PROPIEDADES CUALITATIVAS DE ESQUEMAS NUMÉRICOS DE APROXIMACIÓN DE ECUACIONES DE DIFUSIÓN Y DE DISPERSION

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Outline

1. Introduction
2. Schrödinger Equation
   - A conservative scheme
   - Two-grid method
   - A viscous scheme
   - Fully discrete schemes
3. Wave Equation
4. An observability problem
5. Conclusions

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Basic elements of classical numerical analysis

Consider the finite difference approximation

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h & = & 0, \ t > 0, \\ u^h(0) & = & \varphi^h. \end{cases}$$

Here u^h stands for the infinite vector unknown $\{u_n^h\}_{n\in\mathbb{Z}}$, $u_j^h(t)$ being the approximation of the solution at the node $x_j = jh$, and Δ_h being the classical second order finite difference approximation of $\partial_x^2 u$:

$$\Delta_h u^h = \frac{1}{h^2}(u_{j+1}^h - 2u_j^h + u_{j-1}^h)$$

- The scheme is consistent + stable in $L^2(\mathbb{R})$ and, accordingly, it is also convergent.
- The same convergence result holds for semilinear equations

$$\begin{cases} iu_t + u_{xx} = f(u), & t > 0, x \in \mathbb{R} \\ u(0,x) = \varphi(x), & x \in \mathbb{R}. \end{cases}$$

  provided that the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz.
- The proof is completely standard and only requires the L^2-conservation property of the continuous and discrete equation.

- But the nonlinear Schrödinger equation is also well-posed for some nonlinearities that grow superlinearly at infinity.
- This well-posedness result may not be proved simply as a consequence of the L^2-conservation property. The dispersive properties of the LSE play a key role.
- Accordingly, one may not expect to prove convergence of the numerical schemes without similar dispersive estimates, that should be uniform on the mesh-size parameter $h \to 0$.

Qualitative properties

- Decay rates
- Dispersion
- Propagation of the energy
- Space-time estimates
- Any property that is essential in the well-posedness of the continuous problem

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Linear Schrödinger Equation

$$\begin{cases} iu_t + \Delta u = 0, \ x \in \mathbb{R}^d, \ t \neq 0, \\ u(0,x) = \varphi(x), \ x \in \mathbb{R}^d, \end{cases}$$

Conservation of the L^2-norm

$$\|S(t)\varphi\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\mathbb{R}^d)}$$

Dispersive estimate

$$|S(t)\varphi(x)| \leq \frac{1}{(4\pi|t|)^{d/2}} \|\varphi\|_{L^1(\mathbb{R}^d)}$$

Space time estimates

The α-admissible pairs

$$\frac{1}{q}=\alpha\left(\frac{1}{2}-\frac{1}{r}\right)$$

Strichartz estimates for $d/2$-admissible pairs (q,r)

$$\|S(\cdot)\varphi\|_{L^q(\mathbb{R},\,L^r(\mathbb{R}^d))}\leq C(q,r)\|\varphi\|_{L^2(\mathbb{R}^d)}$$

Space time estimates

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Local Smoothing effect

$$\sup_{x_0,R} \frac{1}{R}\int_{B(x_0,R)}\int_{-\infty}^{\infty} |(-\Delta)^{1/4}e^{it\Delta}\varphi|^2 dt dx \leq C\|\varphi\|^2_{L^2(\mathbb{R}^d)}$$

Nonlinear Schrödinger Equation

$$\begin{cases} iu_t + \Delta u = |u|^p u, \; x \in \mathbb{R}^d, \; t \neq 0 \\ u(0,x) = \varphi(x), \; x \in \mathbb{R}^d \end{cases}$$

- Initial data in $L^2(\mathbb{R}^d)$
  - Tsutsumi '87 - Global existence for $p < 4/d$

$$u \in C(\mathbb{R}, \, L^2(\mathbb{R}^d) \cap L^q_{loc}(\mathbb{R}, \, L^r(\mathbb{R}^d))$$

  - $p = 4/d$ - small initial data

A first numerical scheme for NSE

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = |u^h|^2 u^h, \quad t \neq 0, \\ u^h(0) = \varphi^h. \end{cases}$$

Questions

- Does u^h converge to the solution of NSE?
- Is u^h uniformly bounded in $L^q_{loc}(\mathbb{R}, l^r(h\mathbb{Z}^d))$?
- Local Smoothing ?

Tools

- Semidiscrete Fourier transform

$$\widehat{v}(\xi) = (\mathcal{F}_h v)(\xi) = h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} e^{-i\xi\cdot\mathbf{j}h} v_{\mathbf{j}},\ \xi \in [-\pi/h, \pi/h]^d$$

- Oscillatory integrals, Van der Corput Lemma, Fourier multipliers, Poisson Integrals
- Previous ideas of Keel & Tao '98, Kenig, Ponce & Vega '91, Christ & Kiselev '01, Constantin & Saut '89, Gigante & Soria '02 ...

