

Uniform Boundary Observability of a Two-Grid Method for the 2d-Wave Equation

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Joint work with Enrique Zuazua

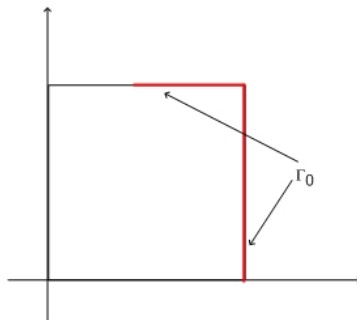
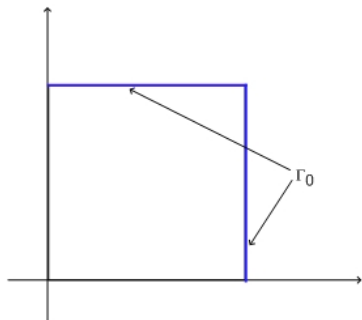


The control problem

$$\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times (0, T), \\ u = v \mathbf{1}_{\Gamma_0}(x) & \text{on } \Gamma \times (0, T), \\ u(0, x) = y^0(x), \quad u_t(0, x) = y^1(x) & \text{in } \Omega, \end{cases}$$

Ω is the unit square $\Omega = (0, 1) \times (0, 1)$

$\Gamma_0 = \{(x_1, 1) : x_1 \in (0, 1)\} \cup \{(1, x_2) : x_2 \in (0, 1)\}$.



The control problem

Given $T > 2\sqrt{2}$ and $(y^0, y^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control function $v \in L^2((0, T) \times \Gamma_0)$ such that the solution $u = u(t, x)$ satisfies

$$u(T) = u_t(T) = 0.$$



The adjoint problem

Wave equation on the unit square with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } Q = \Omega \times (0, T), \\ u = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ u(x, T) = u^0(x), \quad u_t(x, T) = u^1(x) & \text{in } Q = \Omega. \end{cases}$$

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \Rightarrow$$

$$u \in C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega)).$$

Conservation of energy

$$E(t) = \frac{1}{2} \int_{\Omega} [|u_t(x, t)|^2 + |\nabla u(x, t)|^2] dx$$



Observability Inequality

For any $T > 2\sqrt{2}$ there exists $C(T) > 0$ such that

$$\|(u^0, u^1)\|_{H_0^1(\Omega) \times L^2(\Omega)}^2 \leq C(T) \int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt$$

for any $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ where u is the solution of adjoint problem with final data (u^0, u^1) .



CONSTRUCTION OF THE CONTROL

Once the observability inequality is known the control is easy to characterize. Following J.L. Lions's HUM (Hilbert Uniqueness Method), the control is

$$v(t) = \partial_n u,$$

where u is the solution of the adjoint system corresponding to final data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ minimizing the functional

$$J(u_0, u_1) = \frac{1}{2} \int_0^T \int_{\Gamma_0} |\partial_n u(t)|^2 + \int_{\Omega} y^0 u_t(0) dx - \langle y^1, u(0) \rangle_{H^{-1} \times H_0^1}$$

in the space $H_0^1(\Omega) \times L^2(\Omega)$.



Discrete control problems

The idea of approximating the HUM control v for the continuous problem is to consider the following discrete problem

$$\begin{cases} u_h'' - \Delta_h u_h = 0 & \text{in } \Omega_h \times (0, T), \\ u_h = v_h \mathbf{1}_{\Gamma_{0h}} & \text{on } \Gamma_h \times (0, T), \\ u_h(0) = y_h^0, \partial_t u_h(0) = y_h^1 & \text{in } \Omega_h. \end{cases}$$

where the initial data (y_h^0, y_h^1) are approximations of (y^0, y^1) .



