CONVERGENCE OF A TWO-GRID METHOD ALGORITHM FOR THE CONTROL OF THE WAVE EQUATION

LIVIU I. IGNAT AND ENRIQUE ZUAZUA

Abstract. We analyze the boundary observability of the finite-difference space semi-discretizations of the 2-d wave equation in the square. We prove the uniform (with respect to the mesh size) boundary observability for the solutions obtained by a two-grid preconditioner, introduced by Glowinski [6]. Our method uses previously known uniform observability inequality for low frequency solutions and a diadic spectral time decomposition. The method can be applied in any space dimension and for more general domains. As a consequence we prove the convergence of the two-grid boundary controls.

1. Introduction

Let us consider consider the wave equation

$$
\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times (0,T), \\ u = v \mathbf{1}_{\Gamma_0}(x) & \text{on } \Gamma \times (0,T), \\ u(0,x) = u^0(x),\ u_t(0,x) = u^1(x) & \text{in } \Omega, \end{cases} \tag{1}
$$

where Ω is the unit square $\Omega = (0,1) \times (0,1)$ of $\mathbb{R}^2$ and its boundary $\Gamma = \Gamma_0 \cup \Gamma_0$:

$$
\begin{cases} \Gamma_0 = \{(x_1, 1) :\ x_1 \in (0,1)\} \cup \{(1, x_2) :\ x_2 \in (0,1)\}, \\ \\ \Gamma_1 = \{(x_1, 0) :\ x_1 \in (0,1)\} \cup \{(0, x_2) :\ x_2 \in (0,1)\}. \end{cases}
$$

In equation (1), $u = u(t,x)$ is the state, $'$ is the time derivative and v is a control function which acts on on the system through the boundary Γ_1. Classical results of existence and uniqueness of solutions of nonhomogeneous evolution equations (see for instance [15]) show that for any $v \in L^2((0,T) \times \Gamma_0)$ and $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ equation (1) has a unique weak solution $(u, u') \in C([0,T], L^2(\Omega) \times H^{-1}(\Omega))$.

Concerning the controllability of the above system the following exact controllability result is well known (see [14]): *Given $T > 2\sqrt{2}$ and $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists a control function $v \in L^2((0,T) \times \Gamma_0)$ such that the solution $u = u(t,x)$ of (1) satisfies*

$$
u(T, \cdot) = u_t(T, \cdot) = 0.
$$

In fact, given $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ a control function v of minimal L^2-norm may be obtained by Hilbert Uniqueness Method (HUM). The HUM method introduced by J.-L. Lions offers a way to solve this problem and another multi-dimensional similar problems. This method reduces the exact controllability to an equivalent *observability* property for the adjoint problem:

$$
\begin{cases} u'' - \Delta u = 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma \times (0,T), \\ u(T,x) = u^0(x),\ u_t(T,x) = u^1(x) & \text{in } \Omega. \end{cases} \tag{2}
$$

The observability property is the following: *For any* $T > 2\sqrt{2}$ *there exists* $C(T) > 0$ *such that*

$$\|(u^0, u^1)\|_{H_0^1(\Omega)\times L^2(\Omega)} \leq C(T)\int_0^T \int_{\Gamma_0} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma dt \tag{3}$$

for any $(u^0, u^1) \in H_0^1(\Omega)\times L^2(\Omega)$ *where* u *is the solution of* (2) *with initial data* (u^0, u^1). The condition imposed on T is due to the fact that the velocity of propagation of waves is one and then any perturbation of the initial data needs some time in order to arrive at the observation zone. The minimal time $2\sqrt{2}$ coincides with the diameter of the domain, which is the one that needs the largest time to reach the boundary. Observe that (2) is time invariant so we can consider the observability inequality for the homogeneous wave equation with initial state at $t = 0$ instead of final state at $t = T$.

The main objective of this paper is to introduce a convergent numerical approximation of the control problem (1) using the HUM approach. This is equivalent that a similar to (3) inequality holds at the semi-discrete level and in addition is independent with respect to the mesh size h.

Trough the paper we deal with the two-dimensional case but all the arguments we present here work also in any space dimension.

