Uniform Boundary Observability of a Two-Grid Method for the 2d-Wave Equation

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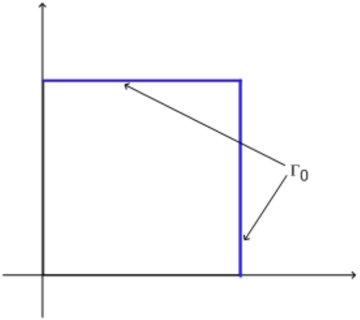
Joint work with Enrique Zuazua

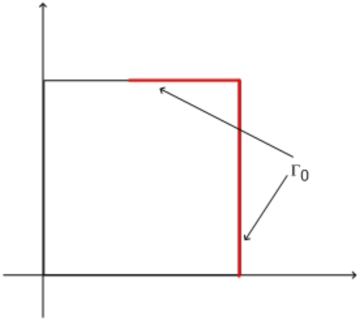
The control problem

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times (0,T), \\ u = v\mathbf{1}_{\Gamma_0}(x) & \text{on } \Gamma \times (0,T), \\ u(0,x) = y^0(x), \ u_t(0,x) = y^1(x) & \text{in } \Omega, \end{cases}$$

Ω is the unit square $\Omega = (0,1) \times (0,1)$

$\Gamma_0 = \{(x_1, 1) : \ x_1 \in (0,1)\} \cup \{(1, x_2) : \ x_2 \in (0,1)\}$.





The control problem

Given $T > 2\sqrt{2}$ and $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control function $v \in L^2((0,T) \times \Gamma_0)$ such that the solution $u = u(t,x)$ satisfies

$$u(T) = u_t(T) = 0.$$

The adjoint problem

Wave equation on the unit square with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } Q = \Omega \times (0,T), \\ u = 0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ u(x,T) = u^0(x), \ u_t(x,T) = u^1(x) & \text{in } Q = \Omega. \end{cases}$$

$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \Rightarrow$

$$u \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)).$$

Conservation of energy

$$E(t) = \frac{1}{2} \int_\Omega [|u_t(x,t)|^2 + |\nabla u(x,t)|^2] dx$$

Observability Inequality

For any $T > 2\sqrt{2}$ there exists $C(T) > 0$ such that

$$\|(u^0, u^1)\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq C(T)\int_0^T \int_{\Gamma_0} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma dt$$

for any $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ where u is the solution of adjoint problem with final data (u^0, u^1).

CONSTRUCTION OF THE CONTROL

Once the observability inequality is known the control is easy to characterize. Following J.L. Lions's HUM (Hilbert Uniqueness Method), the control is

$$v(t) = \partial_n u,$$

where u is the solution of the adjoint system corresponding to final data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ minimizing the functional

$$J(u_0, u_1) = \frac{1}{2} \int_0^T \int_{\Gamma_0} |\partial_n u(t)|^2 + \int_\Omega y^0 u_t(0) dx - \langle y^1, u(0) \rangle_{H^{-1} \times H_0^1}$$

in the space $H_0^1(\Omega) \times L^2(\Omega)$.

Discrete control problems

The idea of approximating the HUM control v for the continuous problem is to consider the following discrete problem

$$\begin{cases} u_h'' - \Delta_h u_h = 0 & \text{in } \Omega_h \times (0,T), \\ u_h = v_h \mathbf{1}_{\Gamma_{0h}} & \text{on } \Gamma_h \times (0,T), \\ u_h(0) = y_h^0, \ \partial_t u_h(0) = y_h^1 & \text{in } \Omega_h. \end{cases}$$

where the initial data (y_h^0, y_h^1) are approximations of (y^0, y^1).