A conservative scheme

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = 0, & t > 0, \\ u^h(0) = \varphi^h. \end{cases}$$

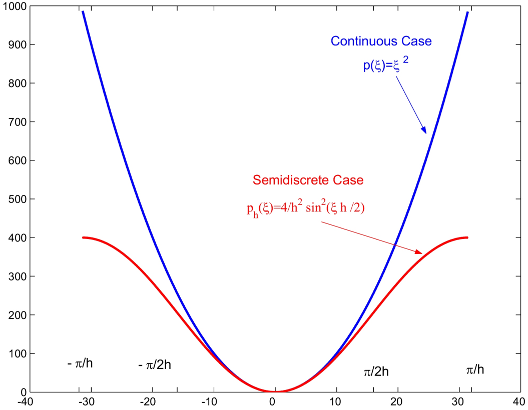
In the Fourier space the solution $\widehat{u}^h$ can be written as

$$\widehat{u}^h(t,\xi) = e^{itp_h(\xi)}\widehat{\varphi}^h(\xi),\ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d,$$

where

$$p_h(\xi) = \frac{4}{h^2}\sum_{k=1}^{d}\sin^2\left(\frac{\xi_k h}{2}\right).$$

The two symbols in dimension one



- Lack of uniform $l^1 \to l^\infty$: $\xi = \pm\pi/2h$
- Lack of uniform local smoothing effect: $\xi = \pm\pi/h$

Theorem

Let $T > 0$, $r_0 \geq 1$ *and* $r > r_0$. *Then*

$$\sup_{h>0,\,\varphi\in l^r(h\mathbb{Z}^d)} \frac{\|S^h(T)\varphi\|_{l^r(h\mathbb{Z}^d)}}{\|\varphi\|_{l^{r_0}(h\mathbb{Z}^d)}} = \infty$$

and

$$\sup_{h>0,\,\varphi\in l^r(h\mathbb{Z}^d)} \frac{\|S^h(\cdot)\varphi\|_{L^1((0,T),l^r(h\mathbb{Z}^d))}}{\|\varphi\|_{l^{r_0}(h\mathbb{Z}^d)}} = \infty.$$

Proof.

Wave packets concentrated at $(\pi/2h)^d$ □

Theorem

Let be $T > 0$, $s > 0$ and $q \geq 1$. Then

$$\sup_{h>0,\varphi\in l^q(h\mathbb{Z}^d)} \frac{h^d \sum_{|\mathbf{j}|h\leq 1} |((-\Delta_h)^s S^h(T)\varphi)_{\mathbf{j}}|^2}{\|\varphi\|^2_{l^q(h\mathbb{Z}^d)}} = \infty \tag{1}$$

and

$$\sup_{h>0,\varphi\in l^q(h\mathbb{Z}^d)} \frac{h^d \sum_{|\mathbf{j}|h\leq 1} \int_0^T |((-\Delta_h)^s S^h(t)\varphi)_{\mathbf{j}}|^2 dt}{\|\varphi\|^2_{l^q(h\mathbb{Z}^d)}} = \infty. \tag{2}$$

Proof.

Wave packets concentrated at $(\pi/h)^d$ □

Filtering initial data

Initial data supported far from $(\pm\pi/2h)^d$

- $\|S^h(t)\varphi\|_{l^\infty(h\mathbb{Z}^d)} \lesssim \frac{1}{|t|^{d/2}}\|\varphi\|_{l^1(h\mathbb{Z}^d)}$
- Strichartz like estimates: $\|S^h(\cdot)\varphi\|_{L^q(\mathbb{R},l^r(h\mathbb{Z}^d))} \lesssim \|\varphi\|_{l^2(h\mathbb{Z}^d)}$

