

Uniform Boundary Observability of a Two-Grid Method for the 2d-Wave Equation

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Joint work with Enrique Zuazua

Wave equation on the unit square with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } Q = \Omega \times (0, T), \\ u(0) = 0 & \text{on } \Sigma = \partial\Omega \times (0, T), \\ u(x, 0) = u^0(x), \quad u_t(x, 0) = u^1(x) & \text{in } Q = \Omega. \end{cases} \quad (1)$$

$$(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega) \Rightarrow u \in C([0, T], H_0^1(\Omega)) \cap C^1([0, T], L^2(\Omega)).$$

Conservation of Energy

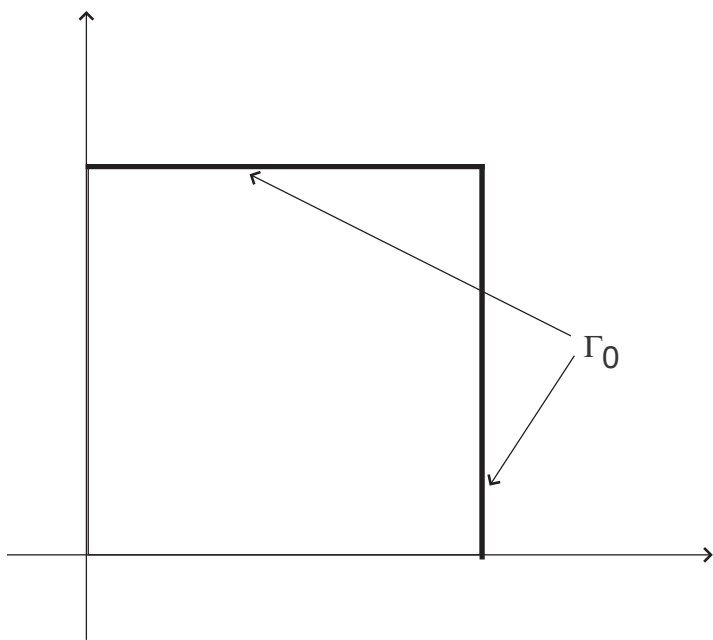
$$E(t) = \frac{1}{2} \int_{\Omega} [|u_t(x, t)|^2 + |\nabla u(x, t)|^2] dx \quad (2)$$

Observability Inequality

For $T > 2\sqrt{2}$ there exists $C(T) > 0$ such that

$$E(0) \leq C(T) \int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \quad (3)$$

$$\Gamma_0 = \{(x_1, 1) : x_1 \in (0, 1)\} \cup \{(1, x_2) : x_2 \in (0, 1)\}.$$



Semi-discretization of the wave equation:

$$\left\{ \begin{array}{l} u''_{jk} - \frac{u_{j+1,k} + u_{j-1,k} - 2u_{jk}}{h^2} - \frac{u_{j,k+1} + u_{j,k-1} - 2u_{jk}}{h^2} = 0, \\ 0 < t < T, \quad j = 0, \dots, N; \quad k = 0, \dots, N, \\ u_{jk} = 0, \quad 0 < t < T, \quad j = 0, \dots, N+1; \quad k = 0, \dots, N+1, \\ u_{jk}(0) = u_{jk}^0, \quad u'_{jk}(0) = u_{jk}^1, \quad j = 0, \dots, N+1; \quad k = 0, \dots, N+1. \end{array} \right. \quad (4)$$

Discrete energy is preserved

$$E_h(t) = \frac{h^2}{2} \sum_{j,k=0}^N \left[|u'_{jk}(t)|^2 + \left| \frac{u_{j+1,k}(t) - u_{jk}(t)}{h} \right|^2 + \left| \frac{u_{j,k+1}(t) - u_{jk}(t)}{h} \right|^2 \right].$$

Discrete version of the energy observed on the boundary

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \sim \int_0^T \left[h \sum_{j=1}^N \left| \frac{u_{jN}}{h} \right|^2 + h \sum_{k=1}^N \left| \frac{u_{Nk}}{h} \right|^2 \right] dt.$$

