

A two-grid approximation scheme for nonlinear Schrödinger equations: Dispersive properties and convergence.

Un schéma de discrétisation bi-maille pour les équations de Schrödinger non-linéaires : Propriétés dispersives et convergence

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Abstract

We introduce a two-grid finite difference approximation scheme for the free Schrödinger equation. This scheme is shown to converge and to possess appropriate dispersive properties as the mesh-size tends to zero. A careful analysis of the Fourier symbol shows that this occurs because the two-grid algorithm (consisting in projecting slowly oscillating data into a fine grid) acts, to some extent, as a filtering one. We show that this scheme converges also in a class of nonlinear Schrödinger equations whose well-posedness analysis requires the so-called Strichartz estimates. This method provides an alternative one to the one introduced by the authors in [3] using numerical viscosity. *To cite this article: Liviu I. Ignat, Enrique Zuazua, C. R. Acad. Sci. Paris, Ser. I.*

Résumé

On introduit une méthode bi-maille semi-discrète en différences finies pour l'approximation numérique de l'équation de Schrödinger. On démontre la convergence L^2 du schéma et des propriétés de dispersivité uniformes par rapport au pas du maillage. Une analyse soignée en Fourier du symbole du schéma (consistant essentiellement à projeter des données lentes sur un maillage fin) montre que l'algorithme bi-maille agit comme un filtre des hautes fréquences. On montre aussi la convergence du schéma dans une classe d'équations non-linéaires dont l'étude dans le cas continu nécessite des inégalités de Strichartz. Cette méthode donne une approche alternative à celle introduite par les auteurs dans [3] à l'aide d'un schéma avec viscosité numérique. *Pour citer cet article : Liviu I. Ignat, Enrique Zuazua, C. R. Acad. Sci. Paris, Ser. I.*

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Version française abrégée

On introduit un schéma bi-maille en différences finies pour l'approximation numérique de l'équation de Schrödinger. Ce travail est motivé par [3] où l'on a constaté que le schéma conservatif classique semi-discret en différences finies ne possède pas les propriétés de dispersivité uniforme (par rapport au pas du maillage), nécessaires pour garantir la convergence pour des équations non-linéaires. Dans [3] on avait introduit un schéma dissipatif, incluant un terme de viscosité numérique et on avait démontré la convergence du schéma. La viscosité numérique avait comme but de dissiper les hautes fréquences numériques qui étaient la cause du manque de dispersivité.

Dans cette note on introduit une méthode bi-maille qui, étant de caractère conservatif, joue le même rôle en filtrant les hautes fréquences. Dans le cas de l'équation de Schrödinger libre la méthode consiste simplement à projeter les données lentes d'un maillage épais dans un maillage fin, le rapport entre les deux mailles étant convenablement choisi $1/4$. Cette méthode est inspirée de celle introduite par R. Glowinski dans [2] pour l'approximation numérique des contrôles frontière de l'équation des ondes, dont l'efficacité est maintenant bien établie, la convergence ayant été démontrée récemment dans [6].

Dans le cas linéaire, moyennant une analyse de Fourier, et en utilisant des arguments analogues à ceux de la théorie continue ([1],[4],[5] et [8]), on montre des propriétés de dispersivité (inégalités de Strichartz) uniformes par rapport au pas du maillage, la convergence L^2 de la méthode étant standard (consistance + stabilité).

Dans le cas non-linéaire on doit introduire une discrétisation soignée de la non-linéarité, permettant de garder ses propriétés de cancellation au niveau des estimations d'énergie, et qui ne fournit qu'une source d'oscillations lentes, admissibles dans notre schéma bi-maille. Ceci étant fait, on démontre la convergence du schéma dans le cas non-linéaire.