Initial data supported far from $(\pm\pi/h)^d$

- Gain of $1/2$ local space derivative

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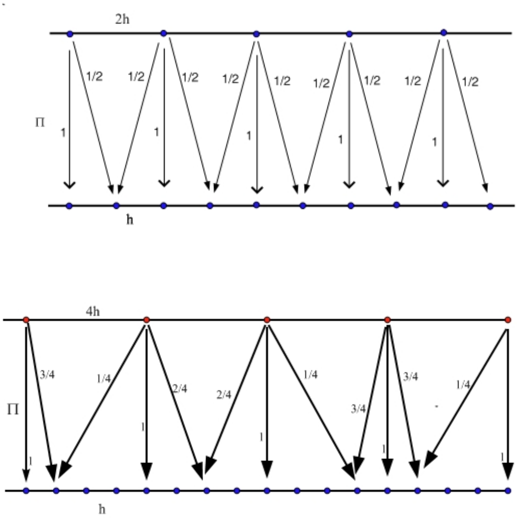
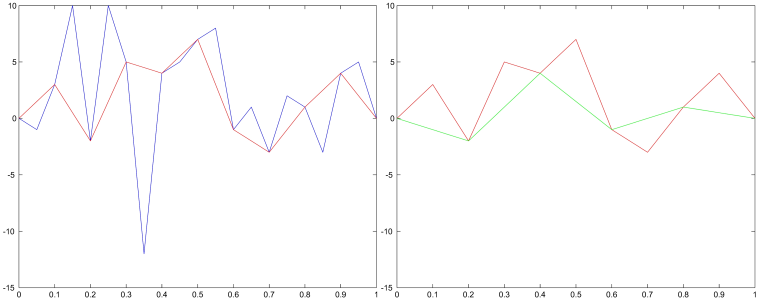
Two-grid method, Glowinski '90




10
5
0
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0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
10
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0
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Expansion and restriction operators

I-multilinear interpolator on $4h\mathbb{Z}^d$

$\widetilde{\Pi} : l^2(4h\mathbb{Z}^d) \to l^2(h\mathbb{Z}^d)$ defined by $(\widetilde{\Pi} f)_{\mathbf{j}} = (I f)_{\mathbf{j}}, \quad \mathbf{j} \in \mathbb{Z}^d$

$\widetilde{\Pi}^* : l^2(h\mathbb{Z}^d) \to l^2(4h\mathbb{Z}^d)$: $(\widetilde{\Pi} f, g)_{l^2(h\mathbb{Z}^d)} = (f, \widetilde{\Pi}^* g)_{l^2(4h\mathbb{Z}^d)}$

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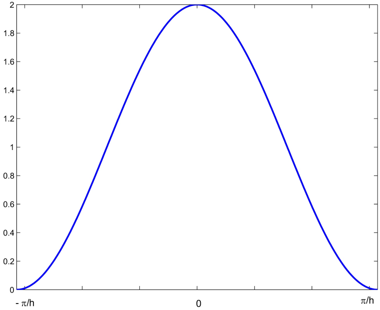
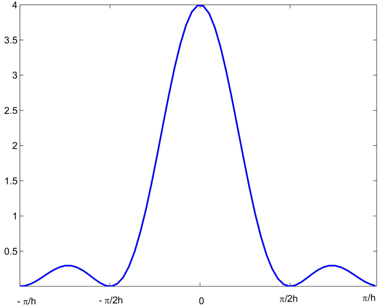
Explicit expressions

$$(\widetilde{\Pi} f)_{4j+r} = \frac{4-r}{4} f_{4j} + \frac{r}{4} f_{4j+4}, \; j \in \mathbb{Z}, \; r \in \{0,1,2,3\}$$

$$(\widetilde{\Pi}^* g)_{4j} = \sum_{r=0}^{3} \frac{4-r}{4} g_{4j+r} + \frac{r}{4} g_{4j-4+r}, \; j \in \mathbb{Z}.$$

Fourier analysis





$$\widehat{\widetilde{\Pi}\psi}(\xi) = 2^d \widehat{\psi}(\xi) \prod_{k=1}^{d} \cos^2\left(\frac{\xi_k h}{2}\right), \quad \psi \in l^2(2h\mathbb{Z}^d)$$

$$\widehat{\widetilde{\Pi}\psi}(\xi) = 4^d \widehat{\psi}(\xi) \prod_{k=1}^{d} \cos^2(\xi_k h) \cos^2\left(\frac{\xi_k h}{2}\right), \quad \psi \in l^2(4h\mathbb{Z}^d)$$

Key estimates

Dispersive estimate

$$\|e^{it\Delta_h}\widetilde{\Pi}\varphi\|_{l^\infty(h\mathbb{Z}^d)} \leq C(d,p)|t|^{-d/2}\|\widetilde{\Pi}\varphi\|_{l^1(h\mathbb{Z}^d)}$$

Local Smoothing

$$\int_I \int_{|x|<R} |(-\Delta)^{1/4} I_*(e^{it\Delta_h}\widetilde{\Pi}\varphi^h)|^2 dx dt \leq C(R,I) \int_{[-\pi/h,\pi/h]^d} |\widehat{\widetilde{\Pi}\varphi^h}(\xi)|^2 d\xi.$$

Proof: Careful application of Kenig, Ponce and Vega '91 results

Application to a nonlinear problem with $L^2(\mathbb{R}^d)$ initial data

$$\begin{cases} iu_t + \Delta u = |u|^p u, \ t > 0, \\ u(0,x) = \varphi(x), \ x \in \mathbb{R}^d, \end{cases}$$