To fix the ideas let us consider the finite-difference semi-discretization of (2). Given $N \in \mathbb{N}$ we set $h = 1/(N + 1)$, $\Omega_h = \Omega \cap h\mathbb{Z}^2$ and $\Gamma_h = \Gamma \cap h\mathbb{Z}^2$. In the same manner we define Γ_{0h} and Γ_{1h}. The finite-difference semi-discretization of (2) is as follows:

$$\begin{cases} u_h'' - \Delta_h u_h = 0 & \text{in } \Omega_h \times [0, T], \\ u_h = 0, & \text{on } \Gamma_h \times (0, T) \\ u_h(0) = u_h^0,\ u_h'(0) = u_h^1 & \text{in } \Omega_h. \end{cases} \tag{4}$$

To simplify the presentation we will avoid the subscript h in the notation of the solution u_h unless will be necessary. Let us now introduce the *discrete energy* associated with system (4):

$$\mathcal{E}_h(t) = \frac{h^2}{2}\sum_{j,k=0}^{N}\left[|u_{jk}'(t)|^2 + \left|\frac{u_{j+1,k}(t) - u_{jk}(t)}{h}\right|^2 + \left|\frac{u_{j,k+1}(t) - u_{jk}(t)}{h}\right|^2\right]. \tag{5}$$

It is easy to see that the energy remains constant in time, i.e.

$$\mathcal{E}_h(t) = \mathcal{E}_h(0),\ \forall\ 0 < t < T \tag{6}$$

for every solution of (4).

Following [1] the discrete version of the energy observed on the boundary is given by:

$$\int_0^T \int_{\Gamma_0} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma dt \sim \int_0^T \left[h\sum_{j=1}^{N}\left|\frac{u_{jN}}{h}\right|^2 + h\sum_{k=1}^{N}\left|\frac{u_{Nk}}{h}\right|^2\right] dt.$$

In the following for any $j = 1, ..., N$ and $k = 1, ..., N$, we denote

$$(\partial_n^h u)_{j,N+1} := -\frac{u_{jN}}{h},\ (\partial_n^h u)_{N+1,k} := -\frac{u_{Nk}}{h}.$$

Also, in order to simplify the presentation, we will use the following notation

$$\int_{\Gamma_{0h}} |\partial_n^h u|^2 d\Gamma_{0h} := h\sum_{j=1}^{N}\left|\frac{u_{jN}}{h}\right|^2 + h\sum_{k=1}^{N}\left|\frac{u_{Nk}}{h}\right|^2. \tag{7}$$

The discrete version of (3) is then an inequality of the form

$$\mathcal{E}_h(0) \leq C_h(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h u|^2 d\Gamma_{0h} dt. \tag{8}$$

System (4) being finite, for all $T > 0$ and $h > 0$ there exists a constant $C_h(T)$ such that inequality (8) holds for all the solutions of equation (2). But, as it was proved in [20], for all $T > 0$ the best constant $C_h(T)$ necessarily blows-up as $h \to 0$. More precisely the blow-up of the observability constant in (8) is due to to solutions of (2) of the form $u = \exp(it\sqrt{\lambda})\varphi$, λ being a sufficiently large eigenvalue of the eigenvalue problem associated to the system (2). These high frequencies eigenfunctions are such that the energy concentrated on the boundary Γ_{0h} is asymptotically smaller than the total energy. In the one-dimensional case the observability constant $C_h(T)$ blows-up exponentially. This was proved by Micu [17] by using the explicit expression of a biorthogonal sequence of functions to the underlying time complex-exponentials. This phenomenon was already observed by R. Glowinski et al. in [6], [8] and [9], in connection with the exact boundary controllability of the wave equation and the numerical implementation of the HUM method.

Several techniques have been introduced as possible remedies to the high frequencies spurious oscillations: Tychonoff regularization [6], filtering of the high frequencies [11], [20], [21], mixed finite elements [7], [4], [5] and the two-grid algorithm [18], [16].