1. Introduction

Let us to consider the 1-d linear Schrödinger Equation (LSE) in the whole line

$$iu_t + u_{xx} = 0, x \in \mathbb{R}, t > 0; \quad u(0, x) = \varphi(x), x \in \mathbb{R}. \quad (1)$$

Its solution is given by $u(t) = S(t)\varphi$, where $S(t) = e^{it\Delta}$ is the free Schrödinger operator which defines a unitary transformation group in $L^2(\mathbb{R})$. The conservation of the L^2 -norm $\|u\|_{L^2(\mathbb{R})} = \|\varphi\|_{L^2(\mathbb{R})}$, together with the classical estimate $|u(t, x)| \leq (4\pi|t|)^{-1/2} \|\varphi\|_{L^1(\mathbb{R})}$, leads by interpolation to the following $L^{p'} - L^p$ result: $\|S(t)\varphi\|_{L^p(\mathbb{R})} \leq t^{-(\frac{1}{2} - \frac{1}{p})} \|\varphi\|_{L^{p'}(\mathbb{R})}$, for all $p \geq 2$ and $t \neq 0$. More refined space-time estimates known as the *Strichartz inequalities* show that, in addition to the decay of the solution as $t \rightarrow \infty$, the linear semigroup $S(t)$ satisfies $\|S(\cdot)\varphi\|_{L^q(\mathbb{R}, L^r(\mathbb{R}))} \leq C \|\varphi\|_{L^2(\mathbb{R})}$ for suitable values of q and r , the so-called admissible pairs satisfying $4/q = 1/2 - 1/r$. Also a local gain of $1/2$ space derivate occurs in $L^2_{x,t}$. These properties are not only relevant for a better understanding of the dynamics of the linear system but also to derive well-posedness results for nonlinear Schrödinger equations ([1],[8]).

In this note we introduce a two-grid finite-difference semi-discretization scheme that reproduces these properties, uniformly with respect to the mesh-size. Let us first consider the finite-difference conservative numerical scheme

$$i \frac{du^h}{dt} + \Delta_h u^h = 0, t \in \mathbb{R}; \quad u^h(0) = \varphi^h. \quad (2)$$

Here u^h stands for the infinite vector unknown $\{u_j^h\}_{j \in \mathbb{Z}}$, $u_j(t)$ being the approximation of the solution at the node $x_j = jh$, and Δ_h the classical second order finite difference approximation of ∂_x^2 : $(\Delta_h u)_j =$

$$(u_{j+1} - 2u_j + u_{j-1})/h^2.$$

This scheme satisfies the classical properties of consistency and stability which imply the L^2 convergence. The same convergence results hold for semilinear equations $iu_h + u_{xx} = f(u)$ (NSE) provided that the nonlinearity f is globally Lipschitz. But, as proved in [8], the NSE is also well-posed for some nonlinearities that grow superlinearly at infinity. We refer to [1] for a survey of the important progresses done in this field. This well-posedness result may not be proved simply as a consequence of the L^2 conservation property. Indeed, the dispersive properties of the LSE play a crucial role. Accordingly, one may not expect to prove convergence of the numerical scheme in this class of nonlinearities without similar dispersive estimates, that should be uniform on the mesh-size parameter $h \rightarrow 0$. As we proved in [3] this conservative scheme (2) fails to have uniform dispersive properties. This is due to the high frequency spurious numerical solutions the scheme (2) introduces.

We remark that there are *slight* but important differences between the symbols of the operators $-\Delta$ and $-\Delta_h : p(\xi) = \xi^2, \xi \in \mathbb{R}$ for $-\Delta$ and $p_h(\xi) = 4/h^2 \sin^2(\xi h/2), \xi \in [\pi/h, \pi/h]$ for $-\Delta_h$. The symbol $p_h(\xi)$ changes convexity at the points $\xi = \pm\pi/2h$ and has critical points also at $\xi = \pm\pi/h$, two properties that the continuous symbol does not fulfill because of its strict convexity.

To compensate this lack of dispersivity we propose a two-grid algorithm (inspired in [2]) and that, to some extent, acts as a filter for those unwanted high frequency components. This method is a natural alternative to the one introduced in [3], based on the use of numerical viscosity.