An approximation

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = \widetilde{\Pi} f(\widetilde{\Pi}^* u^h), \ t \in \mathbb{R} \\ \\ u^h(0) = \widetilde{\Pi}\varphi^h, \end{cases} \tag{3}$$

where $f(u) = |u|^p u$

$$u^h \in G_h \to \widetilde{\Pi}^* u^h \in G_{4h} \to \widetilde{\Pi} f(\widetilde{\Pi}^* u^h) \in G_h$$

Key point

$$(\widetilde{\Pi} f(\widetilde{\Pi}^* u^h), u^h)_{l^2(h\mathbb{Z}^d)} = (f(\widetilde{\Pi}^* u^h), \widetilde{\Pi}^* u^h)_{l^2(4h\mathbb{Z}^d)} \in \mathbb{R}$$

Convergence of the method

Theorem

Let E be the piecewise constant interpolator. The sequence Eu^h satisfies

$$Eu^h \overset{\star}{\rightharpoonup} u \text{ in } L^\infty(\mathbb{R}, L^2(\mathbb{R}^d)),\ Eu^h \rightharpoonup u \text{ in } L^q_{loc}(\mathbb{R}, L^{p+2}(\mathbb{R}^d)),$$

$$Eu^h \to u \text{ in } L^2_{loc}(\mathbb{R}^{d+1}),\ E\widetilde{\Pi} f(\widetilde{\Pi}^* u^h) \rightharpoonup |u|^p u \text{ in } L^{q'}_{loc}(\mathbb{R}, L^{(p+2)'}(\mathbb{R}^d))$$

where u is the unique solution of NSE.

- Main difficulty - passing to the limit in the nonlinear term
- $\ddot\smile$ Local smoothing effect of the linear semigroup for two-grid data $\mapsto$ compactness ...

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A viscous scheme

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = ia(h)\mathrm{sgn}(t)\Delta_h u^h,\ t \neq 0, \\ u^h(0) = \varphi^h, \end{cases}$$

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With $p_h(\xi) = \frac{4}{h^2}(\sum_{k=1}^{d} \sin^2\left(\frac{\xi_k h}{2}\right))$ the solution is given by

$$\begin{aligned}(S^h(t)\varphi)_{\mathbf{j}} &= (e^{it\Delta_h} e^{|t|a(h)\Delta_h}\varphi)_{\mathbf{j}} \\ &= \int_{[-\frac{\pi}{h},\frac{\pi}{h}]^d} e^{-itp_h(\xi)} e^{-|t|a(h)p_h(\xi)} e^{i\mathbf{j}\cdot\xi h}\widehat{\varphi}(\xi)d\xi\end{aligned}$$

Dispersive estimates

Theorem

Let $\alpha > d/2$ and $a(h)$ be a positive function such that

$$\inf_{h>0} \frac{a(h)}{h^{2-\frac{d}{\alpha}}} > 0.$$

There exist positive constants $c(d,\alpha)$ such that

$$\|S^h(t)\varphi\|_{l^\infty(h\mathbb{Z}^d)} \le c(d,p,\alpha)\left[\frac{1}{|t|^{\frac{d}{2}}} + \frac{1}{|t|^\alpha}\right]\|\varphi\|_{l^1(h\mathbb{Z}^d)}$$

holds for all $t \neq 0$, $\varphi \in l^1(h\mathbb{Z}^d)$ and $h > 0$.

Remark

Different behavior at $t \sim 0$ and $t \sim \infty$.

Strichartz-like estimates

Theorem

Let $\alpha \in (d/2, d]$ *and* $a(h)$ *be such that*

$$\inf_{h>0} \frac{a(h)}{h^{2-\frac{d}{\alpha}}} > 0.$$

For any (q, r), $d/2$*-admissible pair and* (q_1, r), α*-admissible pair there exists a positive constant* $C(d, \alpha, r)$ *such that*

$$\|S^h(\cdot)\varphi\|_{L^q(\mathbb{R}, l^r(h\mathbb{Z}^d)) + L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))} \le C(d, \alpha, r)\|\varphi\|_{l^2(h\mathbb{Z}^d)}$$

Application to a nonlinear problem

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = ia(h)\Delta_h u^h + |u^h|^p u^h, & t > 0, \\ u^h(0) = \varphi^h, & \\ i\dfrac{du^h}{dt} + \Delta_h u^h = -ia(h)\Delta_h u^h + |u^h|^p u^h, & t < 0, \end{cases}$$

with $p < 4/d$ and $a(h) = h^{2-d/\alpha(h)}$ such that $\alpha(h) \downarrow d/2$ and $a(h) \to 0$ as $h \downarrow 0$.

