Dilatation structures: geometric and algebraic aspects of differential analysis in metric spaces

Marius Buliga*

Purpose of the visit. To have 1 month time of working and popularizing the ideas described further in a wonderful place for doing research. Discussing these with M. Gromov would be great for me, as well as interacting with people from Paris 11. Using the stay in Paris as a base for short trips to Nancy (for collaboration with W. Bertram), as well as possible other contacts concerning other subjects of research that I currently pursue, for example with LMT Cachan concerning continuum media mechanics, LMS Lille I or LMA Marseille concerning the theory of bipotentials.

Background. The point of view that dilatations can be taken as fundamental objects which induce a differential calculus is relatively well known. The idea is simple: in a vector space $\mathbb{V}$ define the dilatation based at x and of coefficient $\varepsilon > 0$ as the function which associates to y the value $\delta^x_\varepsilon y \ = \ x + \varepsilon(y - x)$. Then for a function $f : \mathbb{V} \to \mathbb{W}$ between vector spaces $\mathbb{V}$ and $\mathbb{W}$ we have:

$$\left(\delta^{f(x)}_{\varepsilon^{-1}} f \delta^x_\varepsilon\right)(u) \ = \ f(x) + \frac{1}{\varepsilon}\left[f(x + \varepsilon(u - x)) - f(x)\right] \quad ,$$

thus the directional derivative of f at x, along $u - x$ appears as:

$$f(x) \ + \ D\, f(x)(u - x) \ = \ \lim_{\varepsilon \to 0} \left(\delta^{f(x)}_{\varepsilon^{-1}} f \delta^x_\varepsilon\right)(u) \quad .$$

Pansu introduced the first really new example of such a differential calculus based on other than usual dilatations: the ones which are associated to a Carnot group. He proved in [17] the potential of what is now called Pansu derivative, by providing an alternative proof of a Margulis rigidity type result, as a corollary of the Rademacher theorem for Lipschitz functions on Carnot groups. The challenge to extend Pansu results to general regular sub-riemannian manifolds, taken by Margulis, Mostow [14] [15], Vodopyanov [18] and others, is difficult because on such general metric space there is no natural underlying algebraic structure, as in the case of Carnot groups, where we have the group operation as a non commutative replacement of the operation of addition in vector spaces.

Another line of research concerns the differential calculus over general base fields and rings, Bertram, Glöckner and Neeb [2]. As the authors explain, it is possible to construct such a differential calculus without using the specific properties of the base field (or ring). They point

*"Simion Stoilow" Institute of Mathematics of the Romanian Academy, PO BOX 1-764,014700 Bucharest, Romania, e-mail: Marius.Buliga@imar.ro

out that differential calculus (integral calculus not included) seems to be a part of analysis which is completely general, based only on elementary results in linear algebra and topology. This differential calculus is a generalization of "classical" calculus in topological vector spaces over general base fields, and even over rings. The operation of vector addition is therefore abelian, modifications being made in relation with the multiplication by scalars.

In fact this is only a part of the nonsmooth calculus encountered in geometric analysis on metric spaces. For a survey see the paper by Heinonen [13]. The objects of interest in nonsmooth calculus as described by Heinonen are spaces of homogeneous type, or metric measured spaces where a generalization of Poincaré inequality is true. In such spaces the differential calculus goes a long way: Sobolev spaces, differentiation theorems, Hardy spaces. It is noticeable that in such a general situation we don't have enough structure to define differentials, but only various constructions corresponding to the norm of a differential of a function. Nevertheless see the remarkable result of Cheeger [8], who proves that to a metric measure space satisfying a Poincaré inequality we can associate an L^∞ cotangent bundle with finite dimensional fibers. Other important works are David, Semmes [9], where spaces with arbitrary small neighborhoods containing similar images of the whole space are studied, or David, Semmes [10], where they study rectifiability properties of subsets of $\mathbb{R}^n$ with arbitrary small neighbourhoods containing "big pieces of bi-Lipschitz images" of the whole subset.

Dilatation structures on metric spaces. Introduced in [3], they describe the approximate self-similarity properties of a metric space. A dilatation structure is a notion related, but more general, to groups and differential structures.

The basic objects of a dilatation structure are dilatations (or contractions, or homotheties). The axioms of a dilatation structure set the rules of interaction between different dilatations. A dilatation structure induces a metric tangent bundle with group operations in each fiber (tangent space to a point), which make it (the tangent space) into a conical group. Conical groups generalize Carnot groups. The affine geometry of conical groups was then studied in [4], in the spirit of Bertram "affine algebra", which is commutative according to our point of view. In such a geometry incidence relations are no longer relevant, being replaced by algebraic axioms concerning dilatations.

We tried in [5] to find an intrinsic frame in which sub-Riemannian geometry would be a model, inspired mainly by the last section of the paper by Bellaïche [1] and the intrinsic point of view of Gromov [12]. In [5] it is shown that regular sub-riemannian manifolds admit dilatation structures constructed via normal frames. In that paper we tried to minimize the contribution of classical differential calculus in the proof of the basic results in sub-riemannian geometry. In [7] we introduce length dilatation structures on metric spaces, tempered dilatation structures and coherent projections and explore the relations between these objects and the Radon-Nikodym property and Gamma-convergence of length functionals. Then we show that the main properties of sub-riemannian spaces can be obtained from pairs of length dilatation structures, the first being a tempered one and the second obtained via a coherent projection. Thus we get an intrinsic, synthetic, axiomatic description of sub-riemannian geometry, which transforms the classical construction of a Carnot-Carathéodory distance on a regular sub-riemannian manifold into a model for this abstract sub-riemannian geometry.