As proved by Zuazua [20], inequality (8) holds uniformly in a class of *low frequencies* (initial data where spurious high modes have been filtered). In the Section 2 we will make this concept precise. The main result of Section 2 states that, once (8) holds uniformly for a class of *low frequencies*, it also holds for all solutions in a larger class with their energy controlled by their projection on the previous *low frequencies* components. The method we employ is different from the ones of [18, 16] and consists in using the already well known observability inequality for a class of *low frequency* data and a dyadic time spectral decomposition of the solutions. The two-grid method that will precise below enters in this class.

The two-grid method was proposed by Glowinski [8] and consists in using a coarse and a fine grid, and interpolating the initial data in (4) from the coarse G_c grid to the fine one G_f. This method eliminates the short wave-length component of the initial conditions (u_h^0, u_h^1), component that is responsable for the blow-up of constant $C_h(T)$ in (8).

In the one-dimensional case, the two-grid method was analyzed by Negreanu and Zuazua in [18] with a discrete multiplier approach. The authors also proved the convergence of the method as $h \to 0$ for $T > 4$. In a recent work, Mehrenberer and Loreti [16], used a fine extension of Ingham's inequality to improve the minimal time of uniform observability $T > 2\sqrt{2}$. However as far as we know there is no proof of the uniform observability in the two-dimensional case. The main goal of this paper is to give the first complete proof of the uniform observability inequality in the multi-dimensional case.

In contrast with the strategy adopted in [18] where the authors consider the ratio between the size of the grids 1/2, we choose the quotient to be 1/4. This is done for merely technical reasons and one may expect the same result should hold when the ratio of the grids is 1/2. Our method works in any dimension by choosing a smaller quotient of the meshes.

This idea of considering the quotient of the grids to be 1/4 has been used successfully in [10] when proving dispersive estimates for conservative semi-discrete approximation schemes of the Schrödinger equation. When diminishing the ratio between grids, the filtering that the two-grid algorithm introduces concentrates the solutions of the numerical problem on lower and lower frequencies for which the velocity of propagation becomes closer and closer to that of the continuous wave equation. It is therefore natural to expect that proving the uniform observability will be easier for smaller grid ratios.

We will introduce the space V^h of functions defined on the fine grid G^h as linear interpolation of functions defined on the coarse grid G^{4h}. In Section 3 we prove that (8) holds uniformly for all $T>4$, in the class of initial data $V^h\times V^h$.

Once the observability inequality for solutions of (4) with initial data in $V^h\times V^h$ has been proved, we introduce a numerical scheme for problem (1). This will be done by considering approximations (u_h^0,u_h^1) of initial data belonging to the space $V^h\times V^h$. By using the HUM method in the semi-discrete setting we construct semi-discrete control functions v_h that approximate the control function v in (1). In Section 5 we prove the convergence of these functions v_h towards the continuous one v.

2. The observability problem

To make our statements precise, let us consider the eigenvalue problem associated to (4):

$$\left\{\begin{array}{rl} -\Delta_h\varphi_h=\lambda\varphi_h & \text{in}\,\Omega_h,\\ \varphi_h=0 & \text{on}\,\Gamma_h. \end{array}\right. \tag{9}$$

Denoting $\Lambda_N:=[1,N]^2\cap\mathbb{Z}^2$, the eigenvalues and eigenvectors of system (9) are

$$\lambda_{\mathbf{k}}(h)=\frac{4}{h^2}\left[\sin^2\left(\frac{k_1\pi h}{2}\right)+\sin^2\left(\frac{k_2\pi h}{2}\right)\right],\ \mathbf{k}=(k_1,k_2)\in\Lambda_N$$

and $\{\varphi_h^{\mathbf{k}}\}_{\mathbf{k}\in\Lambda_N}$:

$$\varphi_h^{\mathbf{k}}(\mathbf{j})=\sin(j_1k_1\pi h)\sin(j_2k_2\pi h),\ \mathbf{k}=(k_1,k_2)\in\Lambda_N,\ \mathbf{j}=(j_1,j_2)\in[0,N+1]^2\cap\mathbb{Z}^2.$$

The vectors $\{\varphi_h^{\mathbf{k}}\}_{\mathbf{k}\in\Lambda_N}$ form a basis for the functions defined on G^h and vanishing on its boundary. Any real function ϕ_h defined on the grid G^h admits the Fourier representation:

$$\phi_h=\sum_{\mathbf{k}\in\Lambda_N}\widehat{\phi}_h(\mathbf{k})\varphi_h^{\mathbf{k}},\ \widehat{\phi}(\mathbf{k})\in\mathbb{R}.$$

