Qualitative Properties of Numerical Approximations of the Heat Equation

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The Heat Equation

$$\begin{cases} u_t - \Delta u = 0 & x \in \mathbf{R}, t > 0, \\ u(0,x) = \varphi(x) & x \in \mathbf{R}. \end{cases} \quad (1)$$

Decay Properties of the Linear Semigroup $e^{t\Delta}$

Theorem 1. *Let $1 \le q \le p \le \infty$. Then there exists a positive constant $c(p,q)$ such that*

$$\|e^{t\Delta}\varphi\|_{L^p(\mathbf{R})} \le c(p,q) t^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{p})} \|\varphi\|_{L^q(\mathbf{R})}. \quad (2)$$

$p = q$

$p = \infty, q = 1$

The solution

$$e^{t\Delta}\varphi = G(t) * \varphi$$

The Fundamental Solution

$$G(t,x) = (4\pi t)^{-1/2} \exp\left(-\frac{|x|^2}{4t}\right)$$

Time Decay of $G(t,x)$

$$\|G(t)\|_{L^p(\mathbf{R})} \le C_p t^{-\frac{1}{2}(1-\frac{1}{p})}$$

Consider the finite difference approximation

$$\begin{cases} \frac{du^h}{dt} - \Delta_h u^h &= 0, \quad t > 0, \\ u^h(0) &= \varphi^h. \end{cases} \tag{3}$$

Here u^h stands for the infinite vector unknown $\{u_j^h\}_{j \in \mathbf{Z}},$, $u_j(t)$ being the approximation of the solution at the node $x_j = jh$, and Δ_h being the classical second order finite difference approximation of ∂_x^2:

$$(\Delta_h u)_j = \frac{1}{h^2}[u_{j+1} - 2u_j + u_{j-1}].$$

Has the numerical approximation (8) the same decay rates as the continuous model?

Motivation: There are numerical schemes based numerical viscosity

Example: To recover the dispersive properties of the Schrödinger Equation we add numerical viscosity : L. Ignat and E. Zuazua, CRAS, 529–534, 2005

The simpler scheme

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = 0, & t > 0, \\ u^h(0) = \varphi^h. \end{cases} \tag{4}$$

is replaced by

$$\begin{cases} i\frac{du^h}{dt} + \Delta_h u^h = ia(h)\Delta_h u^h, & t > 0, \\ u^h(0) = \varphi^h, \end{cases}$$

where $a(h) > 0$ is such that

$$a(h) \to 0$$

as $h \to 0$.

Remark: The solution is a combination of the linear semigroups generated by (8) and (4).

Preliminaries

- For $1 \leq p < \infty$ we define the spaces $l_h^p(\mathbf{Z})$ as

$$l_h^p(\mathbf{Z}) = \{(\varphi_j)_{j\in\mathbf{Z}} \in \mathbf{C}^{\mathbf{Z}} : h \sum_{j\in\mathbf{Z}} |\varphi_j|^p < \infty\}.$$

$$l_h^p(\mathbf{Z}, |x|^m) = \{\{\varphi_j\}_{j\in\mathbf{Z}} : h \sum_{j\in\mathbf{Z}} |jh|^{pm}|\varphi_j|^p < \infty\}$$

- The discrete convolution $((a\star b)_j)_{j\in\mathbf{Z}}$ of two sequences $(a_j)_{j\in\mathbf{Z}}$ and $(b_j)_{j\in\mathbf{Z}}$ is defined by

$$(a \star b)_j = h \sum_{n\in\mathbf{Z}} a_{j-n}b_n = h \sum_{n\in\mathbf{Z}} a_n b_{j-n}$$

- For $v \in l_h^2(\mathbf{Z})$ we define the semidiscrete Fourier transform

$$\widehat{v}(\xi) = h \sum_{j \in \mathbf{Z}} e^{-ijh\xi} v_j, \qquad \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$$

- $$v_j = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} e^{ijh\xi} \widehat{v}(\xi) d\xi, \quad j \in \mathbf{Z}.$$

- $$\|\widehat{v}\|_{L^2(-\pi/h,\pi/h)} = \sqrt{2\pi} \|v\|_{l_h^2(\mathbf{Z})}.$$

