A nonlocal convection-diffusion equation

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Joint work with Julio Rossi (Buenos Aires)

Few words about local diffusion problems

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u(0) = u_0. \end{cases}$$

For any $u_0 \in L^1(\mathbb{R})$ the solution $u \in C([0,\infty), L^1(\mathbb{R}^d))$ is given by:

$$u(t,x) = (H(t,\cdot) * u_0)(x)$$

where

$$H(t,x) = (4\pi t)^{d/2} \exp\left(-\frac{|x|^2}{4t}\right)$$

Smoothing effect

$$u \in C^\infty((0,\infty), \mathbb{R}^d)$$

Decay of solutions, $1 \leq p \leq q \leq \infty$:

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \lesssim t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|u_0\|_{L^p(\mathbb{R}^d)}$$

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Refined asymptotics

Zuazua& Duoandikoetxea, CRAS '92
For all $\varphi \in L^p(\mathbb{R}^d, 1+|x|^k)$

$$u(t,\cdot) \sim \sum_{|\alpha|\leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int f(x) x^\alpha dx \right) D^\alpha H(t,\cdot) \qquad \text{in } L^q(\mathbb{R}^d)$$

for some p, q, k

A linear nonlocal problem

E. Chasseigne, M. Chaves and J. D. Rossi, *Asymptotic behavior for nonlocal diffusion equations*, J. Math. Pures Appl., 86, 271–291, (2006).

$$\begin{cases} u_t(x,t) &= J*u - u(x,t) = \int_{\mathbb{R}^d} J(x-y)u(y,t)\,dy - u(x,t), \\ &= \int_{\mathbb{R}^d} J(x-y)(u(y,t)-u(x,t))dy \\ u(x,0) &= u_0(x), \end{cases}$$

where $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radial function with $\int_{\mathbb{R}^N} J(r)dr = 1$

P. Fife. *Some nonclassical trends in parabolic and parabolic-like evolutions*. Trends in nonlinear analysis, 153–191, Springer, Berlin, 2003.

• $u(x,t)$ - the density of a single population at the point x at time t

• $J(x-y)$ - the probability distribution of jumping from y to x

Then

• $(J*u)(x,t) = \int_{\mathbb{R}^d} J(y-x)u(y,t)\,dy$ is the rate at which individuals are arriving to x from all other places and $-u(x,t) = -\int_{\mathbb{R}^d} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving x to travel to all other sites.

Thus in the absence of external or internal sources, the density u satisfies the nonlocal equation (4).

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Some applications

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Heat equation and nonlocal diffusion

- Similarities
  - bounded stationary solutions are constant
  - a maximum principle holds for both of them

- Difference
  - there is no regularizing effect in general

  The fundamental solution can be decomposed as

$$w(x,t) = e^{-t}\delta_0(x) + v(x,t), \qquad (1)$$

with $v(x,t)$ smooth

$$\begin{aligned} S(t)\varphi = e^{-t}\varphi + v * \varphi &= \text{smooth as initial data} + \text{smooth part} \\ &= \text{no smoothing effect} \end{aligned}$$

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$$= \text{no smoothing effect}$$

• If $\hat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha), \xi \sim 0$, the asymptotic behavior is the same as the one for solutions of the evolution given by the $\alpha/2$ fractional power of the laplacian:

$$\lim_{t\to+\infty} t^{d/\alpha} \max_x |u(x,t) - v(x,t)| = 0,$$

where v is the solution of $v_t(x,t) = -A(-\Delta)^{\alpha/2} v(x,t)$ with initial condition $v(x,0) = u_0(x)$.

• The asymptotic profile is given by

$$\lim_{t\to+\infty} \max_y \left| t^{d/\alpha} u(yt^{1/\alpha}, t) - (\int_{\mathbb{R}^d} u_0) G_A(y) \right| = 0,$$

where $G_A(y)$ satisfies $\hat{G}_A(\xi) = e^{-A|\xi|^\alpha}$.

Other results on the linear problem

- I.L. Ignat and J.D. Rossi, *Refined asymptotic expansions for nonlocal diffusion equations*, submitted to Journal of Evolution Equations.
- I.L. Ignat and J.D. Rossi, *Asymptotic behaviour for a nonlocal diffusion equation on a lattice*, Accepted in Zeitschrift für Angewandte Mathematik und Physik (ZAMP).