Semi-discretization of the adjoint problem

$$\begin{cases} u''_{jk} - (\Delta_h u)_{jk} = 0, \ 0 < t < T, \ j = 0, ..., N; \ k = 0, ..., N, \\ u_{jk} = 0, \ 0 < t < T, \ j = 0, ..., N+1; \ k = 0, ..., N+1, \\ u_{jk}(0) = u^0_{jk}, \ u'_{jk}(0) = u^1_{jk}, \ j = 0, ..., N+1; \ k = 0, ..., N+1. \end{cases} \quad (1)$$

Discrete energy is preserved

$$E_h(t) = \frac{h^2}{2} \sum_{j,k=0}^{N} \left[|u'_{jk}(t)|^2 + \left| \frac{u_{j+1,k}(t) - u_{jk}(t)}{h} \right|^2 + \left| \frac{u_{j,k+1}(t) - u_{jk}(t)}{h} \right|^2 \right]$$

Discrete version of the energy observed on the boundary

$$\int_0^T \int_{\Gamma_0} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma dt \sim \int_0^T \left[h\sum_{j=1}^N \left|\frac{u_{jN}}{h}\right|^2 + h\sum_{k=1}^N \left|\frac{u_{Nk}}{h}\right|^2 \right] dt.$$

Notation

$$\int_{\Gamma_{0h}} |\partial_n^h u|^2 d\Gamma_{0h} := h\sum_{j=1}^N \left|\frac{u_{jN}}{h}\right|^2 + h\sum_{k=1}^N \left|\frac{u_{Nk}}{h}\right|^2. \quad (2)$$

Question

$$E_h(0) \leq C_h(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \overline{u}|^2 d\Gamma_{0h} dt?$$

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Answer: YES

BUT, FOR ALL $T > 0$ (!!!!!)

$$C_h(T) \to \infty, \ h \to 0.$$

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Answer: YES
BUT, FOR ALL $T > 0$ (!!!!!)

$$C_h(T) \to \infty, \ h \to 0.$$

Following the same arguments as in the continuous case, the control function v_h diverges

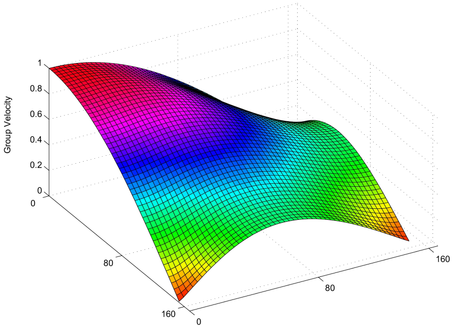
$$\|v_h\|_{L^2((0,T)\times\Gamma_{0h})} \to \infty$$

Group Velocity

$u(t,x) = e^{i(\omega t - \xi x)} \to \omega_h(\xi) = \pm(\sin^2(\frac{\xi_1 h}{2}) + \sin^2(\frac{\xi_2 h}{2}))^{1/2}$

Group velocity $C_h(\xi) = \nabla_\xi \omega_h(\xi)$

$$C_h(\xi) = \frac{1}{2}(\sin(\xi_1 h), \sin(\xi_2 h))/(\sin^2 \frac{\xi_1 h}{2} + \sin^2 \frac{\xi_2 h}{2})^{1/2}$$



Spectral Analysis

Eigenvalue problem associated to (1)

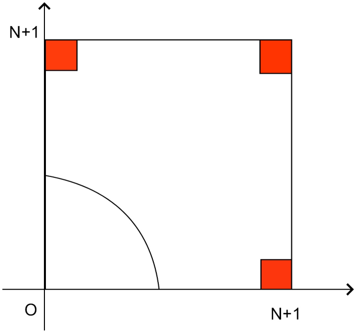
$$\begin{cases} -\dfrac{\varphi_{j+1,k}+\varphi_{j-1,k}-2\varphi_{jk}}{h^2}-\dfrac{\varphi_{j,k+1}+\varphi_{j,k-1}-2\varphi_{jk}}{h^2}=\lambda\varphi_{jk} \\ \quad j=1,...,N;\ k=1,...,N, \\ \varphi_{jk}=0,\ j=0,...,N+1;\ k=0,...,N+1. \end{cases}$$

Eigenvalues: $\lambda_{\mathbf{k}}(h)=\frac{4}{h^2}\left[\sin^2\left(\frac{k_1\pi h}{2}\right)+\sin^2\left(\frac{k_2\pi h}{2}\right)\right],\ \mathbf{k}=(k_1,k_2)$