In [6] we are concerned with dilatation structures on ultrametric spaces. The special case considered is the boundary of the infinite dyadic tree, that is the space of infinite words over the alphabet $X = \{0,1\}$. Self-similar dilatation structures are introduced and studied on this space. We show that on the boundary of the dyadic tree, any self-similar dilatation structure is described by a web of interacting automata.

Purpose of the project. The main idea is that dilatations are really fundamental objects, not only for defining a notion of derivative, but as well for all algebraic structures that we may need. In [4] we have found an algebraic characterization of a conical group, in terms of the properties of the associated dilatation structure. We intend to continue this program along the following lines.

A Lie group is a group endowed with a compatible differential structure. It is also a particular case of symmetric space in the sense of Loos [16]. By combining Loos ideas with dilatation structures it is possible to describe symmetric spaces (in a generalized sense) as dilatation structures with an algebraic property satisfied by the approximative "inv" operators induced by the dilatations (work in preparation).

Further on, as a metric space is nothing but a particular example of a normed groupoid (think about the trivial pair groupoid $X \times X$ and about the distance function $d : X \times X \rightarrow [0, +\infty)$ as being a norm defined on the arrows of the groupoid), it is possible to describe sub-riemannian geometry as the geometry of normed groupoids endowed with dilatation structures (work in progress). This is non-trivial because one may consider other groupoids with norms which have not much in common with a distance function. For example one may consider a group G acting on a space X and the associated action groupoid which has as arrows pairs $(g, x) \in G \times X$. A norm on this groupoid is then a function $d : G \times X \rightarrow [0, +\infty)$. Combining these ideas with the ones concerning coherent projections and abstract sub-riemannian geometry from [7], we might ask what is a sub-riemannian geometry on a normed groupoid? Simpler question: when is a dilatation structure on a normed groupoid linear in the sense of dilatation structure?

The work of Bertram and collaborators concerning generalized projective geometries is a natural development of the ideas concerning generalized affine algebras. This too has a straightforward continuation from the viewpoint of dilatation structures, which is worthy to explore.

There are already a lot of details and collateral work concerning dilatation structures which were left aside in the articles published or submitted. I think it is a good time to start working on a monograph dedicated to the subject.

References

[1] A. The tangent space in sub-riemannian geometry, in: Sub-Riemannian Geometry, A. Bellaïche, J.-J. Risler eds., Progress in Mathematics, **144**, Birchuser, (1996), 1-78

[2] W. Bertram, H. Glöckner, K.-H. Neeb, Differential calculus over general base fields and rings, *Expo. Math.*, **22** (2004) 213-282

[3] M. Buliga, Dilatation structures I. Fundamentals, *J. Gen. Lie Theory Appl.*, **1** (2007), No. 2, 65-95

[4] M. Buliga, Infinitesimal affine geometry of metric spaces endowed with a dilatation structure (2008), to appear in *Houston Journal of Math.*, http://arxiv.org/abs/0804.0135

[5] M. Buliga, Dilatation structures in sub-riemannian geometry, (2007), to appear in "Contemporary Geometry and Topology and Related Topics. Cluj-Napoca, August 19-25, 2007", http://arxiv.org/abs/0708.4298

[6] M. Buliga, Self-similar dilatation structures and automata (2007), Proceedings of the 6-th Congress of Romanian Mathematicians, June 28 - July 4, 2007, Bucharest, Romania, (in press), http://fr.arxiv.org/abs/0709.2224 (2007)

[7] M. Buliga, A characterization of sub-riemannian spaces as length dilatation structures constructed via coherent projections, (2008), available as e-print http://arxiv.org/abs/0810.5042

[8] J. Cheeger, Differentiability of Lipschitz functions on metric measure spaces, *Geom. Funct. Anal.*,**9** (1999), no. 3, 428-517

[9] G. David, S. Semmes, Fractured fractals and broken dreams. Self-similar geometry through metric and measure. Oxford Lecture Series in Mathematics and its Applications, 7. The Clarendon Press, Oxford University Press, New York, (1997)

[10] G. David, S. Semmes,, Analysis of and on uniformly rectifiable sets. Mathematical Surveys and Monographs, 38. American Mathematical Society, Providence, RI, (1993)

[11] M. Gromov, Groups of polynomial growth and expanding maps (with an appendix by Jacques Tits). *Inst. Hautes Études Sci. Publ. Math.*, **53**, 1981, 53-73

[12] M. Gromov, Carnot-Carathéodory spaces seen from within, in the book: Sub-Riemannian Geometry, A. Bellaïche, J.-J. Risler eds., *Progress in Mathematics*, **144**, Birkhäuser, (1996), 79-323

[13] J. Heinonen, Nonsmooth calculus, *Bull. A.M.S.* **44** (2007), 163–232

[14] G.A. Margulis, G.D. Mostow, The differential of a quasi-conformal mapping of a Carnot-Carathéodory space, *Geom. Funct. Analysis*, **8** (1995), 2, 402–433

[15] G.A: Margulis, G.D. Mostow, Some remarks on the definition of tangent cones in a Carnot-Carathéodory space, *J. D'Analyse Math.*, **80** (2000), 299–317

[16] O. Loos, Symmetric spaces. I: General theory. W. A. Benjamin, Inc., New York-Amsterdam 1969

[17] P. Pansu, Métriques de Carnot-Carathéodory at quasiisometries des espaces symetriques de rang un, *Ann. of Math.*, **129** (1989), 1–60

[18] S.K. Vodopyanov, Differentiability of mappings in the geometry of the Carnot manifolds, *Siberian Math. J.*, Vol. **48**, No. 2, (2007), 197-213