In view of this representation, for every $s\in\mathbb{R}$, we will denote by $\mathcal{H}_h^s(\Omega_h)$ the space of all functions defined on the grid G^h, endowed with the norm

$$\|\phi_h\|_{s,h}^2=\sum_{\mathbf{k}\in\Lambda_N}\lambda_{\mathbf{k}}^s(h)|\widehat{\phi}_h(\mathbf{k})|^2.$$

Let us consider $\{\widehat{u}_h^0(\mathbf{j})\}_{\mathbf{j}\in\Lambda_N}$ and $\{\widehat{u}_h^1(\mathbf{j})\}_{\mathbf{j}\in\Lambda_N}$ the coefficients of the initial data (u_h^0,u_h^1) in the basis $\{\varphi_h^{\mathbf{j}}\}_{\mathbf{j}\in\Lambda_N}$. Then the solution of system (4) is given by

$$u_h(t)=\frac{1}{2}\sum_{\mathbf{j}\in\Lambda_N}\left[e^{it\omega_{\mathbf{j}}(h)}\widehat{u}_{\mathbf{j}+}^h+e^{-it\omega_{\mathbf{j}}(h)}\widehat{u}_{\mathbf{j}-}^h\right]\varphi_h^{\mathbf{j}}, \tag{10}$$

where $\omega_{\mathbf{j}}(h)=\sqrt{\lambda_{\mathbf{j}}(h)}$ and

$$\widehat{u}_{\mathbf{j}\pm}^h=\widehat{u}_h^0(\mathbf{j})\pm\frac{\widehat{u}_h^1(\mathbf{j})}{i\sqrt{\lambda_{\mathbf{j}}(h)}}.$$

Using the above notations, the energy is given by

$$\mathcal{E}_h(u_h)=\sum_{\mathbf{j}\in\Lambda_N}\omega_{\mathbf{j}}^2(h)(|\widehat{u}_{\mathbf{j}+}^h|^2+|\widehat{u}_{\mathbf{j}-}^h|^2).$$

Let us introduce the class of filtered solutions of (4) in which the high frequencies have been truncated or filtered. More precisely, for any $0<\gamma\leq 2\sqrt{2}$ we set

$$I_h(\gamma)=\left\{u_h(t)=\sum_{\omega_{\mathbf{j}}(h)\leq\gamma/h}\left[e^{it\omega_{\mathbf{j}}(h)}\widehat{u}^h_{\mathbf{j}+}+e^{-it\omega_{\mathbf{j}}(h)}\widehat{u}^h_{\mathbf{j}-}\right]\varphi^{\mathbf{j}}_h \text{ with } \widehat{u}_{\mathbf{j}+},\widehat{u}_{\mathbf{j}-}\in\mathbb{C}\right\}. \tag{11}$$

This space contains solutions of equation (4) that have been filtered along the level curves of the frequencies $\omega_{\mathbf{j}}(h)$. The class $I_h(\gamma)$ has been intensively used, in connection with the so-called semi-classical analysis, for control problems [12], [2], [13] and the dispersive properties of PDE's [3]. For any solution u_h of equation (4) we denote by $\Pi^h_\gamma u_h$ its projection on the space $I_h(\gamma)$.

The uniform observability in the class $I_h(\gamma)$ has been analyzed in [20] by the multiplier technique. In that article it is shown that for any $0<\gamma<2$ and

$$T>T(\gamma)=\frac{8\sqrt{2}}{4-\gamma^2} \tag{12}$$

there exists $C(\gamma,T)>0$ such that

$$\mathcal{E}_h(u_h)\leq C(\gamma,T)\int_0^T\int_{\Gamma_{0h}}|\partial^h_n u_h(t)|^2 d\Gamma_{0h}dt \tag{13}$$

holds for every solution u of (4) in the class $I_h(\gamma)$ and $h>0$. This observability result will be systematically used along the paper. More than that for $\gamma=2$ and $T>0$ there is no constant $C(T)$ (see [20]) such that (13) holds for all solutions u of (4), uniformly on h:

$$\sup_{u\in I_h(2)}\frac{\mathcal{E}_h(u_h)}{\displaystyle\int_0^T\int_{\Gamma_{0h}}|\partial^h_n u_h(t)|^2 d\Gamma dt}\to\infty,\ h\to 0.$$