$*$ In the critical case $p = 4/d$ the nonlinear term is replaced by $|u^h|^{p(h)} u^h$ with $p(h) = 2/\alpha(h)$.

Convergence of the scheme

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Let E be the piecewise constant interpolator. The sequence Eu^h satisfies

$$Eu^h \overset{\star}{\rightharpoonup} u \text{ in } L^\infty(\mathbb{R}, L^2(\mathbb{R}^d)), \ Eu^h \rightharpoonup u \text{ in } L^s_{loc}(\mathbb{R}, L^{p+2}(\mathbb{R}^d)), \forall\, s < q,$$

$$Eu^h \to u \text{ in } L^2_{loc}(\mathbb{R} \times \mathbb{R}^d), \ |Eu^h|^p Eu^h \rightharpoonup |u|^p u \text{ in } L^{q'}_{loc}(\mathbb{R}, L^{(p+2)'}(\mathbb{R}^d))$$

where u is the unique solution of NSE.

Fully Discrete Schemes

Two level schemes satisfying stability and consistency:

$$U^{n+1} = A_\lambda U^n, n \geq 0$$

where $\lambda = k/h^2$ is keep constant

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Two goals:

1. $l^1 - l^\infty$ decay of solutions
2. local smoothing effect

Fully Discrete Schemes

Two level schemes satisfying stability and consistency:

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where $\lambda = k/h^2$ is keep constant
Two goals:

1. $l^1 - l^\infty$ decay of solutions
2. local smoothing effect

Fourier analysis: A_λ has a symbol $a_\lambda = m_\lambda \exp(i\psi_\lambda)$

Decay properties

Theorem

Let us assume that the symbol a_λ has the following property

$$m_\lambda(\xi_0)=1 \quad \Rightarrow \quad |\psi''_\lambda(\xi_0)|>0 \text{ or } m''_\lambda(\xi_0)\neq 0.$$

Then there is a positive constant $C(\lambda)$ such that

$$\|S_\lambda(n)\varphi\|_{l^\infty(h\mathbb{Z})}\leq C(\lambda)(nk)^{-\frac{1}{2}}\|\varphi\|_{l^1(h\mathbb{Z})}$$

holds for all $n\neq 0$, $h,k>0$.

Local smoothing effect

Theorem

There is a positive s and a constant $C(s,\lambda)$ such that

$$k\sum_{nk\leq 1}\left[h\sum_{|j|h\leq 1}|((-\Delta_h)^{s/2}U^n)_j|^2\right]\leq C(s,\lambda)\left[h\sum_{j\in\mathbb{Z}}|U_j^0|^2\right] \quad (4)$$

holds for all $\varphi\in l^2(h\mathbb{Z})$ and for all $h>0$ if and only if the symbol a_λ satisfies

$$\xi_0\neq 0,\ \psi_\lambda'(\xi_0)=0 \quad\Rightarrow\quad m_\lambda(\xi_0)<1. \quad (5)$$

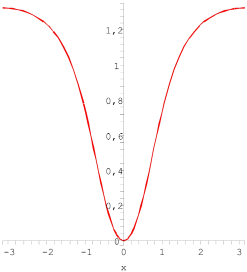
Moreover if (5) holds then $s=1/2$.

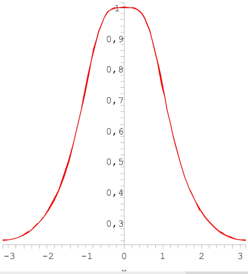
Backward Euler

$$i\frac{U_j^{n+1}-U_j^n}{k}+\frac{U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1}}{h^2}=0,\ n\geq 0,\ j\in\mathbb{Z},$$

$$a_\lambda(\xi)=\frac{1}{1-4i\lambda\sin^2\frac{\xi}{2}}=\frac{\exp(i\arctan(4\lambda\sin^2\frac{\xi}{2}))}{\left(1+16\lambda^2\sin^4\frac{\xi}{2}\right)^{1/2}}$$

The symbols ψ_1 and m_1



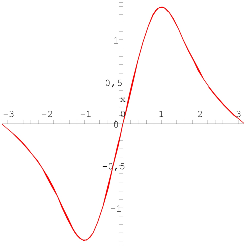


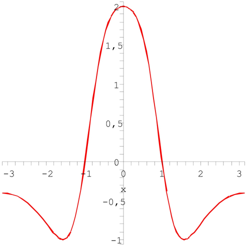
Crank-Nicolson scheme

$$i\frac{U_j^{n+1}-U_j^n}{k}+\frac{U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1}}{2h^2}+\frac{U_{j+1}^{n}-2U_j^{n}+U_{j-1}^{n}}{2h^2}=0$$

$$a_\lambda(\xi)=\frac{1+2i\lambda\sin^2\frac{\xi}{2}}{1-2i\lambda\sin^2\frac{\xi}{2}}=\exp\left(2i\arctan\left(2\lambda\sin^2\frac{\xi}{2}\right)\right)$$

The first two derivatives ψ_1' and ψ_1''





Crank-Nicolson scheme

For any $\lambda \in \mathbb{Q}$

- There is no two-grid algorithm involving the grids $ph\mathbb{Z}$ and $h\mathbb{Z}$ which would provide a $l^1 - l^\infty$ uniform decay
- The involved function ψ''_λ has roots on $[-\pi, \pi]\backslash\pi\mathbb{Q}$, thus cyclothomic polynomials ...

