$$

The relation (11) expresses the fact that the energy behaves near the edge of the crack like the distance from this edge, denoted further by r. The limsup term from the relation (11) is the concentration factor of the elastic energy. It is well known that in the case of linear elasticity, if $(\mathbf{v}, S)$ is a pair displacement-crack such that $div \ \sigma(\mathbf{v}) \ = \ 0$ outside S and $\sigma(\mathbf{v})^+ \mathbf{n} \ = \ \sigma(\mathbf{v})^- \mathbf{n} \ = \ 0$ on S then $\mathbf{v}$ behaves like $\sqrt{r}$ near the edge of the crack, hence the elastic energy behaves like r. That is the reason for which we like to see, for a given function w, the function $CM(u,K)(B)$ as a concentration coefficient of the elastic energy inside the set B. We shall prefer therefore to work with energy concentration coefficients instead of stress concentration coefficients.

In the bidimensional case, if the edge of the crack $\Sigma \cup K$ does not belong to B then $CM(\mathbf{u}, \Sigma)(B)$ equals zero and otherwise equals the elastic energy concentration in the edge. Similarly, if F equals a material constant (like Griffith's constant) times the length of the crack set, then $CF(\Sigma; K, \mathbf{u}^0)(B)$ equals 0 if the edge of the crack does not belong to B, otherwise equals the material constant. Therefore, the inequality (12) is equivalent in the bidimensional case, with the inequality contained in the Irwin fracture criterion, expressing the fact that the energy concentration in the edge of the crack is less or equal to a material constant.

3. The optimal energy balance

In this section we consider a given curve $t \mapsto K_t, \mathbf{u}_t^0$. We are interested in the formulation of the energy balance during a *stable* evolution of the brittle body $t \mapsto \mathbf{u}(t)$. We shall suppose that all time derivatives present further make sense.

The energy given to the body by the rest of the universe in the interval $[0,T]$ is

$$\mathcal{T} = \int_0^T \int_{\partial\Omega} \frac{\partial w}{\partial \mathbf{F}_{ij}}(\nabla \mathbf{u}(t)) \dot{\mathbf{u}}_i^0 \mathbf{n}_j \, \mathrm{d}\mathcal{H}^{n-1} \, \mathrm{d}t$$

In the same interval of time the amount of elastic energy of the body is

$$\mathcal{E} = \int_0^T \frac{d}{dt} \int_{\Omega \setminus K_t} w(\nabla \mathbf{u}(t)) \, \mathrm{d}x \, \mathrm{d}t = \int_{\Omega \setminus K_T} w(\nabla \mathbf{u}(T)) \, \mathrm{d}x - \int_{\Omega \setminus K_0} w(\nabla \mathbf{u}(0)) \, \mathrm{d}x$$

Proposition 3.1. *If for any t K_t is a crack set, the relation (8) is satisfied and we have for $t \leq t'$ $K_t \cup S_t \subset K_{t'} \cup S_{t'}$, then for any time interval the total energy $\mathcal{T}$ is greater then the elastic energy $\mathcal{E}$.*

Proof. Because $\mathbf{u}(t)$ is stable we have the equality:

$$\int_{\partial\Omega} \frac{\partial w}{\partial \mathbf{F}_{ij}}(\nabla \mathbf{u}(t)) \dot{\mathbf{u}}_i^0 \mathbf{n}_j \, \mathrm{d}\mathcal{H}^{n-1} = \int_{\Omega \setminus K_t} \frac{\partial w}{\partial \mathbf{F}}(\nabla \mathbf{u}(t)) \cdot \nabla \dot{\mathbf{v}}(t) \, \mathrm{d}x \tag{13}$$

for any field $\mathbf{v}(t+\tau)$ such that $(\mathbf{v}(t+\tau), S_t)$ is admissible relative to $K_{t+\tau}, \mathbf{u}^0(t+\tau)$. The field $\mathbf{v}$ can be chosen such as $\mathbf{v}(t) = \mathbf{u}(t)$. The right-handed member of the equality (13) can be written as:

$$\int_{\Omega \setminus K_t} \frac{\partial w}{\partial \mathbf{F}}(\nabla \mathbf{u}(t)) \cdot \nabla \dot{\mathbf{v}}(t) \, \mathrm{d}x = \int_{\Omega \setminus K_t} \frac{d}{d\tau} w(\nabla \mathbf{v}(t+\tau))_{|\tau=0} \, \mathrm{d}x \tag{14}$$

Again from the stability assumption and (8) we have, for small τ, the inequality:

$$\int_{\Omega \setminus K_{t+\tau}} w(\nabla \mathbf{u}(t+\tau)) \, \mathrm{d}x \leq \int_{\Omega \setminus K_t} w(\nabla \mathbf{v}(t+\tau)) \, \mathrm{d}x \tag{15}$$

From (15), (14) and (13) we obtain the inequality:

$$\frac{d}{dt} \int_{\Omega \setminus K_t} w(\nabla \mathbf{u}(t)) \, \mathrm{d}x \leq \int_{\partial\Omega} \frac{\partial w}{\partial \mathbf{F}_{ij}}(\nabla \mathbf{u}(t)) \dot{\mathbf{u}}_i^0 \mathbf{n}_j \, \mathrm{d}\mathcal{H}^{n-1}$$

which, combined with the definitions of $\mathcal{T}$ and $\mathcal{E}$, resumes the proof. □

In [3], section 3.3, we have noticed that if $K_t = S_t$ then the difference between $\mathcal{T}$ and $\mathcal{E}$ represents the energy consumed for damaging the body. The optimal energy balance principle, that is the assumption H2 from the introduction, states that the damage of the body occurs with minimal loss of energy communicated to the body by the rest of the universe.

The model that we propose is presented further.

1. The input of the model consists in the initial crack K_0 and the path of imposed boundary deformations $t \mapsto \mathbf{u}^0(t)$.

2. To a curve of crack sets $t \mapsto K_t$, such that $K_t \subset K_{t'}$ if $t \leq t'$ and K_0 agrees with the initial crack set, we associate a curve $t \mapsto (\mathbf{u}_t, S_{\mathbf{u}_t})$ admissible at any moment with respect to $K_t, \mathbf{u}^0(t)$, which minimizes the potential energy $E(\mathbf{v}, S; K_t, \mathbf{u}^0(t))$.
3. The curves $t \mapsto \mathbf{u}_t$ and $t \mapsto K_t$ minimize for any time interval the difference $\mathcal{T} - \mathcal{E}$.

Although it is not among the purposes of this paper, we mention that for any curve $t \mapsto K_t$, there is an associated curve $t \mapsto \mathbf{u}_t$ which satisfies the item 2. from above. We don't know if there are any curves satisfying the item 3; we leave this for further study.

The following proposition is a straight consequence of proposition 3.1 and the considerations from the preceding section connected to the third consequence of the stability assumption (see relation (12)).

Proposition 3.2. *If the curves $t \mapsto \mathbf{u}_t$ and $t \mapsto K_t$ are solutions of the model and we are in the hypothesis of proposition 3.1 then the curves $t \mapsto \mathbf{u}_t$ and $t \mapsto K_t \cup S_{\mathbf{u}_t}$ are also solutions of the model. Moreover, at any moment t we have the inequality:*

$$CM(\mathbf{u}_t, S_{\mathbf{u}_t} \cup K_t)(B) \ \leq \ CF(S_{\mathbf{u}_t} \cup K_t; K, \mathbf{u}^0)(B) \tag{16}$$

for any closed set B.

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