The classical convection-diffusion equation

For $a \in \mathbb{R}^d$ and $q \geq 1$

$$\begin{cases} u_t - \Delta u = a \cdot \nabla(|u|^{q-1}u) \text{ in } (0,\infty) \times \mathbb{R}^d \\ u(0) = u_0 \end{cases}$$

- Asymptotic Behaviour by using

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u|^p dx = -\frac{4(p-1)}{p}\int_{\mathbb{R}^d} |\nabla(|u|^{p/2})|^2 dx.$$

M. Schonbek, *Uniform decay rates for parabolic conservation laws*, Nonlinear Anal., 10(9), 943–956, (1986).

M. Escobedo and E. Zuazua, *Large time behavior for convection-diffusion equations in* $\mathbb{R}^N$, J. Funct. Anal., 100(1), 119–161, (1991).

- M. Schonbek $\rightarrow$ Fourier Splitting Method

Using that $\frac{d}{dt}\int_{\mathbb{R}^d}|u|^2dx \leq -c\int_{\mathbb{R}^d}|\nabla u|^2dx$, define $S(t)=\{\xi : |\xi|\leq (\frac{d}{c(t+1)})^{1/2}\}$. Then

$$\begin{aligned}\frac{d}{dt}\int u^2 &\leq -c\int_S |\xi|^2\hat{u}^2 - c\int_{S^c}|\xi|^2\hat{u}^2\\ &\leq -\frac{d}{t+1}\int_{S^c}\hat{u}^2 = -\frac{d}{t+1}\int u^2 + \frac{d}{t+1}\int_S \hat{u}^2\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}(t+1)^d\int u^2 \leq d(\frac{d}{t+1})^{d-1}\int_S \hat{u}^2 &\leq c(d)(t+1)^{-d/2+1}\|\hat{u}\|_\infty^2\\ &\leq c(d)(t+1)^{-d/2+1}\|u(t)\|_{L^1(\mathbb{R}^d)}\\ &\leq c(d)(t+1)^{-d/2+1}\|u_0\|_{L^1(\mathbb{R}^d)}\end{aligned}$$

• M. Escobedo and E. Zuazua → Energy method

For $d \geq 3$ using the Sobolev inequality $\|v\|_{2d/(d-2)} \leq C(d)\|\nabla v\|_2$ with $v = |u|^{p/2}$ we get

$$\|u\|^p_{pd/(d-2)} \leq C(d)\|\nabla(|u|^{p/2})\|^2_2.$$

Interpolation inequality

$$\|u\|_p \leq \|u\|^{d(p-1)/(2+d(p-1))}_{pd/(d-2)}\|u\|_1^{2/(2+d(p-1))}$$

gives us

$$\|u\|_p^{(d(p-1)+2)/d(p-1)} \leq c(p,d)\|u\|_1^{2p/d(p-1)}\|\nabla(|u|^{p/2})\|^2_2.$$

Then

$$\frac{d}{dt}\|u(t)\|^p_p + \frac{C}{\|u_0\|_1^{2p/d(p-1)}}\|u(t)\|_p^{p[d(p-1)+2]/d(p-1)} \leq 0$$

.... etc...

Nonlocal Convection-Diffusion

L.I. Ignat and J.D. Rossi, *A nonlocal convection-diffusion equation*, J. Funct. Anal., 251, 399–437, (2007).

$$\begin{cases} u_t(t,x) = (J*u-u)(t,x) + (G*(f(u))-f(u))(t,x), & t>0,\ x\in \\ u(0,x)=u_0(x), & x\in\mathbb{R}^d. \end{cases}$$

- J and G are nonnegatives and verify $\int_{\mathbb{R}^d} J(x)dx = \int_{\mathbb{R}^d} G(x)dx = 1$.
- J smooth, $J\in\mathcal{S}(\mathbb{R}^d)$, radially symmetric
- G smooth, $G\in\mathcal{S}(\mathbb{R}^d)$, but not necessarily symmetric, then the individuals have greater probability of jumping in one direction than in others, provoking a convective effect
- $\widehat{J}(\xi)-1+\xi^2\sim|\xi|^3$, for ξ close to 0.
- $f(u)=|u|^{q-1}u$ with $q>1$

Well-posedness

Theorem

For any $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *there exists a unique global solution*

$$u \in C([0,\infty); L^1(\mathbb{R}^d)) \cap L^\infty([0,\infty); \mathbb{R}^d).$$

If u *and* v *are solutions of* (12) *corresponding to initial data* $u_0, v_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *respectively, then the following contraction property*

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}$$

holds for any $t \geq 0$. *In addition,*

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}.$$

Why not $u_0 \in L^1(\mathbb{R}^d)$?