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Wave Equation

$$\begin{cases} u_{tt} - \Delta u = F & \text{on} \quad \mathbb{R}^{1+d}, \\ u_{|t=0} = u_0, \partial_t u_{|t=0} = u_1 & \text{in} \quad \mathbb{R}^d. \end{cases}$$

Strichartz Estimates for $(d-1)/2$ -wave admissible pairs

$$\|u\|_{L^q_t(\mathbb{R}, L^r(\mathbb{R}^d))} \leq C\big(\|u_0\|_{\dot{H}^s(\mathbb{R}^d)} + \|u_1\|_{\dot{H}^{s-1}(\mathbb{R}^d)} + \|F\|_{L^{\tilde{q}'}_t(\mathbb{R}, L^{\tilde{r}'}(\mathbb{R}^d))}\big)$$

α-wave admissible

$$\begin{cases} 2 \leq q \leq \infty, \ 2 \leq r < \infty, \\ \dfrac{1}{q} \leq \left(\dfrac{1}{2} - \dfrac{1}{r}\right)\alpha \end{cases}$$

Gap Condition

$$\begin{cases} \frac{d}{2} - s = \frac{d}{r} + \frac{1}{q} \\ \dfrac{d}{r} + \dfrac{1}{q} = \dfrac{d}{\tilde{r}'} + \dfrac{1}{\tilde{q}'} - 2 \end{cases}$$

The semidiscrete scheme

$$\begin{cases} \dfrac{d^2u^h}{dt^2} - \Delta_h u^h = F^h, & t > 0, \\ u^h_{\mathbf{j}}(0) = u^h_{0,\mathbf{j}},\ u^h_t(0) = u^h_{1,\mathbf{j}}, & \mathbf{j} \in \mathbb{Z}^d. \end{cases} \tag{6}$$

Similar result for $1/2$ -wave admissible pairs

$$\|u^h\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \lesssim \|u^h_0\|_{\dot{h}^s(h\mathbb{Z}^d)} + \|u^h_1\|_{\dot{h}^{s-1}(h\mathbb{Z}^d)} + \|F^h\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(h\mathbb{Z}^d))}.$$

Why $1/2$?

Continuous case: $\sqrt{-\Delta} \to |\xi|$,

$$\operatorname{rank}(H_{|\xi|}(\xi)) \geq d-1,\ \xi \neq 0$$

$*$ $\widehat{\beta}$ supported far away from zero

$$\|e^{it\sqrt{-\Delta}}\beta\|_{L^\infty(\mathbb{R}^d)} \leq \frac{c(\beta)}{(1+|t|)^{(d-1)/2}}$$

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$*$ $\widehat{\beta}$ supported far away from zero

$$\|e^{it\sqrt{-\Delta}}\beta\|_{L^\infty(\mathbb{R}^d)} \leq \frac{c(\beta)}{(1+|t|)^{(d-1)/2}}$$

Semidiscrete case : $\sqrt{-\Delta_1} \to p(\xi) = (\sum_{k=1}^d \sin^2 \frac{\xi_k}{2})^{1/2}$

$$\mathrm{rank}(H_p(\xi)) \geq 1,\ \xi \neq 0$$

$*$ $\widehat{\beta}$ supported in $[-\pi, \pi]$ far away from zero

$$\|e^{it\sqrt{-\Delta_1}}\beta\|_{l^\infty(\mathbb{Z}^d)} \leq \frac{c(\beta)}{(1+|t|)^{1/2}}$$

Outline

1. Introduction
2. Schrödinger Equation
   - A conservative scheme
   - Two-grid method
   - A viscous scheme
   - Fully discrete schemes
3. Wave Equation
4. An observability problem
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The observability problem

Wave equation on the unit square with Dirichlet boundary conditions

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } Q = \Omega \times (0,T), \\ u(0) = 0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x) & \text{in } \Omega. \end{cases}$$