The method is roughly as follows. We consider two meshes: the coarse one of size $4h$, $h > 0$, $4h\mathbb{Z}$, and the fine one, $h\mathbb{Z}$, of size $h > 0$. The method relies basically on solving the finite-difference semi-discretization (2) on the fine mesh $h\mathbb{Z}$, but only for slow data, interpolated from the coarse grid $4h\mathbb{Z}$. As we shall see, the $1/4$ ratio between the two meshes is important to guarantee the convergence of the method. This particular structure of the data cancels the two pathologies of the discrete symbol mentioned above. Indeed, a careful Fourier analysis of those initial data shows that their discrete Fourier transform vanishes quadratically at the points $\xi = \pm\pi/2h$ and $\xi = \pm\pi/h$. As we shall see, this suffices to recover the dispersive properties of the continuous model.

2. Fourier Analysis of Slowly Oscillating Sequences

In this section we obtain explicit properties of the discrete Fourier transform of slowly oscillating sequences (SOS). The SOS on the fine grid $h\mathbb{Z}$ are those which are obtained from the coarse grid $4h\mathbb{Z}$ by an interpolation process. Obviously there is a one to one correspondence between the coarse grid sequences and the space $\mathbb{C}_4^{h\mathbb{Z}} = \{\psi \in \mathbb{C}^{h\mathbb{Z}} : \text{supp } \psi \subset 4h\mathbb{Z}\}$.

We define the space of slowly oscillating sequences on the fine grid as the following subspace of $\mathbb{C}^{h\mathbb{Z}}$:

$$V_4^h = \{\phi \in \mathbb{C}^{h\mathbb{Z}} : \phi((4j+r)h) = \frac{4-r}{4}\psi(4jh) + \frac{r}{4}\psi((4j+4)h), \psi \in \mathbb{C}_4^{h\mathbb{Z}}, j \in \mathbb{Z}, r = \overline{0, 3}\}. \quad (3)$$

We consider the projection and extension operators $\Pi : \mathbb{C}^{h\mathbb{Z}} \rightarrow \mathbb{C}_4^{h\mathbb{Z}}, E : \mathbb{C}_4^{h\mathbb{Z}} \rightarrow \mathbb{C}^{h\mathbb{Z}}$ by

$$(\Pi\phi)((4j+r)h) = \phi((4j+r)h)\delta_{4r}, \forall j \in \mathbb{Z}, r = \overline{0, 3}, \phi \in \mathbb{C}^{h\mathbb{Z}} \quad (4)$$

and

$$(E\psi)((4j+r)h) = \frac{4-r}{4}\psi(4jh) + \frac{r}{4}\psi((4j+4)h), \forall j \in \mathbb{Z}, r = \overline{0, 3}, \psi \in \mathbb{C}_4^{h\mathbb{Z}} \quad (5)$$

where δ is the Kronecker's symbol. We remark that $E : \mathbb{C}_4^{h\mathbb{Z}} \rightarrow V_4^h$ and $\Pi : V_4^h \rightarrow \mathbb{C}_4^{h\mathbb{Z}}$ are bijective linear maps satisfying $\Pi E = I_{\mathbb{C}_4^{h\mathbb{Z}}}$, $E\Pi = I_{V_4^h}$, where I_X denotes the identity operator on X . We now define $\tilde{\Pi} = E\Pi : \mathbb{C}^{h\mathbb{Z}} \rightarrow V_4^h$, which acts as a smoothing operator and associates to each sequence on the fine

grid a sequence which is slowly oscillating. As we said above the restriction of this operator to V_4^h is the identity. Concerning the discrete Fourier transform of a SOS by means of explicit computations one can prove that:

Lemma 2.1 *Let $\phi \in l^2(h\mathbb{Z})$. Then*

$$\widehat{\widetilde{\Pi}\phi}(\xi) = 4 \cos^2(\xi h) \cos^2(\xi h/2) \widehat{\Pi\phi}(\xi). \quad (6)$$

Remark 1 A simpler construction may be done interpolating $2h\mathbb{Z}$ sequences. We then get $\widehat{\widetilde{\Pi}\phi}(\xi) = 2 \cos^2(\xi h/2) \widehat{\Pi\phi}(\xi)$. This cancels the spurious numerical solutions at the frequencies $\pm\pi/h$, but not at $\pm\pi/2h$. In this case, as we proved in [3], the Strichartz estimates fail to be uniform on h . Thus we rather choose the ratio between grids to be $1/4$.

