Uniform Boundary Observability of a Two-Grid Method for the 2d-Wave Equation

Liviu Ignat
Universidad Autónoma de Madrid, Spain
liviu.ignat@uam.es,

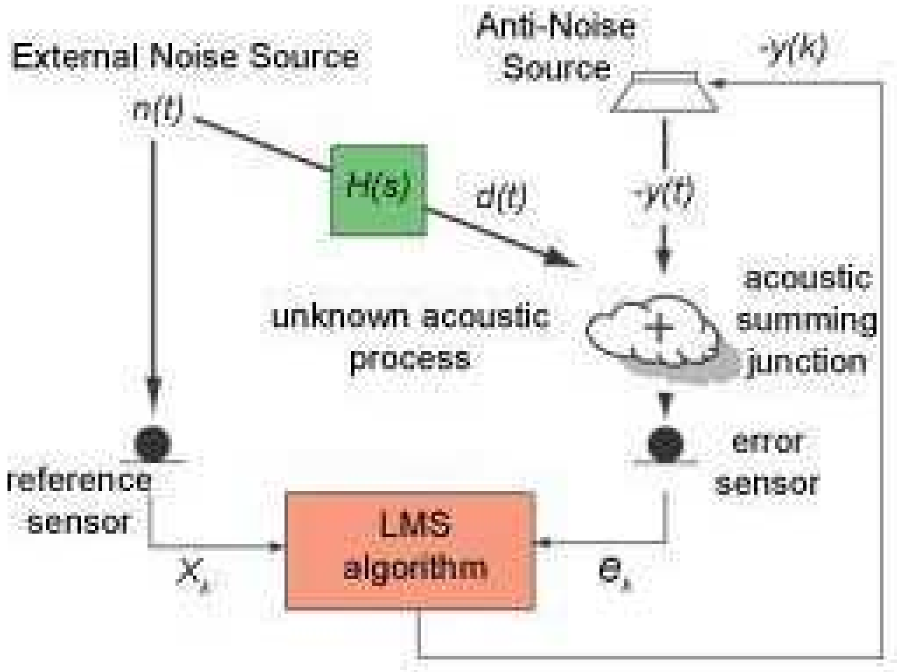
IMAR, Bucharest, 12 April 2006

Joint work with Enrique Zuazua

IS THE CONTROL OF WAVES AND, MORE PARTICULARLY, OF THE WAVE EQUATION RELEVANT?

The answer is, definitely, YES.

- Noise reduction in cavities and vehicles.



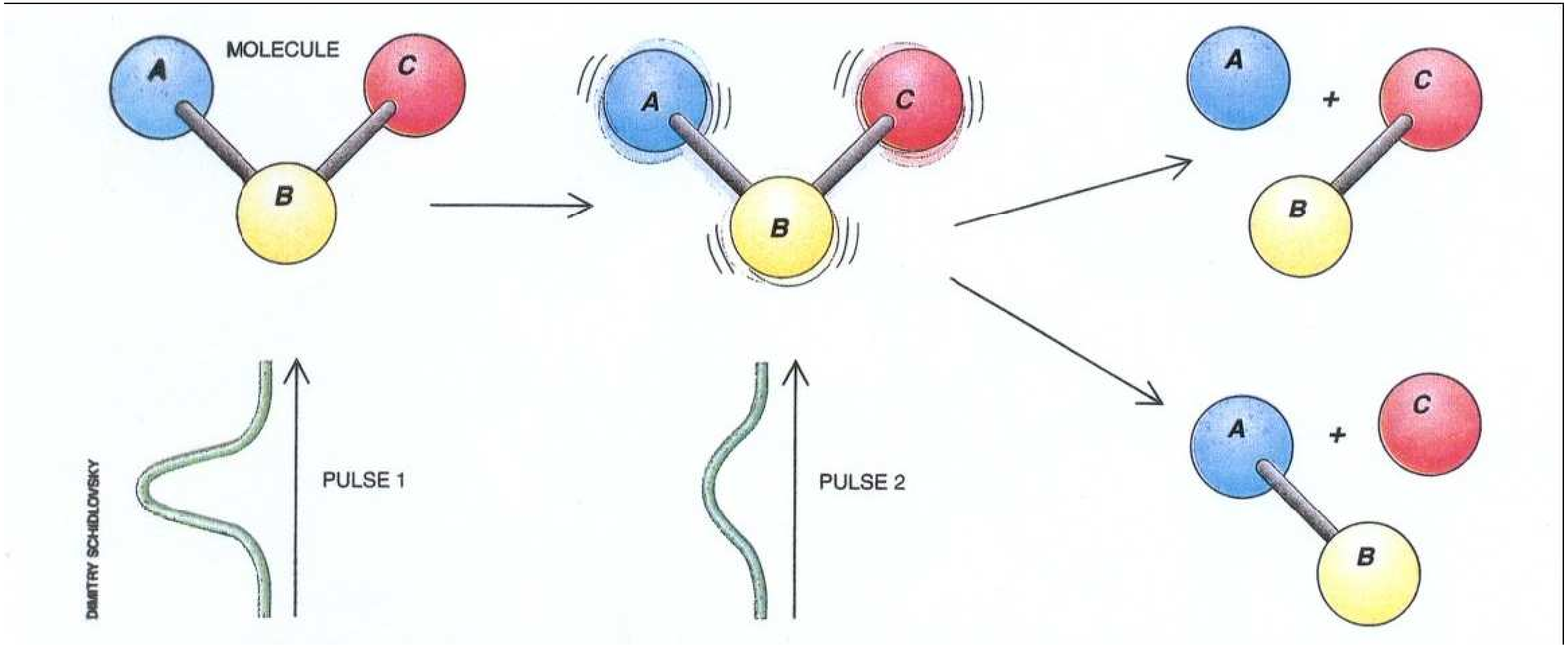
Closed-loop control diagram.

http://www.ind.rwth-aachen.de/research/noise reduction.html

- Quantum control and Computing.