Back to the Semidiscrete Heat Equation

$$\begin{cases} \dfrac{du^h}{dt} - \Delta_h u^h &= 0, \quad t > 0, \\ u^h(0) &= \varphi^h. \end{cases}$$

Exact solution

$$u^h(t) = K_t^h \star \varphi^h$$

Fundamental Solution : K_t^h

$$\widehat{K_t^h}(\xi) = e^{-\frac{4t}{h^2}\sin^2(\frac{\xi h}{2})}, \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$$

$$(K_t^h)_j = \frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-\frac{4t}{h^2}\sin^2(\frac{\xi h}{2})} e^{ijh\xi} d\xi$$

Decay Properties $\Leftrightarrow$ Decay Rates of Fundamental Solution

Theorem 2. *If $1 \le p \le \infty$, then there is a positive constant $C(p)$ independent of h, such that*

$$\|K_t^h\|_{l_h^p(\mathbf{Z})} \le C(p)t^{-\frac{1}{2}(1-\frac{1}{p})}, \ \forall \ t > 0. \qquad (5)$$

Proof Ideas: Interpolation between the cases $p = 1$ and $p = \infty$.

$p = \infty$: $\frac{4}{h^2}\sin^2(\frac{\xi h}{2}) \sim |\xi|^2, \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]$

$p = 1$:

Lemma 1. *(Carlson - Beurling Inequality) Assume that $a \in L^2(\mathbf{R})$, $a' \in L^2(\mathbf{R})$. Then $\hat{a} \in L^1(\mathbf{R})$ and*

$$\|\hat{a}\|_{L^1(\mathbf{R})} \le (2\|a\|_{L^2(\mathbf{R})}\|a'\|_{L^2(\mathbf{R})})^{\frac{1}{2}} \qquad (6)$$

More about the fundamental solution K_t^h

$$(K_t^h)_j = \frac{e^{-\frac{2t}{h^2}}}{\pi h} I_j \left(\frac{2t}{h^2} \right)$$

$I_\nu(x)$ is the modified Bessel function

Using the properties of Bessel's function we prove

Lemma 2. *For any positive numbers h, t and for all integers j, $(K_t^h)_j$ is positive. Also the map*

$$j \to (K_t^h)_j$$

is increasing for $j \leq 0$ and decreasing for $j \geq 0$.

Notation : the discrete gradient

$$(\nabla_h^+ f)_j = \frac{f_{j+1} - f_j}{h}$$

Theorem 3. *If $1 \le p \le \infty$, then there is a positive constant $C(p)$ independent of h, such that*

$$\|\nabla_h^+ K_t^h\|_{l_h^p(\mathbf{Z})} \le C(p) t^{-\frac{1}{2}(1-\frac{1}{p})}, \ \forall \ t > 0. \tag{7}$$

A finer Analysis of the long time behavior

Zuazua and Duandikoethea (C.R.Acad. Sci. Paris, 1992, 693-698) show that if $u(x,t)$ is solution of the Heat Equation with initial data $\varphi \in L^1(\mathbf{R}, 1+|x|^{m+1})$ then

$$\|u(\cdot,t) - \sum_{0\le\alpha\le m} \frac{(-1)^\alpha}{\alpha!}\left(\int \varphi(x) x^\alpha dx\right) D^\alpha G(\cdot,t)\|_{L^p} \le t^{-\frac{1}{2}(m+2-\frac{1}{p})} \|\varphi\|_{L^1(|x|^{m+1})}$$

where

$$G(x,t) = (4\pi t)^{-1/2} \exp\left(-\frac{|x|^2}{4t}\right)$$

is the fundamental solution of the heat equation.