Semi-discretization of the adjoint problem

$$\begin{cases} u_{jk}'' - (\Delta_h u)_{jk} = 0, & 0 < t < T, \quad j = 0, \dots, N; \quad k = 0, \dots, N, \\ u_{jk} = 0, & 0 < t < T, \quad j = 0, \dots, N+1; \quad k = 0, \dots, N+1, \\ u_{jk}(0) = u_{jk}^0, \quad u_{jk}'(0) = u_{jk}^1, & j = 0, \dots, N+1; \quad k = 0, \dots, N+1. \end{cases} \quad (1)$$

Discrete energy is preserved

$$E_h(t) = \frac{h^2}{2} \sum_{j,k=0}^N \left[|u_{jk}'(t)|^2 + \left| \frac{u_{j+1,k}(t) - u_{jk}(t)}{h} \right|^2 + \left| \frac{u_{j,k+1}(t) - u_{jk}(t)}{h} \right|^2 \right]$$



Discrete version of the energy observed on the boundary

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \sim \int_0^T \left[h \sum_{j=1}^N \left| \frac{u_{jN}}{h} \right|^2 + h \sum_{k=1}^N \left| \frac{u_{Nk}}{h} \right|^2 \right] dt.$$

Notation

$$\int_{\Gamma_{0h}} |\partial_n^h u|^2 d\Gamma_{0h} := h \sum_{j=1}^N \left| \frac{u_{jN}}{h} \right|^2 + h \sum_{k=1}^N \left| \frac{u_{Nk}}{h} \right|^2. \quad (2)$$



Question

$$E_h(0) \leq C_h(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \bar{u}|^2 d\Gamma_{0h} dt?$$



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Answer: **YES**

BUT, FOR ALL $T > 0$ (!!!!!)

$$C_h(T) \rightarrow \infty, \quad h \rightarrow 0.$$



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Following the same arguments as in the continuous case, the control function v_h diverges

$$\|v_h\|_{L^2((0,T) \times \Gamma_{0h})} \rightarrow \infty$$

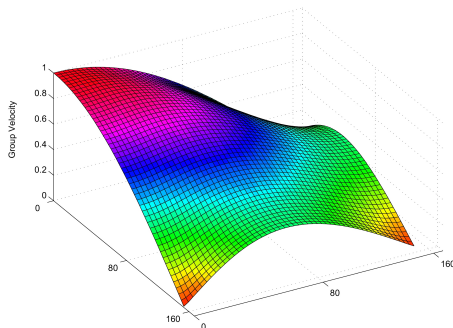


Group Velocity

$$u(t, x) = e^{i(\omega t - \xi x)} \rightarrow \omega_h(\xi) = \pm(\sin^2(\frac{\xi_1 h}{2}) + \sin^2(\frac{\xi_2 h}{2}))^{1/2}$$

$$\text{Group velocity } C_h(\xi) = \nabla_{\xi} \omega_h(\xi)$$

$$C_h(\xi) = \frac{1}{2}(\sin(\xi_1 h), \sin(\xi_2 h))/(\sin^2 \frac{\xi_1 h}{2} + \sin^2 \frac{\xi_2 h}{2})^{1/2}$$



Eigenvalue problem associated to (1)

$$\begin{cases} -\frac{\varphi_{j+1,k} + \varphi_{j-1,k} - 2\varphi_{jk}}{h^2} - \frac{\varphi_{j,k+1} + \varphi_{j,k-1} - 2\varphi_{jk}}{h^2} = \lambda\varphi_{jk} \\ j = 1, \dots, N; \quad k = 1, \dots, N, \\ \varphi_{jk} = 0, \quad j = 0, \dots, N+1; \quad k = 0, \dots, N+1. \end{cases}$$

Eigenvalues:

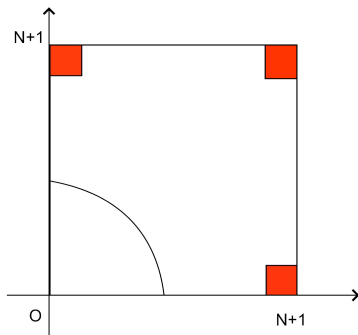
$$\lambda_{\mathbf{k}}(h) = \frac{4}{h^2} \left[\sin^2 \left(\frac{k_1 \pi h}{2} \right) + \sin^2 \left(\frac{k_2 \pi h}{2} \right) \right], \quad \mathbf{k} = (k_1, k_2)$$