3. Estimates of the Linear Semigroup

As we proved in [3], there is no gain (uniformly in h) of integrability of the linear semigroup $e^{it\Delta_h}$. However, there are subspaces of $\mathbb{C}^{h\mathbb{Z}}$, namely V_4^h , where the linear semigroup has appropriate decay properties, uniformly on $h > 0$. The main results we get are the following

Theorem 3.1 *Let $p \geq 2$. The following properties hold :*

- i) $\|e^{it\Delta_h} \widetilde{\Pi}\varphi\|_{l^p(h\mathbb{Z})} \lesssim |t|^{-1/2(1/p-1/p')} \|\widetilde{\Pi}\varphi\|_{l^{p'}(h\mathbb{Z})}$ for all $\varphi \in l^{p'}(h\mathbb{Z})$, $h > 0$ and $t \neq 0$.
- ii) For every sequence $\varphi \in l^2(h\mathbb{Z})$, the function $t \rightarrow e^{it\Delta_h} \widetilde{\Pi}\varphi$ belongs to $L^q(\mathbb{R}, l^r(h\mathbb{Z})) \cap C(\mathbb{R}, l^2(h\mathbb{Z}))$ for every admissible pair (q, r) . Furthermore $\|e^{it\Delta_h} \widetilde{\Pi}\varphi\|_{L^q(\mathbb{R}, l^r(h\mathbb{Z}))} \lesssim \|\widetilde{\Pi}\varphi\|_{l^2(h\mathbb{Z})}$ uniformly on $h > 0$.
- iii) Let $(q, r), (\tilde{q}, \tilde{r})$ be two admissible pairs. Then $\|\int_{s < t} e^{i(t-s)\Delta_h} \widetilde{\Pi}F(s) ds\|_{L^q(\mathbb{R}, l^r(h\mathbb{Z}))} \lesssim \|\widetilde{\Pi}F\|_{L^{\tilde{q}}(\mathbb{R}, l^{\tilde{r}}(h\mathbb{Z}))}$ for all $F \in L^{\tilde{q}}(\mathbb{R}, l^{\tilde{r}}(h\mathbb{Z}))$, uniformly in $h > 0$.

Sketch of the Proof. Using that $e^{it\Delta_h} = e^{i(t/h^2)\Delta_1}$, by scaling we can assume that $h = 1$. By (6) we obtain

$$(e^{it\Delta_1} \widetilde{\Pi}\varphi)_j = \int_{-\pi}^{\pi} 4 \cos^2(\xi) \cos^2(\xi/2) e^{-4it \sin^2(\xi/2)} \widehat{\Pi\varphi}(\xi) e^{ij\xi} d\xi =_{def} (T(t)(\Pi\varphi))_j.$$

It is sufficient to show that $T(t)$ maps $l^2(\mathbb{Z})$ to $l^2(\mathbb{Z})$ and $l^1(\mathbb{Z})$ to $l^\infty(\mathbb{Z})$ with appropriate norm decay in t . The case $p = 2$ follows by Plancherel's identity. In the case $p = 1$ we write $T(t)$ as a convolution operator $T(t)\psi = K^t * \psi$ where $\widehat{K^t}(\xi) = 4e^{-4it \sin^2 \xi/2} \cos^2 \xi \cos^2(\xi/2)$. It remains to prove that $\|K^t\|_{l^\infty(\mathbb{Z})} \lesssim 1/\sqrt{t}$. Using that $(4 \sin^2(\xi/2))'' = 2 \cos(\xi)$, by [5] (Corollary 2.9, p. 46) we obtain

$$\|K^t\|_{l^\infty(\mathbb{Z})} \lesssim \frac{1}{\sqrt{t}} (\| |\cos(\xi)|^{3/2} \cos^2(\xi/2) \|_{L^\infty([-\pi, \pi])} + \int_{-\pi}^{\pi} | |\cos(\xi)|^{3/2} \cos^2(\xi/2)' | d\xi) \lesssim \frac{1}{\sqrt{t}}.$$