What we can say about the semidiscrete case ?

$$\begin{cases} \dfrac{du^h}{dt} - \Delta_h u^h &= 0, \quad t > 0, \\ u^h(0) &= \varphi^h. \end{cases} \tag{8}$$

A Discrete Result

There is a function $K^h(t,x)$ such that

$$\left\| u^h(t) - \sum_{\alpha=0}^{m} \frac{(-1)^\alpha M_\alpha \varphi^h}{\alpha!} \frac{\partial^\alpha K^h(t,\cdot)}{\partial x^\alpha} \right\|_{l^q(h\mathbb{Z})} \leq C(m) t^{-\frac{1}{2}(m+2-\frac{1}{q})} \|\varphi^h\|_{l^1(h\mathbb{Z},|x|^{m+1})} \tag{9}$$

for all $\varphi^h \in l^1(h\mathbb{Z},|x|^{m+1})$ and $t > 0$, uniformly in $h > 0$.

$$M_0 \varphi^h = h \sum_{n\in\mathbb{Z}} \varphi_n^h$$

$$M_\alpha \varphi^h = h \sum_{n\in\mathbf{Z}} (nh)^\alpha \varphi_n^h, \alpha \geq 1$$

More about the function $K_t^h(x)$

$$K^h(t,x) = \frac{1}{2\pi}\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-\frac{4t}{h^2}\sin^2(\frac{\theta h}{2})} e^{ix\theta} d\theta. \tag{10}$$

$$K^h(t, jh) = (K_t^h)_j$$

Theorem 4. *For all $m \in \mathbf{N}$ and $1 \leq p \leq \infty$ there is a constant $C(m,p) > 0$ such that*

$$\left\| \frac{\partial^m K^h(t,\cdot)}{\partial x^m} \right\|_{L^p(\mathbf{R})} \leq C(m,p) t^{-(\frac{m+1}{2} - \frac{1}{2p})} \tag{11}$$

for all $h > 0$ and $t > 0$.

Proof of The Discrete Result

$$(u^h(t))_j = (K_t^h \star \varphi^h)_j = h \sum_{n \in \mathbf{Z}} K^h(t,(j-n)h)\varphi_n^h$$

$$\left(u^h(t) - \sum_{\alpha=0}^{m} \frac{(-1)^\alpha M_\alpha \varphi_n^h}{\alpha!} \frac{\partial^\alpha K_t^h}{\partial x^\alpha}\right)_j =$$

$$= h \sum_{n \neq 0} \varphi_n^h \left(K_t^h((j-n)h) - \sum_{\alpha=0}^{m} \frac{(nh)^\alpha}{\alpha!} \frac{\partial^\alpha K_t^h}{\partial x^\alpha}(jh) \right)$$

$$= h \sum_{n \neq 0} \varphi_n^h b_{jn}$$

It suffices to consider the cases $q = \infty$, $q = 1$

Estimates for b_{jn}

$$|b_{jn}| \leq \frac{|nh|^{m+1}}{m!} \left\| \frac{\partial^{m+1} K^h(t, \cdot)}{\partial x^{m+1}} \right\|_{L^\infty(\mathbf{R})}$$

$$|b_{jn}| \leq \frac{|nh|^m}{m!} \int_{\min\{jh,(j-n)h\}}^{\max\{jh,(j-n)h\}} \left| \frac{\partial^{m+1} K^h(t, \cdot)}{\partial x^{m+1}} \right| dx$$

The Convection-Diffusion Equation

$$\begin{cases} u_t - \Delta u = a\nabla(|u|^{q-1}u) \text{ in } (0,\infty)\times \mathbf{R}^N \\ u(0) = u_0 \end{cases} \tag{12}$$

Escobedo, Zuazua, J. Func. Analysis, 1991 :

- Global Existence and Uniqueness for $u_0 \in L^1(\mathbf{R}^N)$
- $q > 1 + \frac{1}{N}$, $u_0 \in L^1(\mathbf{R}^N, 1+|x|) \cap L^\infty(\mathbf{R}^N)$ the large time behavior is given by the heat kernel

$$\left\| u(t) - \left(\int u_0 \right) G(t) \right\|_p \leq t^{\frac{N}{2}(1-\frac{1}{p})} a(t), a(t) \to 0, t \to \infty, p \in [1, \infty]$$

A Semidiscrete Scheme

$$\begin{cases} \dfrac{du_j^h}{dt} = \dfrac{u_{j+1}^h - 2u_j^h + u_{j-1}^h}{h^2} + \dfrac{|u_{j+1}^h|^{q-1}u_{j+1}^h - |u_j^h|^{q-1}u_j^h}{h}, & t > 0\,, j \in \mathbf{Z} \\ u_j^h(0) = \varphi_j^h, & j \in \mathbf{Z}. \end{cases} \tag{13}$$