Eigenvectors: $\bar{\varphi}_{\mathbf{j}}^{\mathbf{k}} = \sin(j_1 k_1 \pi h) \sin(j_2 k_2 \pi h)$

$$\bar{u}(t) = \frac{1}{2} \sum_{\mathbf{k}} \left[e^{it\sqrt{\lambda_{\mathbf{k}}(h)}} \hat{u}_{\mathbf{k}+} + e^{-it\sqrt{\lambda_{\mathbf{k}}(h)}} \hat{u}_{\mathbf{k}-} \right] \bar{\varphi}^{\mathbf{k}}$$



Filtering: Zuazua 99, Multipliers



$$\Pi_{\gamma} u = \frac{1}{2} \sum_{\lambda_{\mathbf{k}}(h) \leq \gamma/h^2} \left[e^{it\sqrt{\lambda_{\mathbf{k}}(h)}} \hat{u}_{\mathbf{k}+} + e^{-it\sqrt{\lambda_{\mathbf{k}}(h)}} \hat{u}_{\mathbf{k}-} \right] \bar{\varphi}^{\mathbf{k}}, \gamma < 4$$

$$E_h(\Pi_{\gamma} u) \leq \int_0^{T(\gamma)} \int_{\Gamma_{0h}} |\partial_n^h(\Pi_{\gamma} u)| d\Gamma_{0h} dt$$

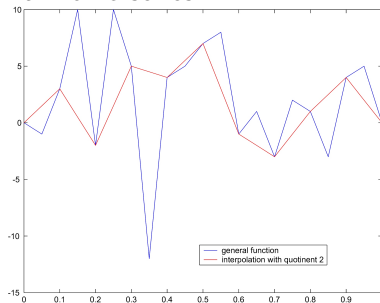


Two-grid algorithm, Glowinski '90

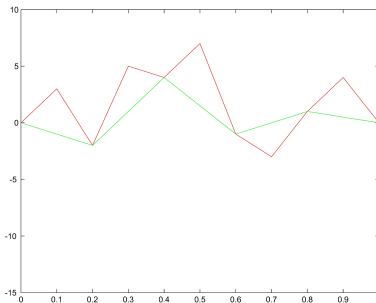
1D-case

$T > 4$ - Negreanu & Zuazua 04 - Multiplies

$T > 2\sqrt{2}$ - Loreti & Mehrenberger 05 - Ingham Inequalities for non harmonic series



$$E_h(u) \leq 2E_h(\Pi_{1/2}u)$$



$$E_h(u) \leq 4E_h(\Pi_{1/4}u)$$



New Idea : Low frequency estimates + Semi-classical decomposition following the level sets of the frequencies

Main Result: Let $\bar{u}$ be a solution of (1) and $\gamma > 0$ be such that

$$E_h(\bar{u}) \leq C E_h(\Pi_\gamma \bar{u}).$$

Let us assume the existence of a time $T(\gamma)$ such that for all $T > T(\gamma)$ there exists a constant $C(T)$, independent of h , such that

$$E_h(\bar{v}) \leq C(\gamma, T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \bar{v}(t)|^2 d\Gamma dt$$