Γ_h as the set of grid points belonging to Γ_0 :

$$\Gamma_h = \{(jh, N+1), j = 1, \dots, N\} \cup \{(N+1, kh), k = 1, \dots, N\}.$$

Notation

$$\int_{\Gamma_h} |\partial_n^h \bar{u}|^2 d\Gamma_h := h \sum_{j=1}^N \left| \frac{u_{jN}}{h} \right|^2 + h \sum_{k=1}^N \left| \frac{u_{Nk}}{h} \right|^2. \quad (5)$$

Question

$$E_h(0) \leq C_h(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \bar{u}|^2 d\Gamma_h dt. \quad (6)$$

Answer: YES

BUT, FOR ALL $T > 0$ (!!!!!)

$$C_h(T) \rightarrow \infty, \quad h \rightarrow 0.$$

Spectral Analysis

Eigenvalue problem associated to (4)

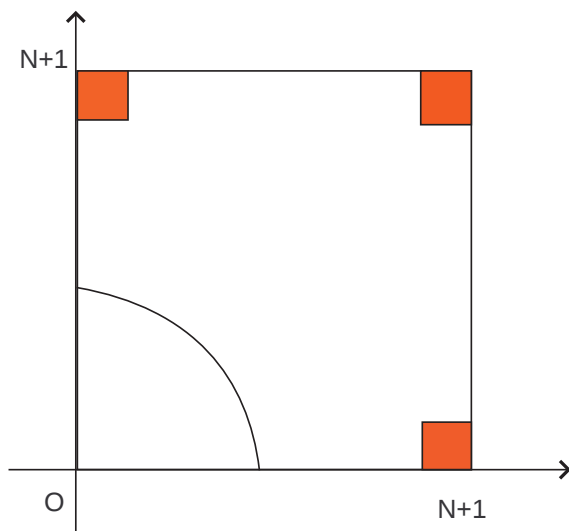
$$\left\{ \begin{array}{l} -\frac{\varphi_{j+1,k} + \varphi_{j-1,k} - 2\varphi_{jk}}{h^2} - \frac{\varphi_{j,k+1} + \varphi_{j,k-1} - 2\varphi_{jk}}{h^2} = \lambda \varphi_{jk} \\ j = 1, \dots, N; \quad k = 1, \dots, N, \\ \varphi_{jk} = 0, \quad j = 0, \dots, N+1; \quad k = 0, \dots, N+1. \end{array} \right. \quad (7)$$

Eigenvalues: $\lambda_{\mathbf{k}}(h) = \frac{4}{h^2} \left[\sin^2 \left(\frac{k_1 \pi h}{2} \right) + \sin^2 \left(\frac{k_2 \pi h}{2} \right) \right], \quad \mathbf{k} = (k_1, k_2)$

Eigenvectors: $\varphi_{\mathbf{j}}^{\mathbf{k}} = \sin(j_1 k_1 \pi h) \sin(j_2 k_2 \pi h)$

$$\bar{u}(t) = \frac{1}{2} \sum_{\mathbf{k}} \left[e^{it\sqrt{\lambda_{\mathbf{k}}(h)}} \hat{u}_{\mathbf{k}+} + e^{-it\sqrt{\lambda_{\mathbf{k}}(h)}} \hat{u}_{\mathbf{k}-} \right] \varphi^{\mathbf{k}}$$