Eigenvectors: $\overline{\varphi}^{\mathbf{k}}_{\mathbf{j}}=\sin(j_1k_1\pi h)\sin(j_2k_2\pi h)$

$$\overline{u}(t)=\frac{1}{2}\sum_{\mathbf{k}}\left[e^{it\sqrt{\lambda_{\mathbf{k}}(h)}}\widehat{u}_{\mathbf{k}+}+e^{-it\sqrt{\lambda_{\mathbf{k}}(h)}}\widehat{u}_{\mathbf{k}-}\right]\overline{\varphi}^{\mathbf{k}}$$

Filtering: Zuazua 99, Multipliers



$$\Pi_\gamma u = \frac{1}{2} \sum_{\lambda_{\mathbf{k}}(h) \leq \gamma/h^2} \left[e^{it\sqrt{\lambda_{\mathbf{k}}(h)}} \widehat{u}_{\mathbf{k}+} + e^{-it\sqrt{\lambda_{\mathbf{k}}(h)}} \widehat{u}_{\mathbf{k}-} \right] \overline{\varphi}^{\mathbf{k}}, \gamma < 4$$

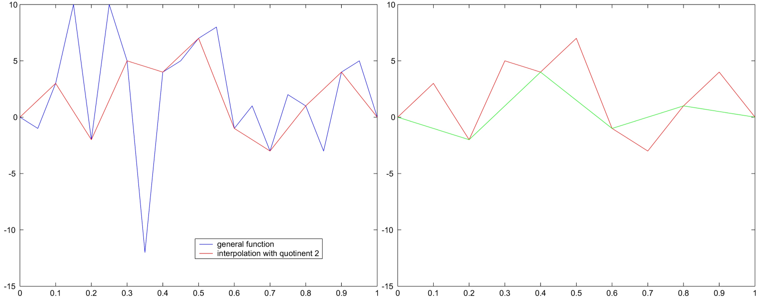
$$E_h(\Pi_\gamma u) \leq \int_0^{T(\gamma)} \int_{\Gamma_{0h}} |\partial_n^h (\Pi_\gamma u)| d\Gamma_{0h} dt$$

Two-grid algorithm, Glowinski '90

1D-case

$T > 4$ - Negreanu & Zuazua 04 - Multiplies

$T > 2\sqrt{3}$ - Loreti & Mehrenberger 05 - Ingham Inequalities for non harmonic series



$E_h(u) \leq 2E_h(\Pi_{1/2}u)$ $E_h(u) \leq 4E_h(\Pi_{1/4}u)$

New Idea : Low frequency estimates + Semi-classical decomposition following the level sets of the frequencies

Main Result: Let $\overline{u}$ be a solution of (1) and $\gamma > 0$ be such that

$$E_h(\overline{u}) \leq C E_h(\Pi_\gamma \overline{u}).$$

Let us assume the existence of a time $T(\gamma)$ such that for all $T > T(\gamma)$ there exists a constant $C(T)$, independent of h, such that

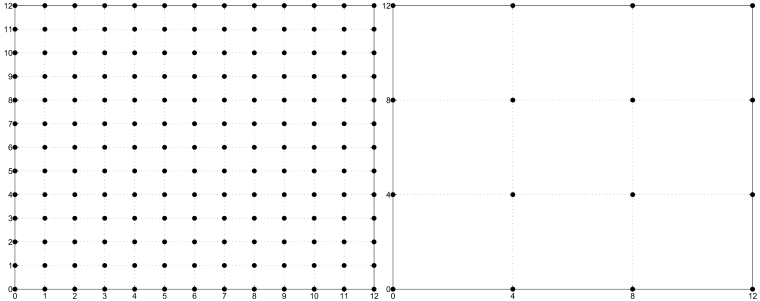
$$E_h(\overline{v}) \leq C(\gamma, T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \overline{v}(t)|^2 d\Gamma dt$$

for all $\overline{v} \in \Pi_\gamma$. Then for all $T > T(\gamma)$ there exists a constant $C_1(T)$, independent of h, such that

$$E_h(\overline{u}) \leq C_1(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \overline{u}|^2 d\Gamma_{0h} dt$$