- For the local problem $\|v(t)\|_\infty \le C\|v_0\|_1 t^{-d/2}$ for any $v_0 \in L^1 \cap L^\infty$, so for any $u_0 \in L^1$ we can choose $u_{0,\epsilon} \in L^1 \cap L^\infty$ with $u_{0,\epsilon} \to u_0$ in L^1 and we can pass to the limit.
- In the nonlocal model, we cannot prove such type of inequality independently of the L^∞-norm of the initial data.
- In the one-dimensional case with $f(u) = |u|^{q-1}u$, $1 \le q < 2$ we have
$$\sup_{u_0 \in L^1(\mathbb{R})} \sup_{t \in [0,1]} \frac{t^{\frac{1}{2}}\|u(t)\|_{L^\infty(\mathbb{R})}}{\|u_0\|_{L^1(\mathbb{R})}} = \infty.$$
- The $L^1 - L^\infty$ regularizing effect is not available for the linear equation $w_t = J * w - w$

$$\begin{cases} w_t(t,x) = (J * w - w)(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ w(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{2}$$

Lemma

The fundamental solution of (2) can be decomposed as

$$S(t,x) = e^{-t}\delta_0(x) + K_t(x), \tag{3}$$

with

$$K_t(x) = e^{-t}(\exp(t\widehat{J}(\xi)) - 1)^{\vee}$$

smooth.

Estimations on the smooth part

Lemma

Let J *be such that* $\widehat{J}(\xi)-1+\xi^2\sim|\xi|^3,$ *as* $\xi\sim 0$*. For any* $p\geq 1$*,* K_t *satisfies:*

$$\|K_t\|_{L^p(\mathbb{R}^d)}\leq c(p,J)\,\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})} \tag{4}$$

for any $t>0$*.*

Remark

Assumning that $J\in\mathcal{S}(\mathbb{R}^d)$ and replacing t with $\langle t\rangle$, K_t has the same behaviour as the heat kernel H_t

Back to the nonlinear problem: Local existence

Banach's Fix Point Theorem for the functional

$$\Phi[u](t) = S(t) * u_0 + \int_0^t S(t-s) * (G * (f(u)) - f(u))(s)\, ds$$

and the space

$$X(T) = C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty([0,T]; \mathbb{R}^d)$$

endowed with the norm

$$\|u\|_{X(T)} = \sup_{t\in[0,T]} \left(\|u(t)\|_{L^1(\mathbb{R}^d)} + \|u(t)\|_{L^\infty(\mathbb{R}^d)} \right).$$

Estimates on the linear semigroup

For any $1 \leq p \leq \infty$

- $$\|S(t) * \varphi\|_p \leq (e^{-t} + \|K_t\|_1)\|\varphi\|_p \leq 3\|\varphi\|_p$$

- $$\|S(t) * G * \varphi - S(t) * \varphi\|_{L^p(\mathbb{R}^d)} \leq C\langle t\rangle^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}}\|\varphi\|_{L^p(\mathbb{R}^d)}. \tag{5}$$

Global existence: control of the L^1 and L^∞ norms

- L^1-norm: easy, $\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)|\,dx \le 0$
- L^∞-norm

$$\frac{d}{dt}\int_{\mathbb{R}^d}(u(t,x)-m)^+dx = I_1(t)+I_2(t)$$

where

$$I_1(t)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x-y)u(t,y)\,\mathrm{sgn}(u(t,x)-m)^+\,dy\,dx - \int_{\mathbb{R}^d}u(t,x)\,\mathrm{sgn}(u(t,x)-m)^+\,dx$$

and

$$I_2(t)=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}G(x-y)f(u)(t,y)\,\mathrm{sgn}(u(t,x)-m)^+\,dy\,dx - \int_{\mathbb{R}^d}f(u)(t,x)\,\mathrm{sgn}(u(t,x)-m)^+\,dx.$$