Filtering: Zuazua 99, Multipliers



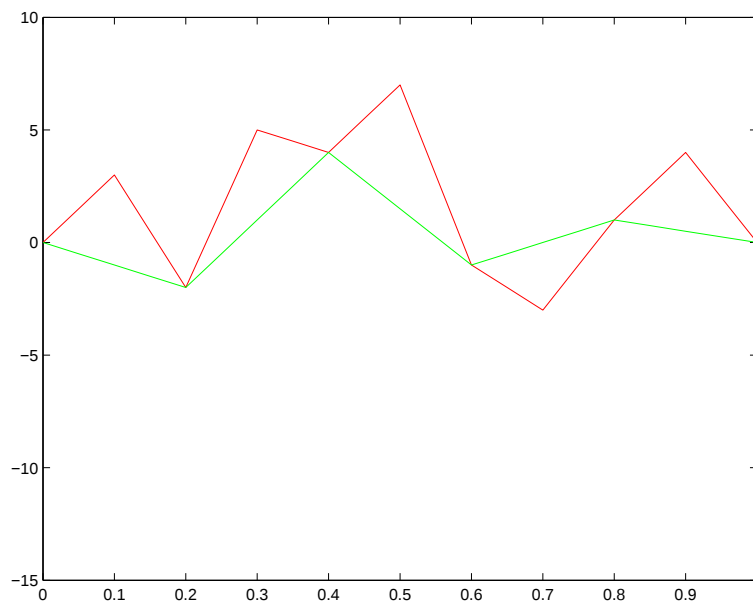
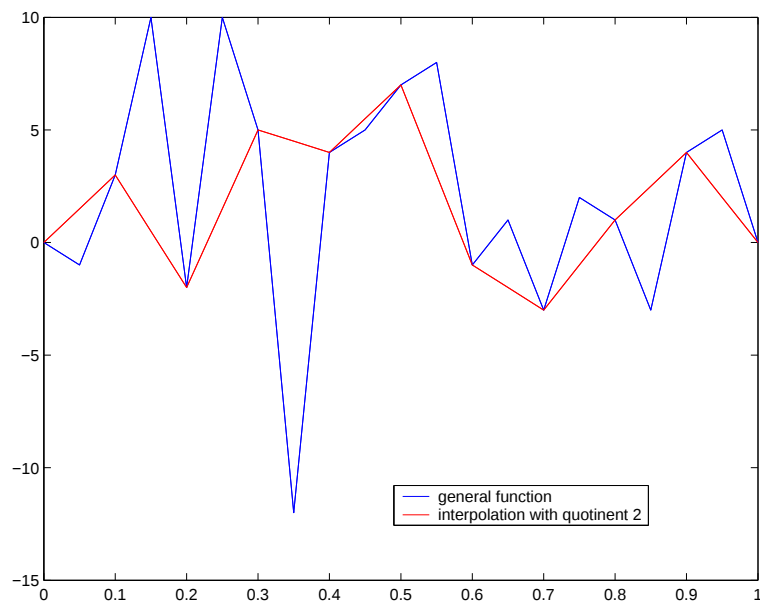
$$\Pi_{\gamma} u = \frac{1}{2} \sum_{\lambda_{\mathbf{k}}(h) \leq \gamma/h^2} \left[e^{it\sqrt{\lambda_{\mathbf{k}}(h)}} \hat{u}_{\mathbf{k}+} + e^{-it\sqrt{\lambda_{\mathbf{k}}(h)}} \hat{u}_{\mathbf{k}-} \right] \bar{\varphi}^{\mathbf{k}}, \gamma < 4$$

$$E_h(\Pi_{\gamma} u) \leq \int_0^{T(\gamma)} \int_{\Gamma_h} |\partial_n^h(\Pi_{\gamma} u)| d\Gamma_h dt$$

1D-case, Two-grid algorithm Glowinski 90

$T > 4$ - Negreanu & Zuazua 04 - Multiplies

$T > 2\sqrt{2}$ - Loreti & Mehrenberger 05 - Ingham Inequalities for non harmonic series



$$E_h(u) \leq 2E_h(\Pi_{1/2}u)$$

$$E_h(u) \leq 4E_h(\Pi_{1/4}u)$$

New Idea : Low frequency estimates + Semi-classical decomposition following the level sets of the frequencies

Main Result: Let $\bar{u}$ be a solution of (4) and $\gamma > 0$ be such that

$$E_h(\bar{u}) \leq C E_h(\Pi_\gamma \bar{u}). \quad (8)$$

Let us assume the existence of a time $T(\gamma)$ such that for all $T > T(\gamma)$ there exists a constant $C(T)$, independent of h , such that

