Uniform Boundary Observability of a Two-Grid Method for the 2d-Wave Equation

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Rome, 27 October 2006
European Conference on Smart Systems

Wave equation on the unit square with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } Q = \Omega \times (0,T), \\ u(0) = 0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{in } Q = \Omega. \end{cases} \tag{1}$$

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \Rightarrow u \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)).$$

Conservation of energy

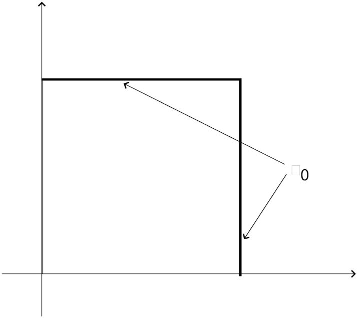
$$E(t) = \frac{1}{2} \int_\Omega [|u_t(x,t)|^2 + |\nabla u(x,t)|^2] dx \tag{2}$$

Observability Inequality

For $T > 2\sqrt{2}$ there exists $C(T) > 0$ such that

$$E(0) \leq C(T) \int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \tag{3}$$

$\Gamma_0 = \{(x_1, 1) : \ x_1 \in (0,1)\} \cup \{(1, x_2) : \ x_2 \in (0,1)\}.$



Semi-discretization of the wave equation

$$\begin{cases} u''_{jk} - (\Delta_h u)_{jk} = 0, \ 0 < t < T, \ j = 0, ..., N; \ k = 0, ..., N, \\ u_{jk} = 0, \ 0 < t < T, \ j = 0, ..., N+1; \ k = 0, ..., N+1, \\ u_{jk}(0) = u^0_{jk}, \ u'_{jk}(0) = u^1_{jk}, \ j = 0, ..., N+1; \ k = 0, ..., N+1. \end{cases} \tag{4}$$

Discrete energy is preserved

$$E_h(t) = \frac{h^2}{2} \sum_{j,k=0}^{N} \left[|u'_{jk}(t)|^2 + \left| \frac{u_{j+1,k}(t) - u_{jk}(t)}{h} \right|^2 + \left| \frac{u_{j,k+1}(t) - u_{jk}(t)}{h} \right|^2 \right]$$

Discrete version of the energy observed on the boundary

$$\int_0^T \int_{\Gamma_0} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma dt \sim \int_0^T \left[h \sum_{j=1}^{N} \left|\frac{u_{jN}}{h}\right|^2 + h \sum_{k=1}^{N} \left|\frac{u_{Nk}}{h}\right|^2 \right] dt.$$

Notation

$$\int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h := h \sum_{j=1}^{N} \left|\frac{u_{jN}}{h}\right|^2 + h \sum_{k=1}^{N} \left|\frac{u_{Nk}}{h}\right|^2 . \quad (5)$$

Question

$$E_h(0) \leq C_h(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt. \tag{6}$$

Answer: YES

BUT, FOR ALL $T > 0$ (!!!!!)

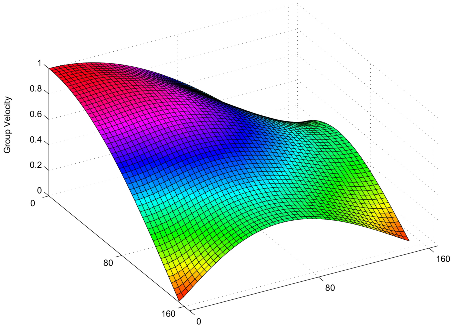
$$C_h(T) \to \infty, \ h \to 0.$$

Group Velocity

$u(t,x) = e^{i(\omega t - \xi x)} \rightarrow \omega_h(\xi) = \pm(\sin^2(\frac{\xi_1 h}{2}) + \sin^2(\frac{\xi_2 h}{2}))^{1/2}$

Group velocity $C_h(\xi) = \nabla_\xi \omega_h(\xi)$

$$C_h(\xi) = \frac{1}{2}(\sin(\xi_1 h), \sin(\xi_2 h))/(\sin^2 \frac{\xi_1 h}{2} + \sin^2 \frac{\xi_2 h}{2})^{1/2}$$



Spectral Analysis

Eigenvalue problem associated to (4)

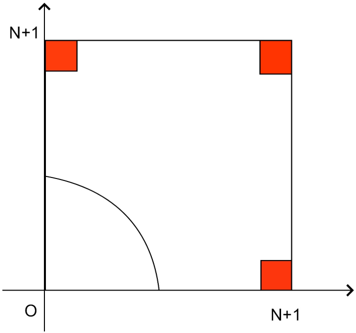
$$\begin{cases} -\dfrac{\varphi_{j+1,k}+\varphi_{j-1,k}-2\varphi_{jk}}{h^2}-\dfrac{\varphi_{j,k+1}+\varphi_{j,k-1}-2\varphi_{jk}}{h^2}=\lambda\varphi_{jk} \\ \quad j=1,...,N;\ k=1,...,N, \\ \varphi_{jk}=0,\ j=0,...,N+1;\ k=0,...,N+1. \end{cases} \qquad (7)$$