Observe that the operators $T(t)$ satisfy $(T(t))^* = T(-t)$ for all real t . As a consequence we obtain $\|T(t)(T(s))^* \psi\|_{l^\infty(\mathbb{Z})} \lesssim |t-s|^{-1/2} \|\psi\|_{l^1(\mathbb{Z})}$, for all $t \neq s$ and $\psi \in l^1(\mathbb{Z})$. We are in the hypothesis of [4] (Theorem 1.2, p. 956). This implies that for all admissible pairs (q, r) and $(\tilde{q}, \tilde{r})$ we get

$$\|T(\cdot)f\|_{L^q(\mathbb{R}, l^r(\mathbb{Z}))} \lesssim \|f\|_{l^2(\mathbb{Z})} \text{ and } \|\int_{s < t} T(t-s)F(s) ds\|_{L^q(\mathbb{R}, l^r(\mathbb{Z}))} \lesssim \|F\|_{L^{\tilde{q}}(\mathbb{R}, l^{\tilde{r}}(\mathbb{Z}))}. \quad (7)$$

Finally we use the definition of $T(t)$ in order to obtain the estimates for $e^{it\Delta_1}$.

Concerning the local smoothing properties we can prove that

Theorem 3.2 *Let $r \in (1, 2]$. Then*

$$\sup_{j \in \mathbb{Z}} \int_{-\infty}^{\infty} \left| (D^{1-1/r} e^{it\Delta_h} \tilde{\Pi} f)_j \right|^2 dt \lesssim \|\tilde{\Pi} f\|_{l^r(h\mathbb{Z})}^2 \quad (8)$$

for all $f \in l^r(h\mathbb{Z})$, uniformly in $h > 0$.

Sketch of the Proof. By scaling we can assume that $h = 1$. Using that $e^{it\Delta_1} \tilde{\Pi} f = T(t)\Pi f$ it is sufficient to prove that $\sup_{j \in \mathbb{Z}} \int_{-\infty}^{\infty} \left| (D^{1-1/r} T(t)\psi)_j \right|^2 dt \lesssim \|\psi\|_{l^r(\mathbb{Z})}^2$. Denoting

$$(T_1(t)\varphi)(x) = \int_{-\pi}^{\pi} e^{-4it \sin^2 \frac{\xi}{2}} \hat{\varphi}(\xi) e^{ix\xi} \cos^2 \xi \cos^2(\xi/2) d\xi, \quad (9)$$

it remains to prove that

$$\sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} \left| (D^{1-1/r} T_1(t)\varphi)(x) \right|^2 dt \lesssim \|\varphi\|_{L^r(\mathbb{R})}^2 \quad (10)$$

for all $\varphi \in L^r(\mathbb{R})$ with $\text{supp } \hat{\varphi} \in [-\pi, \pi]$. Using Sobolev's imbedding the inequality (10) may be reduced to the following one

$$\sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} |(T_1(t)\varphi)(x)|^2 dt \lesssim \|D^{-1/2}\varphi\|_{L^2(\mathbb{R})}^2 \text{ for all } \varphi \in \mathcal{S}(\mathbb{R}). \quad (11)$$

Applying the results of [5] (Theorem 4.1, p. 54) to T_1 we get

$$\sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} |(T_1(t)\varphi)(x)|^2 dt \lesssim \int_{-\pi}^{\pi} \frac{|\hat{f}(\xi)|^2 \cos^4 \xi |\cos(\xi/2)|^3}{|\sin \xi|} d\xi \lesssim \int_{-\pi}^{\pi} \frac{|\hat{f}(\xi)|^2}{|\xi|} d\xi \lesssim \|D^{-1/2} f\|_{L^2(\mathbb{R})}^2. \quad (12)$$

4. A conservative approximation of the NSE

We concentrate on the semilinear NSE equation in $\mathbb{R}$ with repulsive power law nonlinearity :

$$iu_t + \Delta u = |u|^p u, \quad x \in \mathbb{R}, \quad t \in \mathbb{R}; \quad u(0, x) = \varphi(x), \quad x \in \mathbb{R}. \quad (13)$$