An useful lemma

Lemma

Let $\theta \in L^1(\mathbb{R}^d)$ and K be a nonnegative function with mass one. Then for any $\mu \geq 0$ the following hold:

$$\int_{\theta(x)>\mu} \int_{\mathbb{R}^d} K(x-y)\theta(y)\,dy\,dx \leq \int_{\theta(x)>\mu} \theta(x)\,dx. \tag{6}$$

Proof. Step 1. $\mu = 0 \to \mu > 0$. For $\mu = 0$

$$\int_{\theta(x)>0} \int_{\mathbb{R}^d} K(x-y)\theta(y)\,dy\,dx \leq \int_{\theta(x)>0} \int_{\theta(y)>0} K(x-y)\theta(y)\,dy\,dx$$

$$= \int_{\theta(y)>0} \theta(y) \int_{\theta(x)>0} K(x-y)\,dx\,dy \leq \int_{\theta(y)>0} \theta(y) \int_{\mathbb{R}^d} K(x-y)\,dx\,dy$$

$$= \int_{\theta(y)>0} \theta(y)\,dy.$$

Blow-up of the L^∞-norm for L^1-data

Due to the lack of regularizing effect, the $L^\infty(\mathbb{R})$-norm does not get bounded for positive times when we consider initial conditions in $L^1(\mathbb{R})$. This is in contrast to what happens for the local convection-diffusion problem.

Proposition

Let $d = 1$ and $|f(u)| \leq C|u|^q$ with $1 \leq q < 2$. Then

$$\sup_{u_0 \in L^1(\mathbb{R})} \sup_{t \in [0,1]} \frac{t^{\frac{1}{2}} \|u(t)\|_{L^\infty(\mathbb{R})}}{\|u_0\|_{L^1(\mathbb{R})}} = \infty.$$

Assume the above limit to be finite. Then

$$\|u(1)\|_{L^\infty(\mathbb{R})} \geq \|S(1)*u_0\|_{L^\infty(\mathbb{R})} - \Big\| \int_0^1 S(1-s)*(G*(f(u))-f(u))(s)\,ds \Big\|_{L^\infty(\mathbb{R})}$$

and

$$\begin{aligned}
\Big\| \int_0^1 S(1-s)*(G*(f(u))-f(u))(s)\,ds \Big\|_{L^\infty(\mathbb{R})} \\
&\leq \int_0^1 \langle 1-s\rangle^{-\frac{1}{2}} \|f(u(s))\|_{L^\infty(\mathbb{R})}\,ds \\
&\leq C\int_0^1 \|u(s)\|^q_{L^\infty(\mathbb{R})} ds \leq CM^q \|u_0\|^q_{L^1(\mathbb{R})} \int_0^1 s^{-\frac{q}{2}} ds \\
&\leq CM^q \|u_0\|^q_{L^1(\mathbb{R})}.
\end{aligned}$$

The $L^\infty(\mathbb{R})$-norm of the solution at time $t = 1$ satisfies

$$\begin{aligned}\|u(1)\|_{L^\infty(\mathbb{R})} &\geq \|S(1) * u_0\|_{L^\infty(\mathbb{R})} - CM^q\|u_0\|^q_{L^1(\mathbb{R})}\\ &\geq e^{-1}\|u_0\|_{L^\infty(\mathbb{R})} - \|K_1\|_{L^\infty(\mathbb{R})}\|u_0\|_{L^1(\mathbb{R})} - CM^q\|u_0\|^q_{L^1(\mathbb{R})}\\ &\geq e^{-1}\|u_0\|_{L^\infty(\mathbb{R})} - C\|u_0\|_{L^1(\mathbb{R})} - CM^q\|u_0\|^q_{L^1(\mathbb{R})}.\end{aligned}$$

Choosing now a sequence $u_{0,\varepsilon}$ with $\|u_{0,\varepsilon}\|_{L^1(\mathbb{R})} = 1$ and $\|u_{0,\varepsilon}\|_{L^\infty(\mathbb{R})} \to \infty$ we obtain that

$$\|u_{0,\varepsilon}(1)\|_{L^\infty(\mathbb{R})} \to \infty.$$

Convergence to the local problem

Let us consider

$$J_\varepsilon(s) = \frac{1}{\varepsilon^d} J\left(\frac{s}{\varepsilon}\right), \qquad G_\varepsilon(s) = \frac{1}{\varepsilon^d} G\left(\frac{s}{\varepsilon}\right),$$

and the solution $u_\varepsilon(t,x)$ to our convection-diffusion problem rescaled adequately,

$$\begin{cases} (u_\varepsilon)_t(t,x) &= \dfrac{1}{\varepsilon^2}\displaystyle\int_{\mathbb{R}^d} J_\varepsilon(x-y)(u_\varepsilon(t,y)-u_\varepsilon(t,x))\,dy \\ &\qquad + \dfrac{1}{\varepsilon}\displaystyle\int_{\mathbb{R}^d} G_\varepsilon(x-y)(f(u_\varepsilon(t,y))-f(u_\varepsilon(t,x))) \\ u_\varepsilon(x,0) &= u_0(x). \end{cases} \tag{7}$$