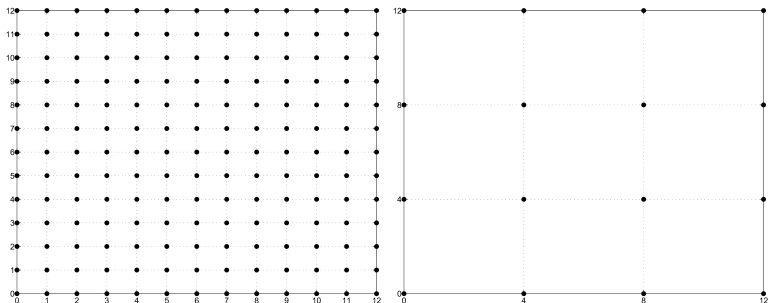
for all $\bar{v} \in \Pi_\gamma$. Then for all $T > T(\gamma)$ there exists a constant $C_1(T)$, independent of h , such that

$$E_h(\bar{u}) \leq C_1(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \bar{u}|^2 d\Gamma_{0h} dt$$



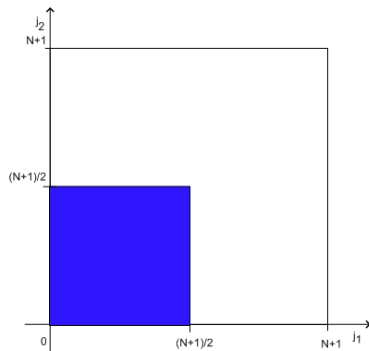
Two-grid Method in $2 - d$

Fine and Coarse Grids G^h and G^{4h} , $N = 11$



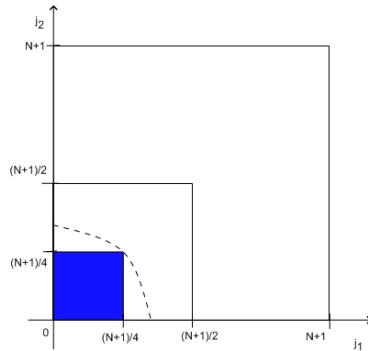
Energy estimates for two-grid data

G^h and G^{2h}



$$E_h(\bar{u}) \leq 4E_h(\Pi_{1/2}^\infty \bar{u})$$

G^h and G^{4h}

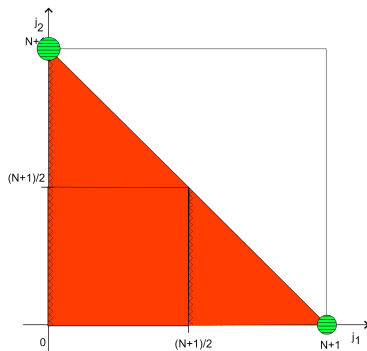
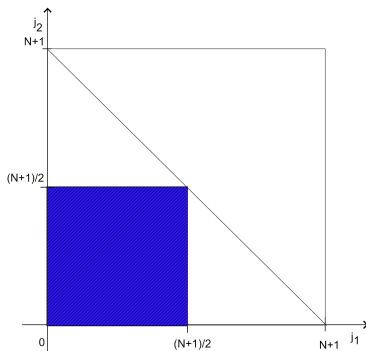


$$E_h(\bar{u}) \leq 4E_h(\Pi_{1/4}^\infty \bar{u})$$



Why not using ratio 1/2 for the two-grids?

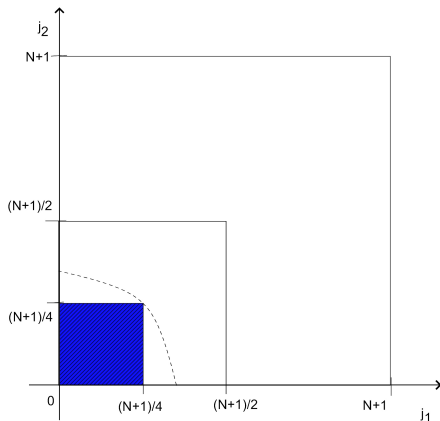
The relevant zone of frequencies intersects a level set of the phase velocity for which the group velocity vanishes at some critical points.