This is a consequence of fact that the presence of the frequencies near the points $(\pi/h,0)$, $(0,\pi/h)$ have group velocity of order h that spend a time of order $1/h$ to reach the boundary and coresspond to eigenvalues with $\omega_{\mathbf{j}}(h)\sim 2/h^2$.

In the following we give a general result that can be applied in a more general setting. We will consider a numerical scheme such that the energy of their solutions is controlled by the energy of its low frequency projection Π^h_γ. We also assume that in the class $I_h(\gamma)$ the observability inequality holds uniformly with respect to the mesh size h. Then the same observability result holds for the solutions of the considered numerical scheme.

Let us fix an positive M. For any $0<\gamma\leq 2\sqrt{2}$ we define $K^h_\gamma(M)$ as the subspace of $\mathcal{H}^1_h(\Omega_h)\times\mathcal{H}^0_h(\Omega_h)$ consisting in all the functions (φ,ψ) such that their norm is controlled by the one of its projection on $I_h(\gamma)$:

$$K^h_\gamma(M)=\{(\varphi,\psi):\|\varphi\|^2_{1,h}+\|\psi\|^2_{0,h}\leq M(\|\Pi^h_\gamma\varphi\|^2_{1,h}+\|\Pi^h_\gamma\psi\|^2_{0,h})\}.$$

We point out that the conservation of energy (6) guarantees that the solutions of equation (4) with initial data $(\varphi,\psi)\in K^h_\gamma(M)$ satisfy

$$\mathcal{E}_h(u)\leq M\mathcal{E}_h(\Pi^h_\gamma u), \tag{14}$$

and thus obviously $(u_h(t),u_h'(t))\in K^h_\gamma(M)$ for any $t\geq 0$.

The main result of this section is given by the following theorem.

Theorem 2.1. *Let be $\gamma > 0$ and M positive. Assume the existence of a time $T(\gamma)$ such that for all $T > T(\gamma)$ there exists a constant $C(T)$, independent of h, such that*

$$\mathcal{E}_h(v) \leq C(\gamma, T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h v(t)|^2 d\Gamma dt \tag{15}$$

holds for all $v \in I_h(\gamma)$. Then for all $T > T(\gamma)$ there exists a constant $C_1(T, \gamma, M)$, such that

$$\mathcal{E}_h(u_h) \leq C_1(\gamma, T, M) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h|^2 d\Gamma_{0h} dt \tag{16}$$

holds for all the solutions u_h of problem (4) with initial data $(u_h^0, u_h^1) \in K_\gamma^h(M)$ and $h > 0$.

Remark 2.1. *Inequalities (14) and (15) show that the uniform boundary observability*

$$\mathcal{E}_h(u_h) \leq C(T) \int_0^T \int_{\Gamma_{0h}} |\partial_n^h \Pi_\gamma^h u_h|^2 d\Gamma_{0h} dt$$

holds in the class $K_\gamma^h(M)$ as well. Unfortunately, the right side term cannot be estimated directly in terms of the energy of the solution u measured at the boundary Γ_{0h}:

$$\int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h|^2 d\Gamma_{0h} dt.$$

A careful analysis is required to show that estimate.

Remark 2.2. *In the proof of the above Theorem we use that the so-called "direct inequality" holds. In fact (see [20]) for any $T > 0$ there exists a constant $C(T)$, independent of h, such that*

$$\int_0^T \int_{\Gamma_{0h}} |\partial_n^h u_h|^2 d\Gamma_{0h} dt \leq C(T) \mathcal{E}_h(u_h). \tag{17}$$

for all solutions u of the semi-discrete system (4) and for all $h > 0$.