Laser control in Quantum mechanical and molecular systems to design coherent vibrational states.

In this case the fundamental equation is the Schrödinger one. Most of the theory we shall develop here applies in this case too. The Schrödinger equation may be viewed as a wave equation with infinite speed of propagation.



P. Brumer and M. Shapiro, Laser Control of Chemical reactions, Scientific American, March, 1995, pp.34-39.

THE 1-D CONTROL PROBLEM

The 1-d wave equation, with Dirichlet boundary conditions, describing the vibrations of a flexible string, with control at one end:

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1,\ 0 < t < T, \\ y(0,t) = 0; y(1,t) = v(t), & 0 < t < T, \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x), & 0 < x < 1 \end{cases}$$

$y = y(x,t)$ is the state and $v = v(t)$ is the control.

The goal is to stop the vibration, i.e. to drive the solution to equilibrium in a given time T: Given initial data $\{y^0(x), y^1(x)\}$ to find a control $v = v(t)$ such that

$$y(x,T) = y_t(x,T) = 0,\ 0 < x < 1.$$

THE 1-D OBSERVATION PROBLEM

The control problem above is equivalent to the following one, on the adjoint wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1, \; 0 < t < T, \\ u(0,t) = 0; u(1,t) = 0, & 0 < t < T, \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x), & 0 < x < 1 \end{cases}$$

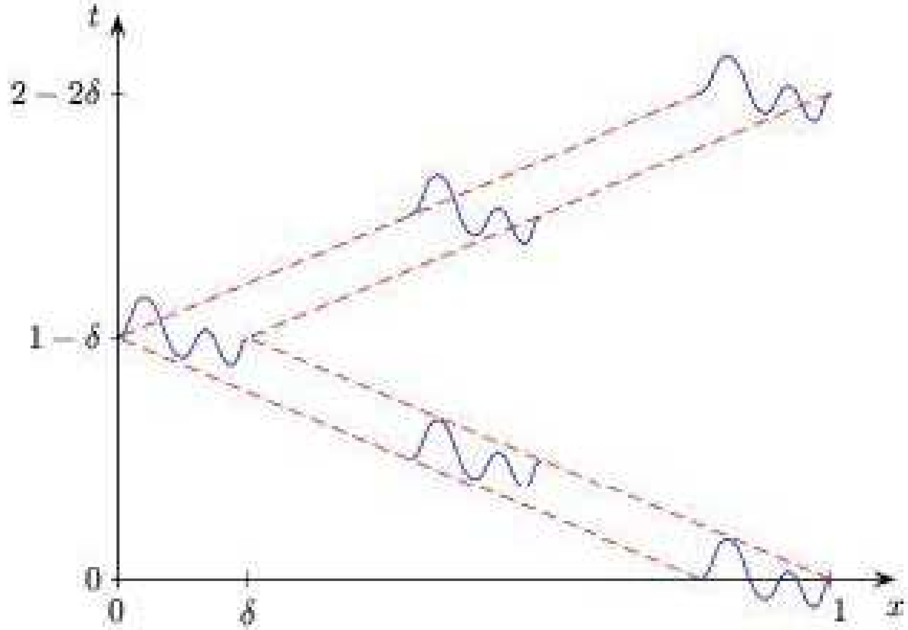
The energy of solutions is conserved in time, i.e.

$$E(t) = \frac{1}{2} \int_0^1 \left[|u_x(x,t)|^2 + |u_t(x,t)|^2 \right] dx = E(0), \; \forall \, 0 \leq t \leq T.$$

The question is then reduced to analyze whether so called observability inequality is true for all solutions u:

$$E(0) \leq C(T) \int_0^T |u_x(1,t)|^2 dt.$$

The answer to this question is easy to guess: The observability inequality holds if and only if $T \geq 2$.



Wave localized at $t = 0$ near the extreme $x = 1$ that propagates with velocity one to the left, bounces on the boundary point $x = 0$ and reaches the point of observation $x = 1$ in a time of the order 2.

CONSTRUCTION OF THE CONTROL:

Once the observability inequality is known the control is easy to characterize. Following J.L. Lions's HUM (Hilbert Uniqueness Method), the control is

$$v(t) = u_x(1,t),$$

where u is the solution of the adjoint system corresponding to initial data $(u_0, u_1) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional

$$J(u_0, u_1) = \frac{1}{2} \int_0^T |u_x(1,t)|^2 dt + \int_0^1 y^0 u^1 dx - \langle y^1, u^0 \rangle_{H^{-1} \times H_0^1}$$

, in the space $H_0^1(0,1) \times L^2(0,1)$.

COERCIVITY OF J = OBSERVABILITY INEQUALITY

CONCLUSION:

The $1-d$ wave equation is controllable from one end, in time 2, twice the length of the interval.

Similar results are true in several space dimensions. The region in which the observation/control applies needs to be large enough to capture all rays of Geometric Optics. (Bardos-Lebeau-Rauch, Burq-Gérard)

THE PROBLEM:

EFFICIENTLY COMPUTE NUMERICALLY THE CONTROL

THE SEMI-DISCRETE PROBLEM: 1-D.