Convergence to the local problem

Theorem

For any $T > 0$, *we have*

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_\varepsilon - v\|_{L^2(\mathbb{R}^d)} = 0,$$

where $v(t,x)$ *is the unique solution to the local convection-diffusion problem* $v_t(t,x) = \Delta v(t,x) + b \cdot \nabla f(v)(t,x)$ *with initial condition* $v(x,0) = u_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ *and* $b = (b_1, ..., b_d)$ *given by*

$$b_j = \int_{\mathbb{R}^d} x_j \, G(x) \, dx, \qquad j = 1, ..., d.$$

Proof of convergence

We write the two problems in the semigroup formulation,

$$u_\varepsilon(t) = S_\varepsilon(t) * u_0 + \int_0^t \frac{S_\varepsilon(t-s) * G_\varepsilon - S_\varepsilon(t-s)}{\varepsilon} * f(u_\varepsilon(s))\,ds$$

and

$$v(t) = H(t) * u_0 + \int_0^t b \cdot \nabla H(t-s) * f(v(s))\,ds.$$

Then

$$\sup_{t\in[0,T]} \|u_\varepsilon(t) - v(t)\|_{L^2(\mathbb{R}^d)} \leq \sup_{t\in[0,T]} I_{1,\varepsilon}(t) + \sup_{t\in[0,T]} I_{2,\varepsilon}(t)$$

where $I_{1,\varepsilon}(t)$ and $I_{2,\varepsilon}(t)$ are given by the difference of linear, respectively nonlinear part.

Estimates on $I_{1,\varepsilon}(t)$

Using the Fourier representation of the two semigroups and applying Lebesgue's Convergence Theorem we get

$$\|S_\varepsilon(t)*u_0-H(t)*u_0\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \left| e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}} - e^{-t\xi^2} \right|^2 |\widehat{u_0}(\xi)|^2 \, d\xi \to 0$$

Lemma (Uniform decay with respect to ε)

There exists a positive constant $C = C(J, G)$ such that

$$\Big\| \Big(\frac{S_\varepsilon(t) * G_\varepsilon - S_\varepsilon(t)}{\varepsilon} \Big) * \varphi \Big\|_{L^2(\mathbb{R}^d)} \leq C\, t^{-\frac{1}{2}} \|\varphi\|_{L^2(\mathbb{R}^d)}$$

holds for all $t > 0$ and $\varphi \in L^2(\mathbb{R}^d)$, uniformly on $\varepsilon > 0$.

Remark

The decay is the same as for ∇H:

$$\|\nabla H * \varphi\|_{L^2(\mathbb{R})} \leq t^{-1/2} \|\varphi\|_{L^2(\mathbb{R})}.$$

Lemma

Let be $T > 0$ and $M > 0$. Then the following

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \int_0^t \left\| \left(\frac{S_\varepsilon(s) * G_\varepsilon - S_\varepsilon(s)}{\varepsilon} - b \cdot \nabla H(s) \right) * \varphi(s) \right\|_{L^2(\mathbb{R}^d)} ds = 0,$$

holds uniformly for all $\|\varphi\|_{L^\infty([0,T];L^2(\mathbb{R}^d))} \leq M$. Here $b = (b_1, ..., b_d)$ is given by

$$b_j = \int_{\mathbb{R}^d} x_j \, G(x) \, dx, \qquad j = 1, ..., d.$$

Long time behaviour of the solutions

Theorem

Let $f = |u|^{q-1}u$ *with* $q > 1$ *and* $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. *Then, for every* $p \in [1, \infty)$ *the solution* u *of equation* (12) *verifies*

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \leq C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}) \langle t \rangle^{-\frac{d}{2}(1-\frac{1}{p})}. \tag{8}$$