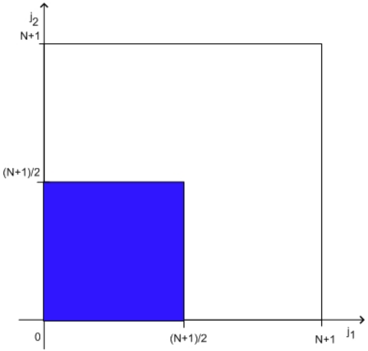
Two-grid Method in $2-d$

Fine and Coarse Grids G^h and G^{4h}, $N = 11$



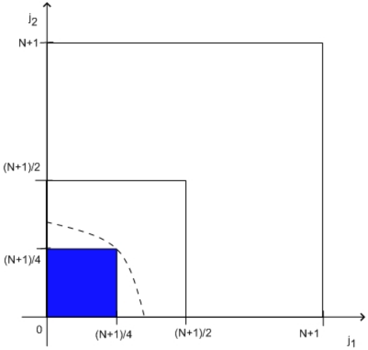
Energy estimates for two-grid data

G^h and G^{2h}



$$E_h(\overline{u}) \leq 4E_h(\Pi^{\infty}_{1/2}\overline{u})$$

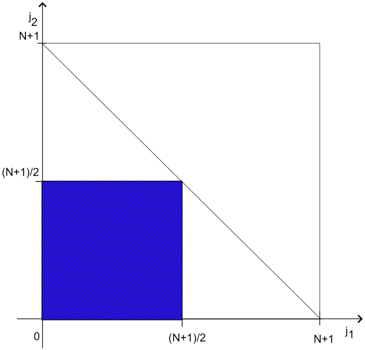
G^h and G^{4h}



$$E_h(\overline{u}) \leq 4E_h(\Pi^{\infty}_{1/4}\overline{u})$$

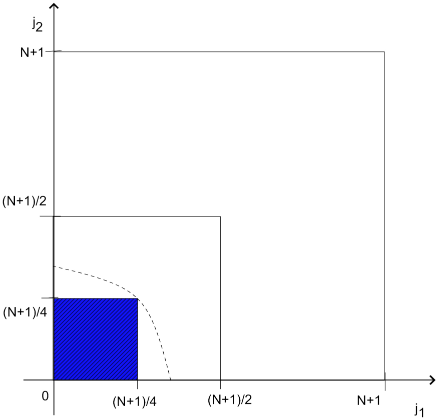
Why not using ratio 1/2 for the two-grids?

The relevant zone of frequencies intersects a level set of the phase velocity for which the group velocity vanishes at some critical points.



$E_h(\overline{u}) \leq 4E_h(\Pi^{\infty}_{1/2}\overline{u}) \leq 4E_h(\Pi_4\overline{u})$

When using the mesh ratio 1/4 this pathology disappears:



$$E_h(\overline{u}) \leq 16 E_h(\Pi^{\infty}_{1/4}\overline{u}) \leq 16 E_h(\Pi_{8\sin^2(\pi/8)}\overline{u})$$

Application to a two-grid method

Theorem

Let be $T > 4$. There exists a constant $C(T)$ such that

$$E_h(\overline{u}) \leq C(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \overline{u}|^2 d\Gamma_{0h} dt$$

holds for all solutions of (1) with $(\overline{u}^0, \overline{u}^1) \in V^h \times V^h$, uniformly on $h > 0$, V^h being the class of the two-grid data obtained with ratio $1/4$.

$T_0 = 4$ is not optimal one.

Its depends by the optimality of the time for the class $\Pi_{8\sin^2(\pi/8)}$

Analysis of the group velocity: expected time $T_0 = \frac{2\sqrt{2}}{\cos(\pi/8)}$

Main difficulty

$$E_h(\overline{u}) \leq 16 E_h(\Pi^{\infty}_{1/4}\overline{u}) \leq C(T)\int_0^T \int_{\Gamma_{0h}} |\partial_n^h \Pi^{\infty}_{1/4}\overline{u}|^2 d\Gamma_h dt.$$

$$\not\Rightarrow$$

$$E_h(\overline{u}) \leq C(T)\int_0^T \int_{\Gamma_{0h}} |\partial_n^h \overline{u}|^2 d\Gamma_h dt$$