As proved in [8], (13) is globally well posed for all $\varphi \in L^2(\mathbb{R})$ and $p \in [0, 4)$. We consider the following semi-discretization

$$i \frac{du^h}{dt} + \Delta_h u^h = \tilde{\Pi} f(u^h), \quad t \in \mathbb{R}; \quad u^h(0) = \tilde{\Pi} \varphi^h, \quad (14)$$

where $f(u^h)$ is a suitable approximation of $|u|^p u$ with $0 < p < 4$. In order to prove the global well-posedness of (14), we need to guarantee the conservation of the $l^2(h\mathbb{Z})$ norm of solutions, a property that the solutions of NSE satisfy. For that the nonlinear term $f(u^h)$ has to be chosen such that satisfies $(\tilde{\Pi} f(u^h), u^h)_{l^2(h\mathbb{Z})} \in \mathbb{R}$. The following holds:

Theorem 4.1 *Let be $p \in (0, 4)$ and $f : \mathbb{C}^{h\mathbb{Z}} \rightarrow \mathbb{C}^{h\mathbb{Z}}$ satisfying $\|\tilde{\Pi} f(u)\|_{l^{(p+2)'}(h\mathbb{Z})} \lesssim \|u\|_{l^{(p+2)'}(h\mathbb{Z})}^p$ and $(\tilde{\Pi} f(u), u)_{l^2(h\mathbb{Z})} \in \mathbb{R}$. Then for every $\varphi^h \in l^2(h\mathbb{Z})$, there exists a unique global solution*

$$u^h \in C(\mathbb{R}, l^2(h\mathbb{Z})) \cap L_{loc}^q(\mathbb{R}; l^{p+2}(h\mathbb{Z})) \quad (15)$$

of (14) which satisfies the following estimates

$$\|u^h\|_{L^\infty(\mathbb{R}, l^2(h\mathbb{Z}))} \leq \|\tilde{\Pi} \varphi\|_{l^2(h\mathbb{Z})} \text{ and } \|u^h\|_{L^q(I, l^{p+2}(h\mathbb{Z}))} \leq c(I) \|\tilde{\Pi} \varphi\|_{l^2(h\mathbb{Z})} \quad (16)$$

where $(q, p+2)$ is an admissible pair and the above constants are independent of h .

Remark 2 The conditions above on the approximation of the nonlinear term, $g(u) = |u|^p u$, are satisfied if one chooses

$$(f(u^h))_{4j} = g\left((u_{4j}^h + \sum_{r=1}^3 \frac{4-r}{4}(u_{4j+r}^h + u_{4j-r}^h))/4\right). \quad (17)$$

With this choice it is easy to check that $\|\tilde{\Pi}f(u^h)\|_{l^{(p+2)'}(h\mathbb{Z})} \leq C\|u^h|^p u^h\|_{l^{(p+2)'}(h\mathbb{Z})}$ with $C > 0$ independently of $h > 0$.

Sketch of the Proof. Local existence and uniqueness are consequence of the Strichartz estimates (Theorem 3.1) and a fixed point argument. By construction $(\tilde{\Pi}f(u^h), u^h)_{l^2(h\mathbb{Z})} \in \mathbb{R}$ as follows

$$\begin{aligned} (\tilde{\Pi}f(u^h), u^h)_{l^2(h\mathbb{Z})} &= h \sum_{r=0}^3 \sum_{j \in \mathbb{Z}} \left(\frac{4-r}{4}(f(u^h))_{4j} + \frac{r}{4}(f(u^h))_{4j+4} \right) \bar{u}_{4j+r}^h \\ &= h \sum_{j \in \mathbb{Z}} (f(u^h))_{4j} \left(\sum_{r=0}^3 \frac{4-r}{4} \bar{u}_{4j+r}^h + \sum_{r=0}^3 \frac{r}{4} \bar{u}_{4j+r-4}^h \right) \\ &= h \sum_{j \in \mathbb{Z}} g\left((u_{4j}^h + \sum_{r=1}^3 \frac{4-r}{4}(u_{4j+r}^h + u_{4j-r}^h))/4\right) (\bar{u}_{4j}^h + \sum_{r=1}^3 \frac{4-r}{4}(\bar{u}_{4j+r}^h + \bar{u}_{4j-r}^h)). \end{aligned}$$

Therefore the discrete energy $h \sum_{j \in \mathbb{Z}} |u_j(t)|^2$ is constant in time. This allows excluding finite-time blow-up.