Remark 2.3. *In Theorem 2.1 we analyze the problem of boundary observability. But, in fact, its proof applied in a much more general context. In particular it can be applied in the problem of internal observability for which the measurement on solutions is done in an open subset ω of the domain Ω.*

The proof of Theorem 2.1 will be postponed until Section 6.

3. A Two-grid Method

In this section we introduce a two-grid method and prove the uniform observability inequality (8) in the class of two-grid data.

The two-grid algorithm we propose is the following: Let N be such that $N \equiv 3 \pmod 4$ and $h = 1/(N+1)$. We introduce a coarse grid of mesh-size $4h$:

$$G^{4h} : x_{\mathbf{j}},\ x_{\mathbf{j}} = 4h\mathbf{j},\ \mathbf{j} \in \left[0, \frac{N+1}{4}\right]^2 \cap \mathbb{Z}^2$$

and a fine one of size h:

$$G^h : y_{\mathbf{j}},\ y_{\mathbf{j}} = \mathbf{j}h,\ \mathbf{j} \in [0, N+1]^2 \cap \mathbb{Z}^2.$$

We consider the space V^h of all functions φ defined on the fine grid G^h as a linear interpolation of the functions ψ defined on the coarse grid G^{4h}.

We define now another class of filtered solutions, better adapted to the spectral analysis of the two-grid functions. For any $0 < \eta \leq 1$ we set

$$J_h(\eta) = \left\{ u_h(t) = \sum_{\|\mathbf{j}\|_\infty \leq \eta(N+1)} \left[e^{it\omega_{\mathbf{j}}(h)} \widehat{u}^h_{\mathbf{j}+} + e^{-it\omega_{\mathbf{j}}(h)} \widehat{u}^h_{\mathbf{j}-} \right] \varphi^{\mathbf{j}}_h \text{ with } \widehat{u}^h_{\mathbf{j}+}, \widehat{u}^h_{\mathbf{j}-} \in \mathbb{C} \right\}, \tag{18}$$

and for any solution u_h of (4) we denote by $\Upsilon^h_\eta u$, its projection on the space $J_h(\eta)$. The class of filtered solutions $I_\gamma(h)$, introduced in Section 2, is obtained through a filtering along the level curves of the $\omega_{\mathbf{j}}(h)$. The second one consists in filtering the range of indices $\mathbf{j}$ to the square of length side $\eta(N+1)$. Observe that, in dimension one there exists a one-to-one correspondence between the two classes. In dimension two, excepting the case $\gamma = 2\sqrt{2}$, that corresponds to $\eta = 1$, there is no one-to-one correspondence. However the two classes can be easily compared with each other.

The second class of data is better adapted to the two-grid data. In fact we will prove that the total energy of a solution u_h of (4) with initial data in the space $V^h \times V^h$ is bounded above by the energy of its projection on the space $J_h(1/4)$:

$$\mathcal{E}_h(u_h) \leq 4\mathcal{E}_h(\Upsilon^h_{1/4} u). \tag{19}$$

Clearly any $\omega_{\mathbf{j}}(h)$ with $\|\mathbf{j}\|_\infty \leq (N+1)/4$ satisfies

$$\omega_{\mathbf{j}}(h) \leq \left(\frac{8}{h^2} \sin^2\left(\frac{\pi}{8}\right) \right)^{1/2} \leq \frac{2\sqrt{2}\sin(\pi/8)}{h},$$

and thus, in view of (19) the energy of the solution u_h is bounded above by the energy of its projection on the space $I_h(2\sqrt{2}\sin(\pi/8))$:

$$\mathcal{E}_h(u) \leq 4\mathcal{E}_h(\Upsilon^h_{1/4} u) \leq 4\mathcal{E}_h(\Pi^h_{2\sqrt{2}\sin(\pi/8)} u),$$

i.e. $(u_h(t), u_h'(t)) \in K^h_\gamma(4)$ with $\gamma = 2\sqrt{2}\sin(\pi/8)$.

The following theorem gives us the uniform boundary observability for the solutions u_h on system (4) with initial data $(u^0_h, u^1_h) \in V^h \times V^h$. This theorem is in fact a consequence of Theorem 2.1.