Eigenvalues: $\lambda_{\mathbf{k}}(h)=\frac{4}{h^2}\left[\sin^2\left(\frac{k_1\pi h}{2}\right)+\sin^2\left(\frac{k_2\pi h}{2}\right)\right],\ \mathbf{k}=(k_1,k_2)$

Eigenvectors: $\overline{\varphi}_{\mathbf{j}}^{\mathbf{k}}=\sin(j_1k_1\pi h)\sin(j_2k_2\pi h)$

$$\overline{u}(t)=\frac{1}{2}\sum_{\mathbf{k}}\left[e^{it\sqrt{\lambda_{\mathbf{k}}(h)}}\widehat{u}_{\mathbf{k}+}+e^{-it\sqrt{\lambda_{\mathbf{k}}(h)}}\widehat{u}_{\mathbf{k}-}\right]\overline{\varphi}^{\mathbf{k}}$$

Filtering: Zuazua 99, Multipliers



$$\Pi_\gamma u = \frac{1}{2} \sum_{\lambda_{\mathbf{k}}(h) \leq \gamma/h^2} \left[e^{it\sqrt{\lambda_{\mathbf{k}}(h)}} \widehat{u}_{\mathbf{k}+} + e^{-it\sqrt{\lambda_{\mathbf{k}}(h)}} \widehat{u}_{\mathbf{k}-} \right] \overline{\varphi}^{\mathbf{k}}, \gamma < 4$$

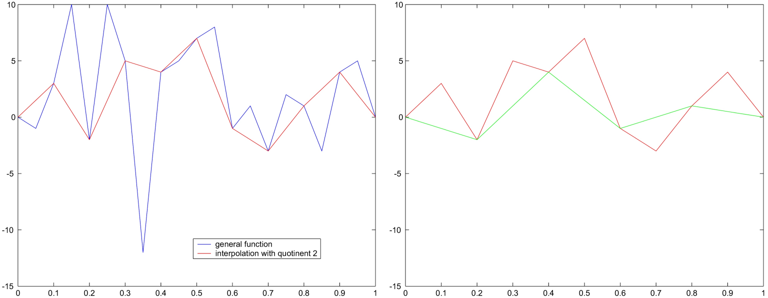
$$E_h(\Pi_\gamma u) \leq \int_0^{T(\gamma)} \int_{\Gamma_h} |\partial_n^h (\Pi_\gamma u)| d\Gamma_h dt$$

Two-grid algorithm, Glowinski '90

1D-case

$T > 4$ - Negreanu & Zuazua 04 - Multiplies

$T > 2\sqrt{2}$ - Loreti & Mehrenberger 05 - Ingham Inequalities for non harmonic series



$E_h(u) \leq 2E_h(\Pi_{1/2}u)$ $\qquad$ $E_h(u) \leq 4E_h(\Pi_{1/4}u)$

New Idea : Low frequency estimates + Semi-classical decomposition following the level sets of the frequencies

Main Result: Let $\overline{u}$ be a solution of (4) and $\gamma > 0$ be such that

$$E_h(\overline{u}) \le C E_h(\Pi_\gamma \overline{u}). \tag{8}$$

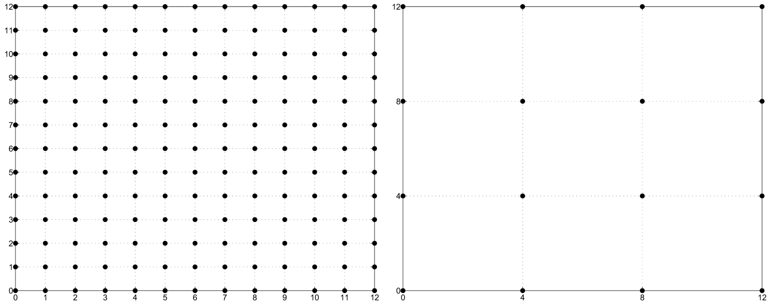
Let us assume the existence of a time $T(\gamma)$ such that for all $T > T(\gamma)$ there exists a constant $C(T)$, independent of h, such that

$$E_h(\overline{v}) \le C(\gamma, T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{v}(t)|^2 d\Gamma dt \tag{9}$$

for all $\overline{v} \in \Pi_\gamma$. Then for all $T > T(\gamma)$ there exists a constant $C_1(T)$, independent of h, such that

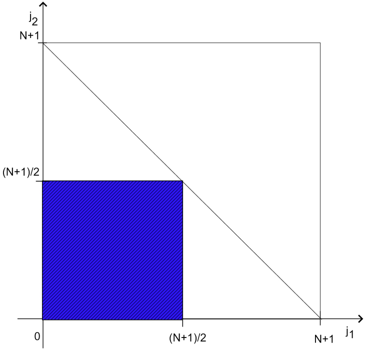
$$E_h(\overline{u}) \le C_1(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt \tag{10}$$

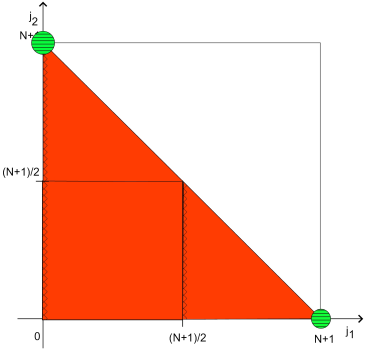
Fine and Coarse Grids G^h and G^{4h}, $N = 11$



Why not using ratio 1/2 for the two-grids?