Conservation of Energy

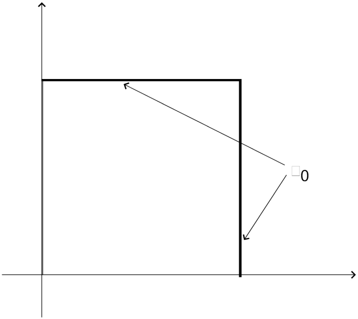
$$E(t) = \frac{1}{2}\int_\Omega [|u_t(x,t)|^2 + |\nabla u(x,t)|^2] dx$$

Observability Inequality

For $T > 2\sqrt{2}$ there exists $C(T) > 0$ such that

$$E(0) \leq C(T) \int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt$$

where $\Gamma_0 = \{(x_1, 1) : \ x_1 \in (0,1)\} \cup \{(1, x_2) : \ x_2 \in (0,1)\}$.



Semi-discretization of the wave equation

$$\begin{cases} u'' - \Delta_h u = 0, \ 0 < t < T \\ u_{jk} = 0, \ 0 < t < T, \ j = 0, ..., N+1; \ k = 0, ..., N+1, \\ u_{jk}(0) = u^0_{jk}, \ u'_{jk}(0) = u^1_{jk}, \ j = 0, ..., N+1; \ k = 0, ..., N+1. \end{cases} \tag{7}$$

Discrete energy is preserved

$$E_h(t) = \frac{h^2}{2} \sum_{j,k=0}^{N} \left[|u'_{jk}(t)|^2 + \left| \frac{u_{j+1,k}(t) - u_{jk}(t)}{h} \right|^2 + \left| \frac{u_{j,k+1}(t) - u_{jk}(t)}{h} \right|^2 \right].$$

Discrete version of the energy observed on the boundary

$$\int_0^T \int_{\Gamma_0} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma dt \sim \int_0^T \left[h \sum_{j=1}^{N} \left|\frac{u_{jN}}{h}\right|^2 + h \sum_{k=1}^{N} \left|\frac{u_{Nk}}{h}\right|^2 \right] dt.$$

Notation

$$\int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h := h \sum_{j=1}^{N} \left|\frac{u_{jN}}{h}\right|^2 + h \sum_{k=1}^{N} \left|\frac{u_{Nk}}{h}\right|^2$$

Question

$$E_h(0) \le C_h(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt.?$$

Answer: YES

Question

$$E_h(0) \leq C_h(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt.?$$

Answer: YES

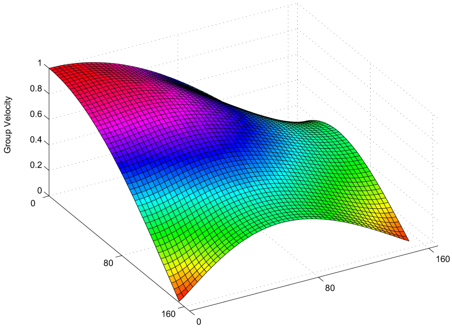
But $C_h(T) \to \infty$ as $h \to 0$

Group Velocity

$u(t,x) = e^{i(\omega t - \xi x)} \rightarrow \omega_h(\xi) = \pm(\sin^2(\frac{\xi_1 h}{2}) + \sin^2(\frac{\xi_2 h}{2}))^{1/2}$

Phase velocity $\frac{\omega_h(\xi)}{\xi}$, Group velocity $C_h(\xi) = \nabla_\xi \omega_h(\xi)$

$$C_h(\xi) = \frac{1}{2}(\sin(\xi_1 h), \sin(\xi_2 h))/(\sin^2 \frac{\xi_1 h}{2} + \sin^2 \frac{\xi_2 h}{2})^{1/2}$$



Spectral analysis

Eigenvalue problem associated to (7)

$$\begin{cases} -\dfrac{\varphi_{j+1,k}+\varphi_{j-1,k}-2\varphi_{jk}}{h^2}-\dfrac{\varphi_{j,k+1}+\varphi_{j,k-1}-2\varphi_{jk}}{h^2}=\lambda\varphi_{jk} \\ \quad j=1,...,N;\ k=1,...,N, \\ \varphi_{jk}=0,\ j=0,...,N+1;\ k=0,...,N+1. \end{cases}$$

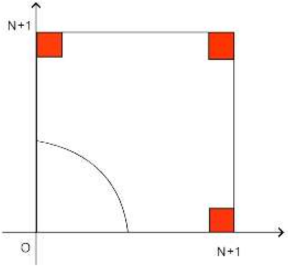
Eigenvalues: $\lambda_{\mathbf{k}}(h)=\frac{4}{h^2}\left[\sin^2\left(\frac{k_1\pi h}{2}\right)+\sin^2\left(\frac{k_2\pi h}{2}\right)\right],\ \mathbf{k}=(k_1,k_2)$