Set $h = 1/(N+1)$ and consider the mesh

$$x_0 = 0 < x_1 < \cdots < x_j = jh < x_N = 1 - h < x_{n+1} = 1,$$

which divides $[0,1]$ into $N+1$ subintervals $I_j = [x_j, x_{j+1}]$, $j = 0, \ldots, N$.

Finite difference semi-discrete approximation of the control problem

$$\begin{cases} y_j'' - \frac{y_{j+1} - 2y_j + y_{j-1}}{h^2} = 0, & j = 1, \ldots, N,, \ 0 < t < T, \\ y_0(t) = 0; y_{N+1}(t) = v(t), & 0 < t < T, \\ y_j(0) = y_j^0, y_j'(0) = y_j^1, & j = 1, \ldots, N \end{cases}$$

and its adjoint

$$\begin{cases} u_j'' - \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = 0, & 0 < t < T, j = 1, \ldots, N, \\ u_0(t) = 0, u_{N+1}(t) = 0 & 0 < t < T, \\ u_j(0) = u_j^0, u_j'(0) = u_j^1, & j = 1, \ldots, N,. \end{cases}$$

The energy of the semi-discrete system (a discrete version of the continuous one)

$$E_h(t) = \frac{h}{2} \sum_{j=0}^{N} \left[|u_j'(t)|^2 + \left| \frac{u_{j+1}(t) - u_j(t)}{h} \right|^2 \right].$$

It is constant in time.

Is the following observability inequality true?

$$E_h(0) \leq C_h(T) \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt$$

$$\left(-\frac{u_N(t)}{h} = \frac{u_{N+1}(t) - u_N(t)}{h} \sim u_x(1,t) \right)$$

YES! It is true for all $h > 0$ and for all time T.

BUT, FOR ALL $T > 0$ (!!!!!)

$$C_h(T) \to \infty, \; h \to 0.$$

THE FOLLOWING CONJECTURES ARE FALSE:

1. The constant $C_h(T)$ blow-up for $T < 2$ as $h \to 0$ since the inequality fails for the wave equation

2. The constant $C_h(T)$ remains bounded for $T \geq 2$ as $h \to 0$ and one recovers in the limit the observability inequality for the wave equation.

CONCLUSION

The classical convergence (consistency+stability) does not guarantee continuous dependence for the observation problem with respect to the discretization parameter.

WHY?

Convergent numerical schemes do reproduce all continuous waves but, when doing that, they create a lot of spurious (non-realistic, purely numerical) high frequency solutions. These spurious solutions destroy the observation properties and are an obstacle for the controls to converge as the mesh-size gets finer and finer.

SPECTRAL ANALYSIS

Eigenvalue problem

$$\begin{cases} -\frac{1}{h^2}[w_{j+1} - 2w_j + w_{j-1}] = \lambda w_j, \;\; j = 1, \dots, N \\ w_0 = w_{N+1} = 0. \end{cases}$$

The eigenvalues are $0 < \lambda_1(h) < ... < \lambda_N(h)$ are

$$\lambda_k^h = \frac{4}{h^2} \sin^2 \left(\frac{k\pi h}{2} \right)$$

and the eigenvectors

$$w_k^h = (w_{k,1}, \dots, w_{k,N})^T : \, w_{k,j} = \sin(k\pi jh), \; k, j = 1, \dots, N.$$

The solutions of the semi-discrete system may be written in Fourier series as follows:

$$\overrightarrow{u} = \sum_{k=1}^{N} \left(a_k \cos\left(t\sqrt{\lambda_k^h}\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(t\sqrt{\lambda_k^h}\right) \right) \overrightarrow{w}_k^h.$$

Compare with the Fourier representation of solutions of the continuous wave equation:

$$u = \sum_{k=1}^{\infty} \left(a_k \cos(k\pi t) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin(k\pi t) \right) \sin(k\pi x)$$

The only relevant difference is that the time-frequencies do not quite coincide, but they converges as $h \to 0$.

Phase and group velocity:

$$u = e^{i(\omega t - \xi x)}$$

ξ-wave number, ω-frequency

dispersion relation $\omega = \omega(\xi) = \pm\xi$ $\omega_h(\xi) = \pm\frac{2}{h}\,\text{sin}(\frac{\xi h}{2}), \xi \in [-\frac{\pi}{h}, \frac{\pi}{h}]$

phase speed $c(\xi) = \frac{\omega(\xi)}{\xi} = \pm 1$ $c_h(\xi) = \pm\frac{2}{\xi h}\,\text{sin}(\frac{\xi h}{2})$

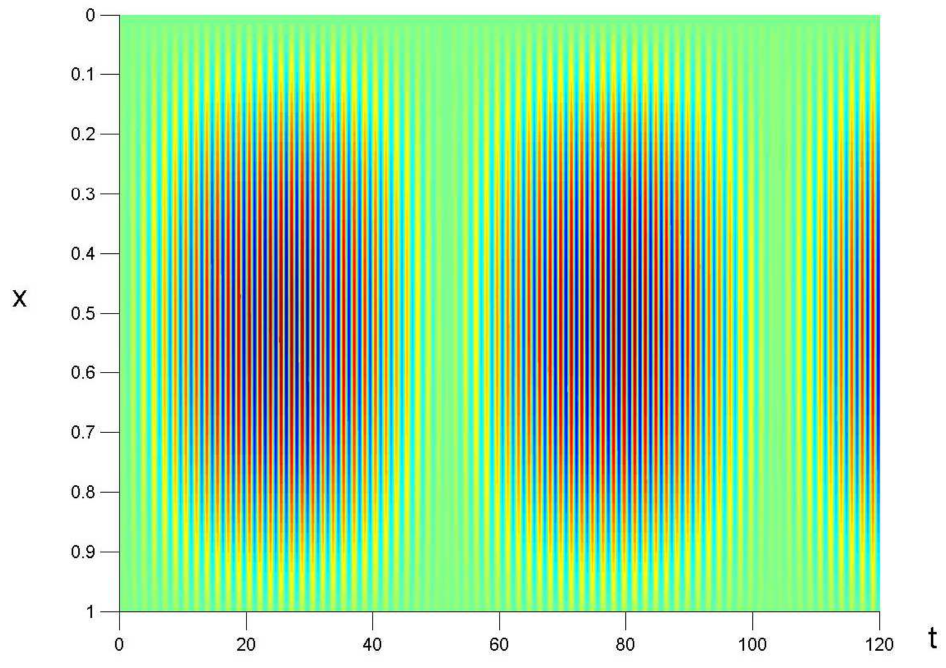
group speed $C(\xi) = \frac{d\omega}{d\xi}(\xi) = \pm 1$ $C_h(\xi) = \text{cos}(\frac{\xi h}{2})$