Sketch of the proof

$$E_h(\overline{u}) \leq CE_h(\Pi_\gamma \overline{u}) \leq C(T)\int_0^T \int_{\Gamma_{0h}} |\partial_n^h \Pi_\gamma \overline{u}|^2 d\Gamma_h dt$$
$$\overset{?}{\leq} C(T)\int_0^T \int_{\Gamma_{0h}} |\partial_n^h \overline{u}|^2 d\Gamma_h dt + LOT$$

Dyadic decomposition or Semi-classical decomposition:

$$P_k\overline{u}(t) = \sum_{\mathbf{j}\in\mathbb{Z}^2} F(c^{-k}\omega_{\mathbf{j}}(h)) \left[e^{it\omega_{\mathbf{j}}(h)}\widehat{u}_+(\mathbf{j}) + e^{-it\omega_{\mathbf{j}}(h)}\widehat{u}_-(\mathbf{j})\right] \overline{\varphi}^{\mathbf{j}}$$

$$E_h(\overline{u}) \lesssim E_h(\Pi_\gamma \overline{u}) \leq \sum_{k=k_0}^{k_h} E_h(P_k \overline{u}) + LOT \tag{3}$$

$$E_h(P_k \overline{u}) \leq C(T,\gamma,\delta) \int_\delta^{T-\delta} \int_{\Gamma_{0h}} |\partial_n^h P_k \overline{u}|^2 d\Gamma_{0h} dt. \tag{4}$$

Combining (3) and (4),

$$E_h(\Pi_\gamma \overline{u}) \leq C(T,\gamma,\delta) \sum_{k=k_0}^{k_h} \int_\delta^{T-\delta} \int_{\Gamma_{0h}} |\partial_n^h P_k \overline{u}|^2 d\Gamma_{0h} dt + LOT.$$

Lebeau and Burq :

$$\sum_{k=k_0}^{k_h} \int_{\delta}^{T-\delta} \int_{\Gamma_{0h}} |\partial_n^h P_k \overline{u}|^2 d\Gamma_{0h} dt$$
$$\leq 2 \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \overline{u}|^2 d\Gamma_{0h} dt + \frac{C(\delta, T)}{c^{2k_0}} E_h(\overline{u})$$

Thus

$$E_h(\overline{u}) \lesssim E_h(\Pi_\gamma \overline{u})$$
$$\leq C(T,\gamma,\delta) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \overline{u}|^2 d\Gamma_{0h} dt + \frac{C(\delta, T)}{c^{2k_0}} E_h(\overline{u}) + LOT$$

$$E_h(\overline{u}) \leq C(T,\gamma,\delta) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \overline{u}|^2 d\Gamma_{0h} dt + LOT.$$

Conclusions

We have developed a quite systematic approach to prove the convergence of the controls obtained by two-grid methods. It relies essentially on the following ingredients:

- A convergent numerical scheme;
- The Fourier decomposition of solutions;
- The conservative nature of the model and the numerical approximation schemes under consideration;
- The uniform (with respect to the mesh size) observability of low frequency solutions.

In these circumstances, the dyadic decomposition argument can then be applied, to yield the uniform observability of the two-grid solutions.

The method we employ can be adapted to

- **Other models:** Schrödinger equations and beam equations
- **Other control mechanisms:** internal observability for which the measurement on solutions is done in an open subset ω of the domain
- **Control of nonlinear wave equations.** In dimension one Zuazua proved the convergence of the two-grid algorithm for semilinear wave equations with globally Lipschitz nonlinearities
- **Fully discrete schemes.**

Limitations of our method

- **More general meshes.** We used intensively Fourier analysis techniques, which is not available for irregular meshes, that require further developments.
- **Dissipative equations.** As we have mentioned above, our analysis is mainly valid for conservative systems. We could also consider the wave equation with a bounded dissipative potential, but the methods we have developed here can not address genuinely dissipative models as the heat equation, viscoelasticity,...
- **Dissipative schemes.** The same can be said about the numerical schemes we have considered. Our analysis applies to both semi-discrete and fully discrete conservative schemes, but not to dissipative ones...

Thanks!!!