$$E_h(\bar{u}) \leq 4E_h(\Pi_{1/2}^\infty \bar{u}) \leq 4E_h(\Pi_4 \bar{u})$$



When using the mesh ratio 1/4 this pathology disappears:



$$E_h(\bar{u}) \leq 16E_h(\Pi_{1/4}^\infty \bar{u}) \leq 16E_h(\Pi_{8 \sin^2(\pi/8)} \bar{u})$$



Application to a two-grid method

Theorem

Let be $T > 4$. There exists a constant $C(T)$ such that

$$E_h(\bar{u}) \leq C(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \bar{u}|^2 d\Gamma_{0h} dt$$

holds for all solutions of (1) with $(\bar{u}^0, \bar{u}^1) \in V^h \times V^h$, uniformly on $h > 0$, V^h being the class of the two-grid data obtained with ratio $1/4$.

$T_0 = 4$ is not optimal one.

Its depends by the optimality of the time for the class $\Pi_{8 \sin^2(\pi/8)}$

Analysis of the group velocity: expected time $T_0 = \frac{2\sqrt{2}}{\cos(\pi/8)}$



Main difficulty

$$E_h(\bar{u}) \leq 16E_h(\Pi_{1/4}^\infty \bar{u}) \leq C(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \Pi_{1/4}^\infty \bar{u}|^2 d\Gamma_h dt.$$

$\Rightarrow$

$$E_h(\bar{u}) \leq C(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \bar{u}|^2 d\Gamma_h dt$$



Sketch of the proof

$$\begin{aligned} E_h(\bar{u}) &\leq C E_h(\Pi_\gamma \bar{u}) \leq C(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \Pi_\gamma \bar{u}|^2 d\Gamma_h dt \\ &\stackrel{?}{\leq} C(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \bar{u}|^2 d\Gamma_h dt + LOT \end{aligned}$$

Dyadic decomposition or Semi-classical decomposition:

$$P_k \bar{u}(t) = \sum_{\mathbf{j} \in \mathbb{Z}^2} F(c^{-k} \omega_{\mathbf{j}}(h)) \left[e^{it\omega_{\mathbf{j}}(h)} \hat{u}_+(\mathbf{j}) + e^{-it\omega_{\mathbf{j}}(h)} \hat{u}_-(\mathbf{j}) \right] \bar{\varphi}^{\mathbf{j}}$$



$$E_h(\Pi_\gamma \bar{u}) \leq \sum_{k=k_0}^{k_h} E_h(P_k \bar{u}) + LOT$$

$$E_h(P_k \bar{u}) \leq C(T, \gamma, \delta) \int_{\delta}^{T-\delta} \int_{\Gamma_{0h}} |\partial_n^h P_k \bar{u}|^2 d\Gamma_{0h} dt.$$

Combining (23) and (23),

$$E_h(\Pi_\gamma \bar{u}) \leq C(T, \gamma, \delta) \sum_{k=k_0}^{k_h} \int_{\delta}^{T-\delta} \int_{\Gamma_{0h}} |\partial_n^h P_k \bar{u}|^2 d\Gamma_{0h} dt + LOT.$$



Lebeau and Burq :

$$\begin{aligned} \sum_{k=k_0}^{k_h} \int_{\delta}^{T-\delta} \int_{\Gamma_{0h}} |\partial_n^h P_k \bar{u}|^2 d\Gamma_{0h} dt \\ \leq 2 \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \bar{u}|^2 d\Gamma_{0h} dt + \frac{C(\delta, T)}{c^{2k_0}} E_h(\bar{u}) \end{aligned}$$

Thus

$$\begin{aligned} E_h(\bar{u}) &\lesssim E_h(\Pi_{\gamma} \bar{u}) \\ &\leq C(T, \gamma, \delta) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \bar{u}|^2 d\Gamma_{0h} dt + \frac{C(\delta, T)}{c^{2k_0}} E_h(\bar{u}) + LOT \\ E_h(\bar{u}) &\leq C(T, \gamma, \delta) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \bar{u}|^2 d\Gamma_{0h} dt + LOT. \end{aligned}$$



Thanks!!!