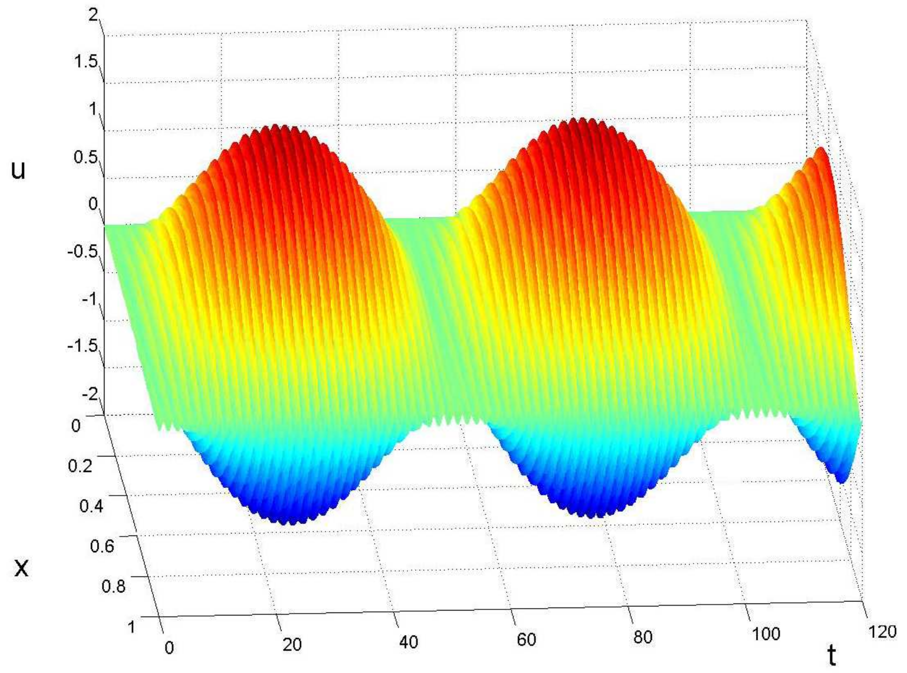
SPURIOUS NUMERICAL SOLUTION

$$\overrightarrow{u} = \exp\left(i\sqrt{\lambda_N(h)}t\right)\overrightarrow{w}_N - \exp\left(i\sqrt{\lambda_{N-1}(h)}t\right)\overrightarrow{w}_{N-1}.$$

Spurious semi-discrete wave combining the last two eigenvectors with very little gap:

$$\sqrt{\lambda_N(h)} - \sqrt{\lambda_{N-1}(h)} \sim h$$





$h = 1/61$, $N = 60$, $0 \leq t \leq 120$. The solution exhibits a time-periodicity property with period τ of the order of $\tau \sim 50$ which contradicts the time-periodicity of period 2 of the wave equation. High frequency wave packets travel at a group velocity $\sim h$.

FIRST REMEDY: FOURIER FILTERING (Zuazua, 99)

To filter the high frequencies, i.e. keep only the components of the solution corresponding to indexes: $k \leq \delta/h$ with $0 < \delta < 1$.

Filtering reestablishes the gap condition, then waves propagate with a speed which is uniform with respect to h and the observability inequality becomes uniform too.

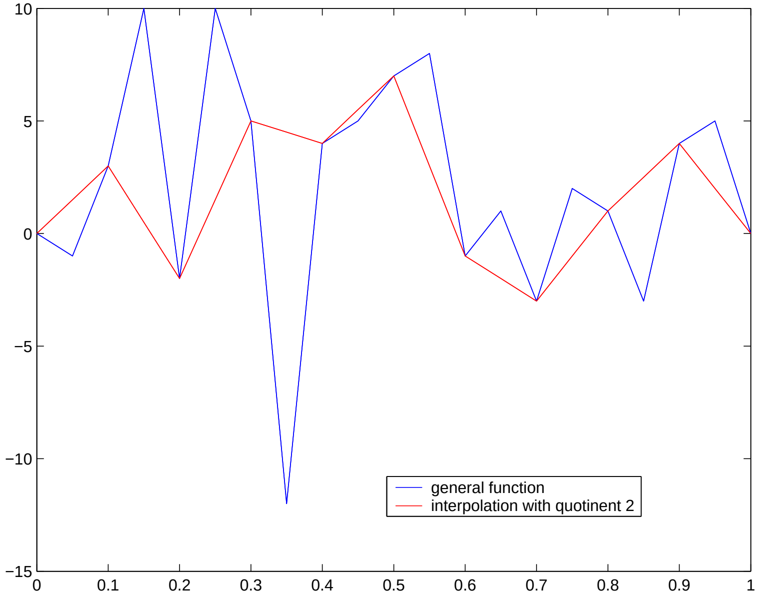
OTHER REMEDIES:

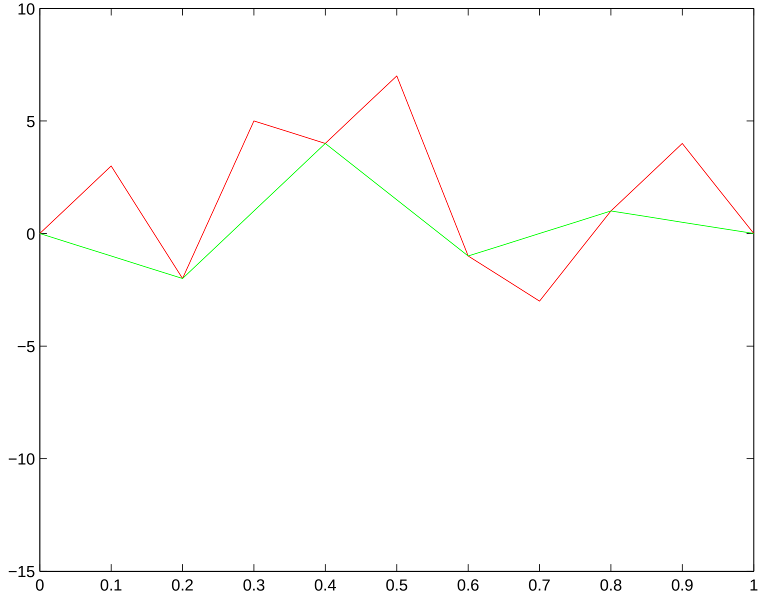
1. mixed finite elements (Glowinski 89, Castro, Micu 2006)

2. Two-grid algorithm Glowinski 90

$T > 4$ - Negreanu & Zuazua 04 - Multiplies

$T > 2\sqrt{2}$ - Loreti & Mehrenberger 05 - Ingham Inequalities for non harmonic series





$E_h(u) \leq 2E_h(\Pi_{1/2}u)$ $\qquad$ $E_h(u) \leq 4E_h(\Pi_{1/4}u)$

Wave equation on the unit square with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } Q = \Omega \times (0,T), \\ u(0) = 0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ u(x,0) = u^0(x),\ \ u_t(x,0) = u^1(x) & \text{in } Q = \Omega. \end{cases} \tag{1}$$

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \Rightarrow u \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)).$$

Conservation of Energy

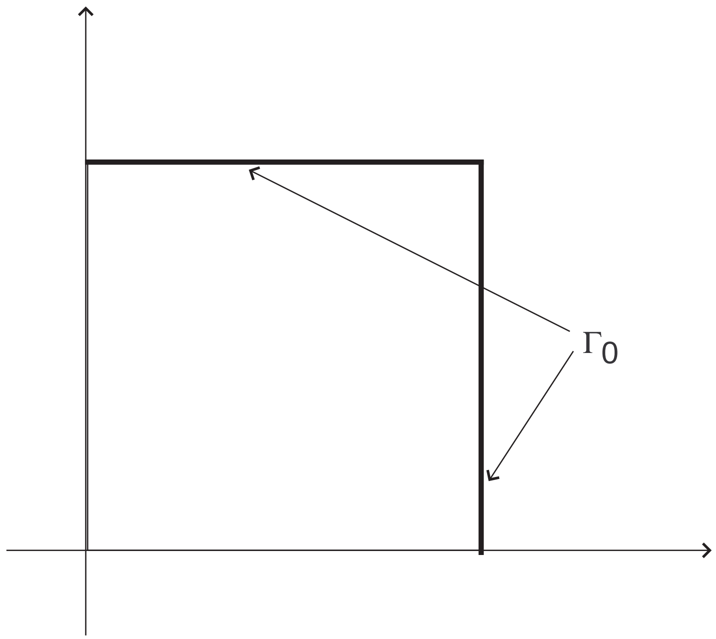
$$E(t) = \frac{1}{2} \int_\Omega [|u_t(x,t)|^2 + |\nabla u(x,t)|^2] dx \tag{2}$$

Observability Inequality

For $T > 2\sqrt{2}$ there exists $C(T) > 0$ such that

$$E(0) \leq C(T) \int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \qquad (3)$$

$\Gamma_0 = \{(x_1, 1) : \ x_1 \in (0,1)\} \cup \{(1, x_2) : \ x_2 \in (0,1)\}.$



Semi-discretization of the wave equation:

$$\begin{cases} u''_{jk} - \dfrac{u_{j+1,k} + u_{j-1,k} - 2u_{jk}}{h^2} - \dfrac{u_{j,k+1} + u_{j,k-1} - 2u_{jk}}{h^2} = 0, \\ \quad 0 < t < T,\ \ j = 0, ..., N;\ \ k = 0, ..., N, \\ u_{jk} = 0,\ \ 0 < t < T,\ \ j = 0, ..., N+1;\ \ k = 0, ..., N+1, \\ u_{jk}(0) = u^0_{jk},\ \ u'_{jk}(0) = u^1_{jk},\ \ j = 0, ..., N+1;\ \ k = 0, ..., N+1. \end{cases} \tag{4}$$

Discrete energy is preserved

$$E_h(t) = \frac{h^2}{2} \sum_{j,k=0}^{N} \left[|u'_{jk}(t)|^2 + \left| \frac{u_{j+1,k}(t) - u_{jk}(t)}{h} \right|^2 + \left| \frac{u_{j,k+1}(t) - u_{jk}(t)}{h} \right|^2 \right].$$