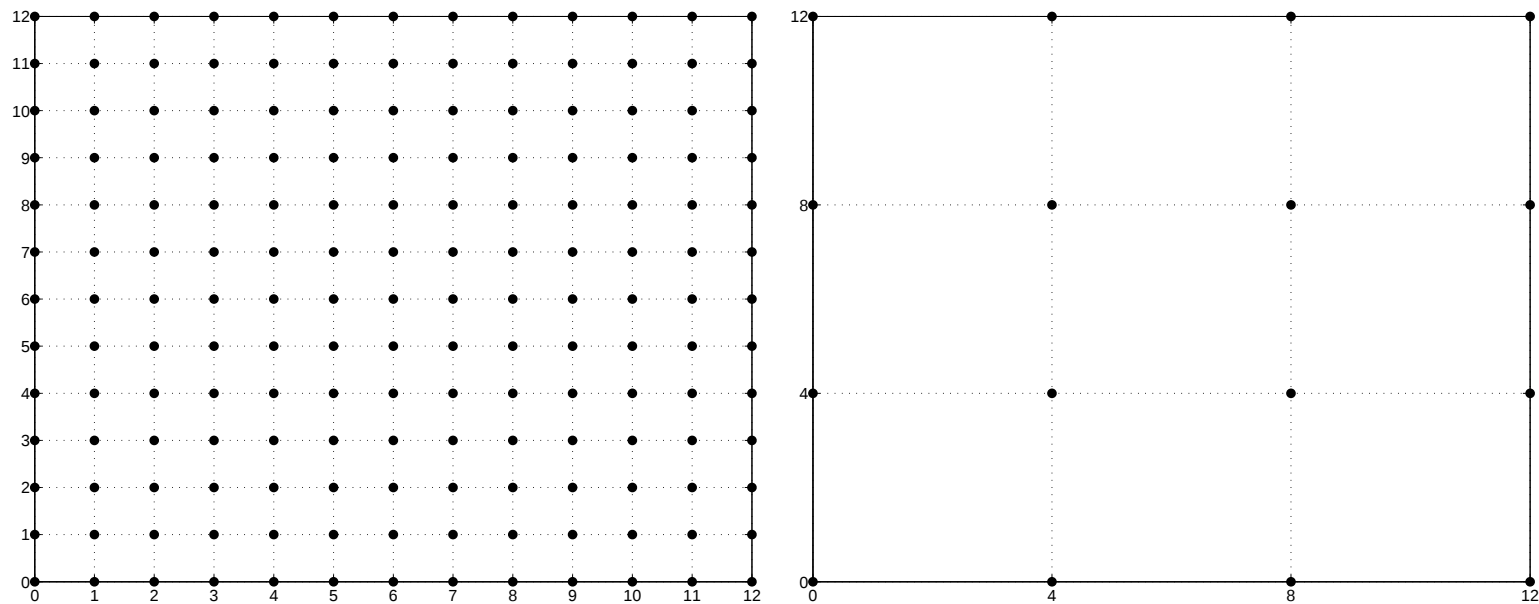
$$E_h(\bar{v}) \leq C(\gamma, T) \int_0^T \int_{\Gamma_h} |\partial_n^h \bar{v}(t)|^2 d\Gamma dt \quad (9)$$

for all $\bar{v} \in I_h(\gamma)$. Then for all $T > T(\gamma)$ there exists a constant $C_1(T)$, independent of h , such that