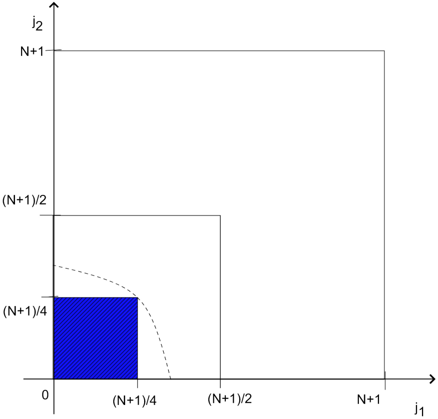
The relevant zone of frequencies intersects a level set of the phase velocity for which the group velocity vanishes at some critical points.





$$E_h(\overline{u}) \le 4E_h(\Pi^{\infty}_{1/2}\overline{u}) \le 4E_h(\Pi_4\overline{u})$$

When using the mesh ratio 1/4 this pathology disappears:



$$E_h(\overline{u}) \leq 16E_h(\Pi^{\infty}_{1/4}\overline{u}) \leq 16E_h(\Pi_{8\sin^2(\pi/8)}\overline{u})$$

Application to a two-grid method

Theorem

Let be $T > 4$. There exists a constant $C(T)$ such that

$$E_h(\overline{u}) \leq C(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt$$

holds for all solutions of (4) with $(\overline{u}^0, \overline{u}^1) \in V^h \times V^h$, uniformly on $h > 0$, V^h being the class of the two-grid data obtained with ratio $1/4$.

$T_0 = 4$ is not optimal one.

Its depends by the optimality of the time for the class $\Pi_{8\sin^2(\pi/8)}$

Analysis of the group velocity: expected time $T_0 = \frac{2\sqrt{2}}{\cos(\pi/8)}$

Main difficulty

$$E_h(\overline{u}) \leq 16 E_h(\Pi^{\infty}_{1/4}\overline{u}) \leq C(T)\int_0^T\int_{\Gamma_h} |\partial_n^h \Pi^{\infty}_{1/4}\overline{u}|^2 d\Gamma_h dt.$$

$$\not\Rightarrow$$

$$E_h(\overline{u}) \leq C(T)\int_0^T\int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt$$

Sketch of the proof

$$E_h(\overline{u}) \leq C E_h(\Pi_\gamma \overline{u}) \leq C(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \Pi_\gamma \overline{u}|^2 d\Gamma_h dt$$
$$\overset{?}{\leq} C(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt + LOT$$

Diadic decomposition or Semi-classical decomposition:

$$P_k \overline{u}(t) = \sum_{\mathbf{j} \in \mathbb{Z}^2} F(c^{-k} \omega_{\mathbf{j}}(h)) \left[e^{it\omega_{\mathbf{j}}(h)} \widehat{u}_+(\mathbf{j}) + e^{-it\omega_{\mathbf{j}}(h)} \widehat{u}_-(\mathbf{j}) \right] \overline{\varphi}^{\mathbf{j}} \qquad (11)$$

$$E_h(\Pi_\gamma \overline{u}) \leq \sum_{k=k_0}^{k_h} E_h(P_k \overline{u}) + LOT \tag{12}$$

$$E_h(P_k \overline{u}) \leq C(T,\gamma,\delta) \int_\delta^{T-\delta} \int_{\Gamma_h} |\partial_n^h P_k \overline{u}|^2 d\Gamma_h dt. \tag{13}$$

Combining (12) and (13),

$$E_h(\Pi_\gamma \overline{u}) \leq C(T,\gamma,\delta) \sum_{k=k_0}^{k_h} \int_\delta^{T-\delta} \int_{\Gamma_h} |\partial_n^h P_k \overline{u}|^2 d\Gamma_h dt + LOT.$$

Lebeau and Burq :

$$\sum_{k=k_0}^{k_h} \int_\delta^{T-\delta} \int_{\Gamma_h} |\partial_n^h P_k \overline{u}|^2 d\Gamma_h dt \leq 2 \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt + \frac{C(\delta, T)}{c^{2k_0}} E_h(\overline{u})$$

Thus

$$E_h(\overline{u}) \lesssim E_h(\Pi_\gamma \overline{u}) \leq C(T, \gamma, \delta) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt + \frac{C(\delta, T)}{c^{2k_0}} E_h(\overline{u}) + LOT$$

$$E_h(\overline{u}) \leq C(T, \gamma, \delta) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt + LOT.$$