Discrete version of the energy observed on the boundary

$$\int_0^T \int_{\Gamma_0} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma dt \sim \int_0^T \left[h \sum_{j=1}^{N} \left|\frac{u_{jN}}{h}\right|^2 + h \sum_{k=1}^{N} \left|\frac{u_{Nk}}{h}\right|^2\right] dt.$$

Γ_h as the set of grid points belonging to Γ_0:

$$\Gamma_h = \{(jh, N+1),\ j = 1, ..., N\} \cup \{(N+1, kh),\ k = 1, ..., N\}.$$

Notation

$$\int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h := h \sum_{j=1}^{N} \left|\frac{u_{jN}}{h}\right|^2 + h \sum_{k=1}^{N} \left|\frac{u_{Nk}}{h}\right|^2. \tag{5}$$

Question

$$E_h(0) \leq C_h(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt. \qquad (6)$$

Answer: YES

The same problem: $C_h(T) \to \infty$ as $h \to 0$

Eigenvalue problem associated to (4)

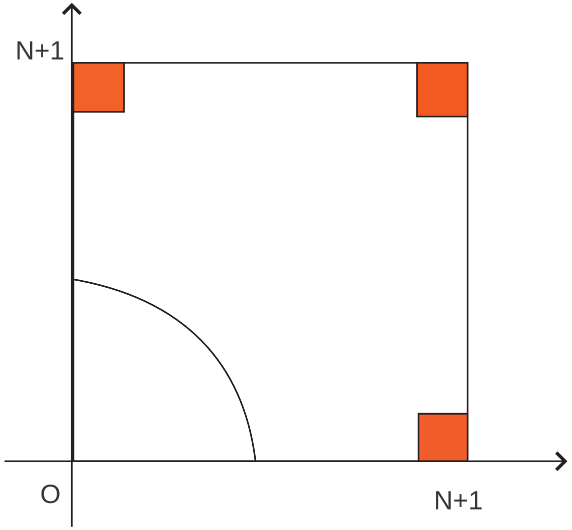
$$\begin{cases} -\dfrac{\varphi_{j+1,k}+\varphi_{j-1,k}-2\varphi_{jk}}{h^2}-\dfrac{\varphi_{j,k+1}+\varphi_{j,k-1}-2\varphi_{jk}}{h^2}=\lambda\varphi_{jk} \\ \quad j=1,...,N;\ \ k=1,...,N, \\ \varphi_{jk}=0,\ \ j=0,...,N+1;\ \ k=0,...,N+1. \end{cases} \tag{7}$$

Eigenvalues: $\lambda_{\mathbf{k}}(h)=\frac{4}{h^2}\left[\sin^2\left(\frac{k_1\pi h}{2}\right)+\sin^2\left(\frac{k_2\pi h}{2}\right)\right],\ \ \mathbf{k}=(k_1,k_2)$

Eigenvectors: $\overline{\varphi}_{\mathbf{j}}^{\mathbf{k}}=\sin(j_1k_1\pi h)\sin(j_2k_2\pi h)$

$$\overline{u}(t)=\frac{1}{2}\sum_{\mathbf{k}}\left[e^{it\sqrt{\lambda_{\mathbf{k}}(h)}}\widehat{u}_{\mathbf{k}+}+e^{-it\sqrt{\lambda_{\mathbf{k}}(h)}}\widehat{u}_{\mathbf{k}-}\right]\overline{\varphi}^{\mathbf{k}}$$

Filtering: Zuazua 99, Multipliers + reduction to $1-d$ case



$$\Pi_\gamma u = \frac{1}{2} \sum_{\lambda_{\mathbf{k}}(h) \leq \gamma/h^2} \left[e^{it\sqrt{\lambda_{\mathbf{k}}(h)}} \widehat{u}_{\mathbf{k}+} + e^{-it\sqrt{\lambda_{\mathbf{k}}(h)}} \widehat{u}_{\mathbf{k}-} \right] \overline{\varphi}^{\mathbf{k}}, \gamma < 4$$

$$E_h(\Pi_\gamma u) \leq \int_0^{T(\gamma)} \int_{\Gamma_h} |\partial_n^h (\Pi_\gamma u)| d\Gamma_h dt$$

New Idea : Low frequency estimates + Semi-classical decomposition following the level sets of the frequencies

Main Result: Let $\overline{u}$ be a solution of (4) and $\gamma > 0$ be such that

$$E_h(\overline{u}) \leq C E_h(\Pi_\gamma \overline{u}). \tag{8}$$

Let us assume the existence of a time $T(\gamma)$ such that for all $T > T(\gamma)$ there exists a constant $C(T)$, independent of h, such that