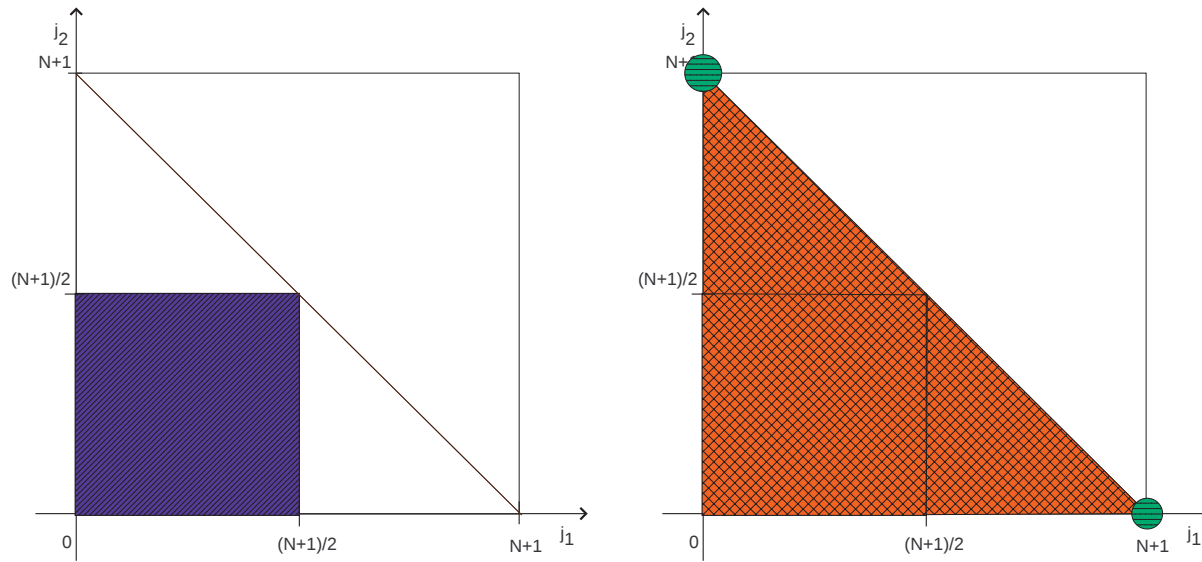
$$E_h(\bar{u}) \leq C_1(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \bar{u}|^2 d\Gamma_h dt \quad (10)$$

Two-grid Method in $2 - d$: G^h and G^{4h}

Fine and Coarse Grids, $N = 11$

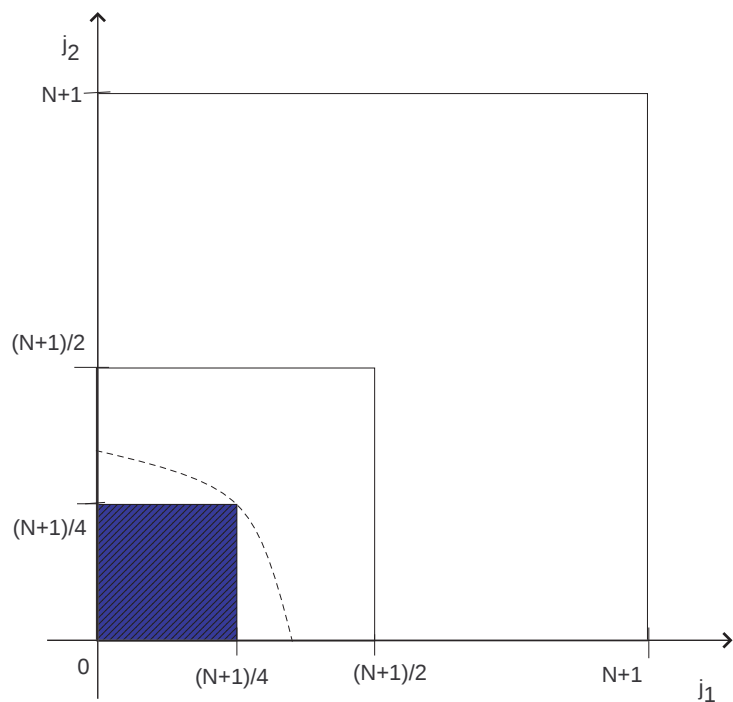


Why not using ratio 1/2 for the two-grids? The relevant zone of frequencies intersects a level set of the phase velocity for which the group velocity vanishes at some critical points.