In the sequel we consider the piecewise constant interpolator I_h . We choose $(\varphi_j^h)_{j \in \mathbb{Z}}$, an approximation of the initial data $\varphi \in L^2(\mathbb{R})$, such that $I_h \tilde{\Pi} \varphi^h \rightharpoonup \varphi$ weakly in $L^2(\mathbb{R})$.

The main convergence result is the following

Theorem 4.2 *Let u^h be the unique solution of (14). Then the sequence $I_h u^h$ satisfies*

$$I_h u^h \xrightarrow{*} u \text{ in } L^\infty(\mathbb{R}, L^2(\mathbb{R})), \quad (18)$$

$$I_h u^h \rightharpoonup u \text{ in } L_{loc}^q(\mathbb{R}, L^{p+2}(\mathbb{R})), \quad (19)$$

$$I_h u^h \rightarrow u \text{ in } L_{loc}^2(\mathbb{R} \times \mathbb{R}), \quad (20)$$

$$I_h \tilde{\Pi} f(u^h) \rightharpoonup |u|^p u \text{ in } L_{loc}^{q'}(\mathbb{R}, L^{(p+2)'(\mathbb{R}))} \quad (21)$$

where u is the unique solution of NSE and $4/q = 1/2 - 1/(p+2)$.

Sketch of the Proof. Using the result of Theorem 3.2 on the gain of $1/(p+2)$ -derivatives we first prove that $\|I_h u^h\|_{L^2(\mathbb{R}, H_{loc}^{1/(p+2)}(\mathbb{R}))} \leq \|\tilde{\Pi} \varphi^h\|_{l_h^2(\mathbb{Z})}$. Then, by a compactness argument (see [7]), we can extract a subsequence converging locally strongly in $L_{x,t}^1$. This, together with the uniform (with respect to h) estimates of Theorem 4.1, suffices to obtain the stated convergence results. In particular, it suffices to pass to the limit in the nonlinear term and to identify the limit as the solution of NSE.

Remark 3 Our method works similarly in the critical case $p = 4$ for small initial data.

Remark 4 The techniques and results of this paper extend to the LSE and NSE in several space dimensions.

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References

- [1] T. Cazenave. *Semilinear Schrödinger Equations*. Courant Lecture Notes, 10, 2003.
- [2] R. Glowinski. *Ensuring well-posedness by analogy: Stokes problem and boundary control for the wave equation*. J. Comput. Phys., (2), **103**, 189–221, 1992.
- [3] L. I. Ignat and E. Zuazua. *Dispersive properties of a viscous numerical scheme for the schrödinger equation*. C. R. Acad. Sci. Paris, Ser. I, 2005 (to appear).
- [4] M. Keel and T. Tao. *Endpoint Strichartz estimates*. Am. J. Math., (5), **120**, 955–980, 1998.
- [5] C. E. Kenig, G. Ponce, and L. Vega. *Oscillatory integrals and regularity of dispersive equations*. Indiana Univ. Math. J., (1), **40**, 33–69, 1991.
- [6] M. Negreanu and E. Zuazua. *Convergence of a multigrid method for the controllability of a 1-d wave equation*. C. R. Math. Acad. Sci. Paris, Ser. I, (5), **338**, 413–418, 2004.
- [7] J. Simon, *Compact sets in the space $L^p(0, T; B)$* . Ann. Mat. Pura Appl., (4), **146**, 65–96, 1987.
- [8] Y. Tsutsumi, *L^2 -solutions for nonlinear Schrödinger equations and nonlinear groups*. Funkc. Ekvacioj, Ser. Int., **30**, 115–125, 1987.