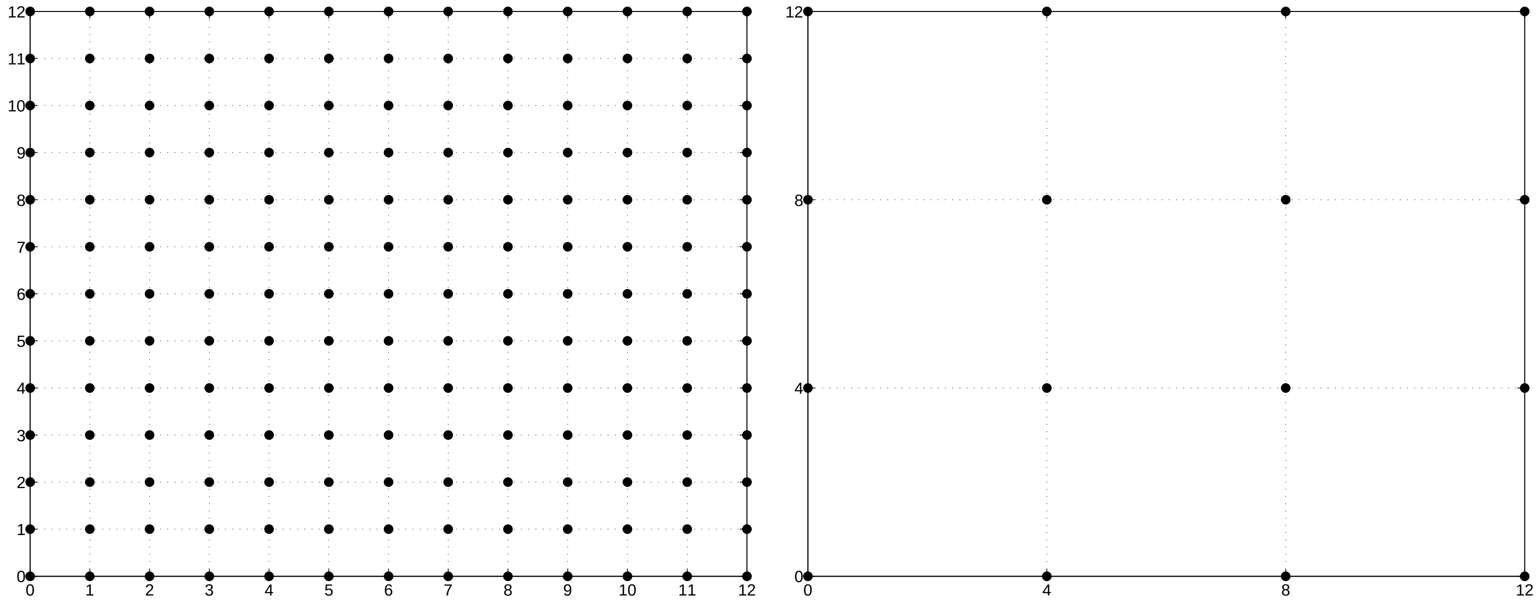
$$E_h(\overline{v}) \leq C(\gamma, T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{v}(t)|^2 d\Gamma dt \tag{9}$$

for all $\overline{v} \in I_h(\gamma)$. Then for all $T > T(\gamma)$ there exists a constant $C_1(T)$, independent of h, such that

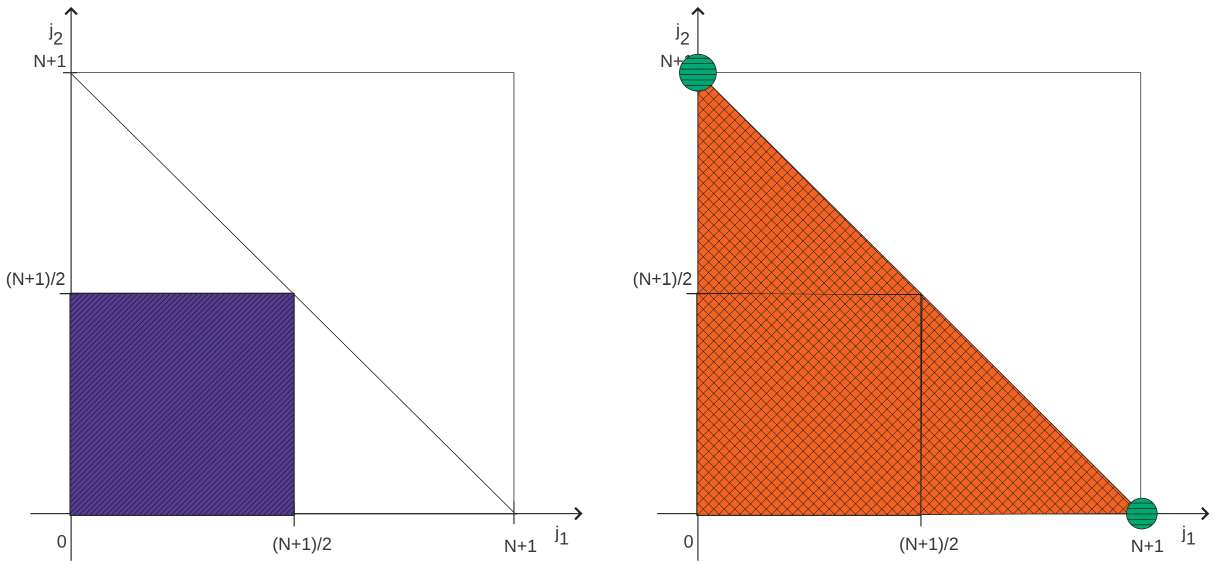
$$E_h(\overline{u}) \leq C_1(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt \tag{10}$$