$$E_h(\bar{u}) \leq 4E_h(\Pi_{1/2}^\infty \bar{u}) \leq 4E_h(\Pi_4 \bar{u})$$

When using the mesh ratio $1/4$ this pathology disappears:



$$E_h(\bar{u}) \leq 16E_h(\Pi_{1/4}^\infty \bar{u}) \leq 16E_h(\Pi_{8\sin^2(\pi/8)} \bar{u})$$

Application to a two-grid method

Theorem 1. Let be $T > 4$. There exists a constant $C(T)$ such that

$$E_h(\bar{u}) \leq C(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \bar{u}|^2 d\Gamma_h dt$$

holds for all solutions of (4) with $(\bar{u}^0, \bar{u}^1) \in V^h \times V^h$, uniformly on $h > 0$, V^h being the class of the two-grid data obtained with ratio $1/4$.

$T_0 = 4$ is not optimal one.

Its depends by the optimality of the time for the class $I_h(8 \sin^2(\pi/8))$

Analysis of the group velocity: expected time $T_0 = \frac{2\sqrt{2}}{\cos(\pi/8)}$

The main difficulty:

$$E_h(\bar{u}) \leq 16E_h(\Pi_{1/4}^\infty \bar{u}) \leq C(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \Pi_{1/4}^\infty \bar{u}|^2 d\Gamma_h dt.$$

$\Rightarrow$

$$E_h(\bar{u}) \leq C(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \bar{u}|^2 d\Gamma_h dt$$

Sketch of the proof

$$\begin{aligned}
 E_h(\bar{u}) &\leq C E_h(\Pi_\gamma \bar{u}) \leq C(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \Pi_\gamma \bar{u}|^2 d\Gamma_h dt \\
 &\stackrel{?}{\leq} C(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \bar{u}|^2 d\Gamma_h dt + LOT
 \end{aligned}$$

Diadic decomposition or Semi-classical decomposition:

$$P_k \bar{u}(t) = \sum_{\mathbf{j} \in \mathbb{Z}^2} F(c^{-k} \omega_{\mathbf{j}}(h)) \left[e^{it\omega_{\mathbf{j}}(h)} \hat{u}_+(\mathbf{j}) + e^{-it\omega_{\mathbf{j}}(h)} \hat{u}_-(\mathbf{j}) \right] \bar{\varphi}^{\mathbf{j}} \quad (11)$$

$$E_h(\Pi_\gamma \bar{u}) \leq \sum_{k=k_0}^{k_h} E_h(P_k \bar{u}) + LOT \quad (12)$$

$$E_h(P_k \bar{u}) \leq C(T, \gamma, \delta) \int_\delta^{T-\delta} \int_{\Gamma_h} |\partial_n^h P_k \bar{u}|^2 d\Gamma_h dt. \quad (13)$$

Combining (12) and (13),

$$E_h(\Pi_\gamma \bar{u}) \leq C(T, \gamma, \delta) \sum_{k=k_0}^{k_h} \int_\delta^{T-\delta} \int_{\Gamma_h} |\partial_n^h P_k \bar{u}|^2 d\Gamma_h dt + LOT.$$

Lebeau and Burq :

$$\sum_{k=k_0}^{k_h} \int_{\delta}^{T-\delta} \int_{\Gamma_h} |\partial_n^h P_k \bar{u}|^2 d\Gamma_h dt \leq 2 \int_0^T \int_{\Gamma_h} |\partial_n^h \bar{u}|^2 d\Gamma_h dt + \frac{C(\delta, T)}{c^{2k_0}} E_h(\bar{u})$$

Thus

$$E_h(\bar{u}) \leq C E_h(\Pi_{\gamma} \bar{u}) \leq C(T, \gamma, \delta) \int_0^T \int_{\Gamma_h} |\partial_n^h \bar{u}|^2 d\Gamma_h dt + \frac{C(\delta, T)}{c^{2k_0}} E_h(\bar{u}) + LOT.$$

$$E_h(\bar{u}) \leq C(T, \gamma, \delta) \int_0^T \int_{\Gamma_h} |\partial_n^h \bar{u}|^2 d\Gamma_h dt + LOT.$$