Theorem 5. *For any initial data $u_0 \in l_h^1(\mathbf{Z})$ there exists a unique solution*

$$u^h \in C^1([0,\infty); l_h^1(\mathbf{Z}))$$

of the problem (13) which satisfies

$$\|u^h(t)\|_{l_h^1(\mathbf{Z})} \leq \|u_0^h\|_{l_h^1(\mathbf{Z})}, \|u^h(t)\|_{l_h^\infty(\mathbf{Z})} \leq \|u_0^h\|_{l_h^\infty(\mathbf{Z})}. \tag{14}$$

Decay of Solutions

Theorem 6. *For every* $p \in [1,\infty]$ *there exists some constants* $C(p)$ *such that*

$$\|u^h(t)\|_{l_h^p(\mathbf{Z})} \leq C(p)t^{-\frac{1}{2}(1-\frac{1}{p})}\|\varphi^h\|_{l_h^1(\mathbf{Z})}, \tag{15}$$

for all $t > 0$*, uniformly on* $h > 0$*.*

Proof.

$$\frac{d}{dt}\left(\sum_{j\in\mathbf{Z}} h|u_j^h|^p\right) \leq -p(\nabla_h^+ u^h)\cdot(\nabla_h^+|u^h|^{p-2}u^h) \leq -c(p)\|\nabla_h^+(|u^h|^{\frac{p}{2}})\|^2_{l_h^2(\mathbf{Z})}. \tag{16}$$

+ Discrete Sobolev Inequalities, Gronwall's Inequality

Asymptotic Behavior

Let $q > 2$, $u_0^h \in l_h^1(\mathbf{Z}, 1+|x|) \cap l_h^\infty(\mathbf{Z})$ and $M_0 = h\sum_{j\in\mathbf{Z}}(u_0^h)_j$. Then

$$\|u^h(t) - M_0 K_t^h\|_{l_h^p(\mathbf{Z})} \leq C_p t^{-\frac{1}{2}(1-\frac{1}{p})} a(t), t > 0$$

uniformly on $h > 0$.

$$a(t) = \begin{cases} t^{-1/2} & q > 3 \\ t^{-1/2}\log(t+2) & q = 3 \\ t^{-(q-2)/2} & 2 < q < 3 \end{cases}$$

Proof :

$$u^h(t) = K_t^h * u_0^h + \int_0^t \nabla^+ K_{t-s}^h |u^h(s)|^{q-1} u^h(s) ds$$

First step

$$\|K_t^h * u_0^h - M_0 K_t^h\|_{l_h^p(\mathbf{Z})} \leq C \|u_0^h\|_{l_h^1(\mathbf{Z},|x|)} t^{-\frac{1}{2}(1-\frac{1}{p})-\frac{1}{2}}$$

by using estimates for the solutions of the discrete heat equation (9)

Second step

$$\| \int_0^t \nabla_h^+ K_{t-s}^h |u^h(s)|^{q-1} u^h(s) ds \|_{l_h^p(\mathbf{Z})} \leq C(\|u_0^h\|_{l_h^1(\mathbf{Z})}, \|u_0^h\|_{l_h^\infty(\mathbf{Z})}) t^{-\frac{1}{2}(1-\frac{1}{p})} a(t)$$

estimations for $\nabla_h^+ K_t^h$ (7)

a priori estimates for u^h (15)

The same results can be proved in the case of

$$\begin{cases} \dfrac{du_j^h}{dt} = (-1)^{m-1}(\Delta_h^m u)_j, & t > 0\,, j \in \mathbf{Z}^d \\ u_j^h(0) = \varphi_j^h, & j \in \mathbf{Z}^d, \end{cases} \tag{17}$$

Future Work

Full discrete schemes

The long time behavior for other nonlinear problems

$$\frac{du_j^h}{dt} = (-1)^{m-1}(\Delta_h^m u)_j + \nabla f(u^h)$$