Two-grid Method in $2-d$: G^h and G^{4h}

Fine and Coarse Grids, $N = 11$

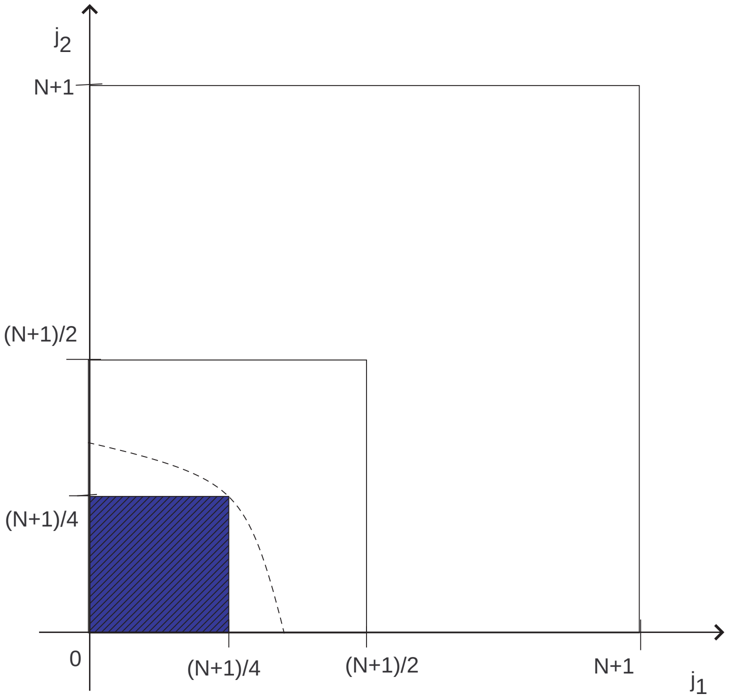


Why not using ratio 1/2 for the two-grids? The relevant zone of frequencies intersects a level set of the phase velocity for which the group velocity vanishes at some critical points.



$$E_h(\overline{u}) \leq 4E_h(\Pi^{\infty}_{1/2}\overline{u}) \leq 4E_h(\Pi_4\overline{u})$$

When using the mesh ratio 1/4 this pathology disappears:



$$E_h(\overline{u}) \leq 16 E_h(\Pi^{\infty}_{1/4}\overline{u}) \leq 16 E_h(\Pi_{8\sin^2(\pi/8)}\overline{u})$$

Application to a two-grid method

Theorem 1. *Let be* $T > 4$. *There exists a constant* $C(T)$ *such that*

$$E_h(\overline{u}) \leq C(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt$$

holds for all solutions of (4) with $(\overline{u}^0, \overline{u}^1) \in V^h \times V^h$, *uniformly on* $h > 0$, V^h *being the class of the two-grid data obtained with ratio* $1/4$.

$T_0 = 4$ is not optimal one.

Its depends by the optimality of the time for the class $I_h(8 \sin^2(\pi/8))$

Analysis of the group velocity: expected time $T_0 = \frac{2\sqrt{2}}{\cos(\pi/8)}$

The main difficulty:

$$E_h(\overline{u}) \leq 16 E_h(\Pi^{\infty}_{1/4}\overline{u}) \leq C(T)\int_0^T \int_{\Gamma_h} |\partial_n^h \Pi^{\infty}_{1/4}\overline{u}|^2 d\Gamma_h dt.$$

$$\not\Rightarrow$$

$$E_h(\overline{u}) \leq C(T)\int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt$$

Sketch of the proof

$$E_h(\overline{u}) \leq CE_h(\Pi_\gamma \overline{u}) \leq C(T)\int_0^T \int_{\Gamma_h} |\partial_n^h \Pi_\gamma \overline{u}|^2 d\Gamma_h dt$$
$$\overset{?}{\leq} C(T)\int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt + LOT$$

Diadic decomposition or Semi-classical decomposition:

$$P_k\overline{u}(t) = \sum_{\mathbf{j}\in\mathbb{Z}^2} F(c^{-k}\omega_{\mathbf{j}}(h)) \left[e^{it\omega_{\mathrm{j}}(h)}\widehat{u}_+(\mathbf{j}) + e^{-it\omega_{\mathrm{j}}(h)}\widehat{u}_-(\mathbf{j})\right] \overline{\varphi}^{\mathbf{j}} \qquad (11)$$

$$E_h(\Pi_\gamma \overline{u}) \leq \sum_{k=k_0}^{k_h} E_h(P_k \overline{u}) + LOT \tag{12}$$

$$E_h(P_k \overline{u}) \leq C(T,\gamma,\delta) \int_\delta^{T-\delta} \int_{\Gamma_h} |\partial_n^h P_k \overline{u}|^2 d\Gamma_h dt. \tag{13}$$

Combining (12) and (13),

$$E_h(\Pi_\gamma \overline{u}) \leq C(T,\gamma,\delta) \sum_{k=k_0}^{k_h} \int_\delta^{T-\delta} \int_{\Gamma_h} |\partial_n^h P_k \overline{u}|^2 d\Gamma_h dt + LOT.$$

Lebeau and Burq :

$$\sum_{k=k_0}^{k_h} \int_\delta^{T-\delta} \int_{\Gamma_h} |\partial_n^h P_k \overline{u}|^2 d\Gamma_h dt \leq 2 \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt + \frac{C(\delta, T)}{c^{2k_0}} E_h(\overline{u})$$

Thus

$$E_h(\overline{u}) \leq C E_h(\Pi_\gamma \overline{u}) \leq C(T, \gamma, \delta) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt + \frac{C(\delta, T)}{c^{2k_0}} E_h(\overline{u}) + LOT.$$

$$E_h(\overline{u}) \leq C(T, \gamma, \delta) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt + LOT.$$

Thanks to Sorin Micu, Univ. din Craiova and Carlos Castro, Univ. Politecnica de Madrid for helpful discussions.

Work Supported by

Fellowship AP2003-2299 MEC-Spain
Grant MTM2005-00714 of the MEC-Spain

More details
E. Zuazua, Propagation, Observation, and Control of Waves Approximated by Finite Difference Methods
Siam Review, vol.47, No.2, 197-243, 2005

For more information
`http://www.uam.es/enrique.zuazua`