A differential inequality

Lemma

Let R and δ be such that the function $\widehat{J}$ satisfies:

$$\widehat{J}(\xi) \leq 1 - \frac{|\xi|^2}{2},\ |\xi| \leq R \ \text{ and } \widehat{J}(\xi) \leq 1-\delta,\ |\xi| \geq R. \qquad (9)$$

Let us assume that for any $t>0$ the function $u:[0,\infty)\times\mathbb{R}^d \to \mathbb{R}$ satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(t,x)|^2\,dx \leq -\frac{c}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(t,x)-u(t,y))^2\,dx\,dy$$

Then there exists a constant $a = d/c\delta$ such that

$$\|u(at)\|_{L^2(\mathbb{R}^d)} \leq \frac{\|u(0)\|_{L^2(\mathbb{R}^d)}}{(t+1)^{\frac{d}{2}}} + \frac{(2\omega_0)^{\frac{1}{2}}(2\delta)^{\frac{d}{4}}}{(t+1)^{\frac{d}{4}}}\|u\|_{L^\infty([0,\infty);\,L^1(\mathbb{R}^d))}.$$

Why not an energy method

If we want to use energy estimates to get decay rates (for example in $L^2(\mathbb{R}^d)$), we arrive easily to

$$\frac{d}{dt}\int_{\mathbb{R}^d}|w(t,x)|^2\,dx=-\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x-y)(w(t,x)-w(t,y))^2\,dx\,dy$$

when we deal with a solution of the linear equation
$w_t=J*w-w$ and to

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)|^2\,dx\le-\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x-y)(u(t,x)-u(t,y))^2\,dx\,dy$$

when we consider the convection-diffusion problem. However, we can not go further since an inequality of the form

$$\left(\int_{\mathbb{R}^d}|u(x)|^p\,dx\right)^{\frac{2}{p}}\le C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x-y)(u(x)-u(y))^2\,dx\,dy$$

is not available for $p>2$ (or we have no idea how to get something useful).

Theorem

Let $f = |u|^{q-1}u$ *with* $q > (d+1)/d$ *and* $u_0 \in L^1(\mathbb{R}^d, 1+|x|) \cap L^\infty(\mathbb{R}^d)$. *For any* $p \in [2,\infty)$ *the following holds*

$$t^{-\frac{d}{2}(1-\frac{1}{p})}\|u(t) - H(t)\int_{\mathbb{R}^d} u_0(x)\,dx\|_{L^p(\mathbb{R}^d)} \leq C(J,G,p,d)\,\alpha_q(t),$$

where

$$\alpha_q(t) = \begin{cases} \langle t\rangle^{-\frac{1}{2}} & \text{if } \quad q \geq (d+2)/d, \\ \langle t\rangle^{\frac{1-d(q-1)}{2}} & \text{if } \quad (d+1)/d < q < (d+2)/d. \end{cases}$$

The proof consists in two ingredients
• Estimate the difference $S(t)*\varphi - MH(t)$, known from our previous paper

$$
\begin{aligned}
\|S(t)*\varphi - MH(t)\|_{L^p(\mathbb{R}^d)} \leq & Ce^{-t}\|\varphi\|_{L^p(\mathbb{R}^d)} \\
& + C\|\varphi\|_{L^1(\mathbb{R}^d,|x|)}\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}, \ t>0,
\end{aligned}
$$

• Show that the contribution from the nonlinear part goes to zero faster than the above difference.

Some Open Problems

- The second term in the asymptotic expansion of the nonlocal convection-diffusion equation
- For the linear part, to try to remove the convolution property of the function J

$$u_t(t,x) = \int_{\mathbb{R}} J(x,y)(u(t,y)-u(t,x))dy$$

under suitable properties of J: symmetric, $\int_{\mathbb{R}} Jdx = \int_{\mathbb{R}} Jdy = 1$, etc...

For a proof we invite to a 

- An energy method which will allow us to recover the decay properties
- Something similar to the viscous Bourger's equation $u_t - u_{xx} = (u^2)_x$ where the asymptotic profile is given by the self-similar solutions
- Until now we treat the case $q > 1 + 1/d$, case when the diffusion play the important role. What it happens when $q < 1 + 1/d$? Is there something similar to the local case, where the convection part gives the asymptotic behaviour (Escobedo, Vazquez, Zuazua, ARMA 1993)

End of the talk

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