Eigenvectors: $\overline{\varphi}^{\mathbf{k}}_{\mathbf{j}}=\sin(j_1k_1\pi h)\sin(j_2k_2\pi h)$

Any solution of problem (7) can be written as

$$\overline{u}(t)=\frac{1}{2}\sum_{\mathbf{k}}\left[e^{it\sqrt{\lambda_{\mathbf{k}}(h)}}\widehat{u}_{\mathbf{k}+}+e^{-it\sqrt{\lambda_{\mathbf{k}}(h)}}\widehat{u}_{\mathbf{k}-}\right]\overline{\varphi}^{\mathbf{k}}$$

Filtering: Zuazua 99



$$\Pi_\gamma u = \frac{1}{2} \sum_{\lambda_{\mathbf{k}}(h) \leq \gamma/h^2} \left[e^{it\sqrt{\lambda_{\mathbf{k}}(h)}} \widehat{u}_{\mathbf{k}+} + e^{-it\sqrt{\lambda_{\mathbf{k}}(h)}} \widehat{u}_{\mathbf{k}-} \right] \overline{\varphi}^{\mathbf{k}}, \gamma < 4$$

$$E_h(\Pi_\gamma u) \leq \int_0^{T(\gamma)} \int_{\Gamma_h} |\partial_n^h(\Pi_\gamma u)| d\Gamma_h dt$$

New idea: Low frequency estimates + Dyadic decomposition following the level sets of the frequencies

Let be $\gamma > 0$ and $\overline{u}$ a solution of (7) such that

$$E_h(\overline{u}) \leq C E_h(\Pi_\gamma \overline{u}).$$

Let us assume the existence of a time $T(\gamma)$ such that for all $T > T(\gamma)$ there exists a constant $C(T)$, independent of h, such that

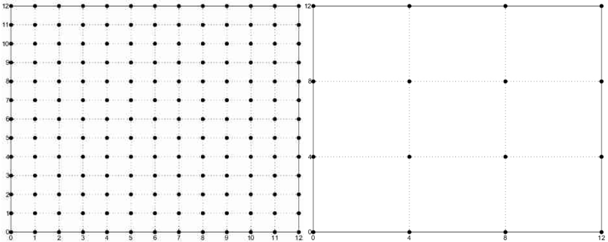
$$E_h(\overline{v}) \leq C(\gamma, T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{v}(t)|^2 d\Gamma dt$$

for all $\overline{v} \in I_h(\gamma)$. Then for all $T > T(\gamma)$ there exists a constant $C_1(T)$, independent of h, such that

$$E_h(\overline{u}) \leq C_1(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt$$

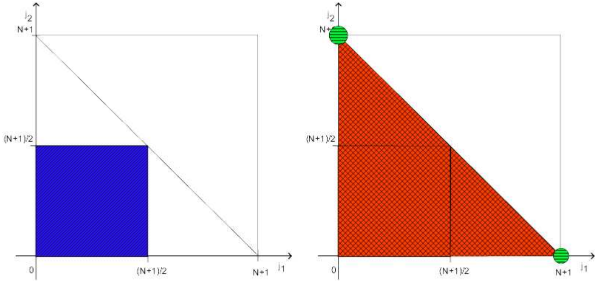
Two-grid Method in $2-d$: G^h and G^{4h}

Fine and Coarse Grids, $N = 11$



Why not using ratio 1/2 for the two-grids?

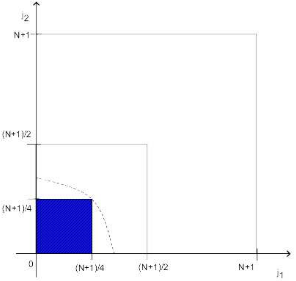
The relevant zone of frequencies intersects a level set of the phase velocity for which the group velocity vanishes at some critical points.



$$E_h(\overline{u}) \leq 4E_h(\Pi^{\infty}_{1/2}\overline{u}) \leq 4E_h(\Pi_4\overline{u})$$

Meshes with quotient $1/4$

When using the mesh ratio $1/4$ this pathology disappears:



$$E_h(\overline{u}) \leq 16 E_h(\Pi^{\infty}_{1/4}\overline{u}) \leq 16 E_h(\Pi_{8\sin^2(\pi/8)}\overline{u})$$

Application to a two-grid method

Theorem

Let be $T > 4$. There exists a constant $C(T)$ such that

$$E_h(\overline{u}) \leq C(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt$$

holds for all solutions of (7) with $(\overline{u}^0, \overline{u}^1) \in V^h \times V^h$, uniformly on $h > 0$, V^h being the class of the two-grid data obtained with ratio $1/4$.

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- MUCH REMAINS TO BE DONE: Complex geometries, variable and irregular coefficients, irregular meshes,...

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THANKS!