Theorem 3.1. *Let $T > 4$. There exists a constant $C(T)$ such that*

$$\mathcal{E}_h(u_h) \leq C(T) \int_0^T \int_{\Gamma_{0h}} |\partial^h_n u_h|^2 d\Gamma_{0h} dt \tag{20}$$

holds for all solutions u_h of (4) with $(u^0_h, u^1_h) \in V^h \times V^h$, uniformly on $h > 0$, V^h being the class of the two-grid data obtained with ratio $1/4$.

Remark 3.1. *The time $T > 4$ is given by the observability time obtained in* [20] *for the class of solutions belonging to $I_h(2\sqrt{2}\sin(\pi/8))$, the smallest class I_h that contains $J_h(1/4)$. We recall that in view of (12) the observability time for the above class of solutions is given by:*

$$T\left(2\sqrt{2}\sin\left(\frac{\pi}{8}\right)\right) = \frac{2\sqrt{2}}{1 - 2\sin^2\left(\frac{\pi}{8}\right)} = \frac{2\sqrt{2}}{\cos\left(\frac{\pi}{4}\right)} = 4.$$

In fact, Theorem 3.1 holds for all $T > T^*$ where T^* is the optimal time for uniform observability in the class $I_h(2\sqrt{2}\sin(\pi/8))$. The estimates $T(2\sqrt{2}\sin(\pi/8)) = 4$ on T^* is not *optimal. The expected minimal time* T^* *is*

$$T^* = \frac{2\sqrt{2}}{\cos(\pi/8)}. \tag{21}$$

This can be easily derived analyzing the group velocity of wave packets in the class $I_h(2\sqrt{2}\sin(\pi/8))$ *(see Trefethen* [19] *and* [21]*). Although the uniform observability in the class* $I_h(2\sqrt{2}\sin(\pi/8))$ *for all* $T > T^*$ *with* T^* *as in (21) is very likely to hold, as far as we known it has not been rigourously proved so far.*

Remark 3.2. *A two-grid algorithm involving the grids* G^h *and* G^{2h} *implies*

$$\mathcal{E}_h(u) \le 2\mathcal{E}_h(\Upsilon^h_{1/2}u) \le 2\mathcal{E}_h(\Pi^h_2 u)$$

for all solutions u *obtained by this method. Indeed, the smallest* γ *such that* $I_h(\gamma)$ *contains all the frequencies* $\omega_{\mathbf{j}}(h)$, $\|\mathbf{j}\|_\infty \le (N+1)/2$ *is* $\gamma = 2$. *Unfortunately, as we pointed before, inequality (15) does not hold in the class* $I_h(2)$. *This is why we choose the ratio between the fine and coarse grid in the two-grid method to be* $1/4$. *This will guarantee that the two hypotheses (14) and (15) are verified.*

Proof of Theorem 3.1. Let u_h be the solution of system (4) with initial data $(u_h^0, u_h^1) \in V^h \times V^h$. Using that $J_h(1/4) \subset I_h(2\sqrt{2}\sin(\pi/8))$ we obtain that

$$\mathcal{E}_h(\Upsilon^h_{1/4}u_h) \le \mathcal{E}_h(\Pi^h_{2\sqrt{2}\sin(\pi/8)}u_h).$$

To apply Theorem 2.1 with $\gamma = 2\sqrt{2}\sin(\pi/8)$ it remains to prove (19). The conservation of energy implies that

$$\mathcal{E}_h(u_h) = \|u_h^0\|^2_{1,h} + \|u_h^1\|^2_{0,h}$$

and

$$\mathcal{E}_h(\Upsilon^h_{1/4}u) = \|\Upsilon^h_{1/4}u_h^0\|^2_{1,h} + \|\Upsilon^h_{1/4}u_h^1\|^2_{0,h}.$$

We make use of the following Lemma, which will be proved in Appendix B.

Lemma 3.1. *For any* $v \in V^h$ *the following holds:*

$$\|v\|_{s,h} \le 2^{(s+1)/2}\|\Upsilon^h_{1/4}v\|_{s,h},\ 0 \le s \le 2. \tag{22}$$

Applying this Lemma to $u_h^0 \in V^h$ and $u_h^1 \in V^h$ we get

$$\|u_h^0\|_{1,h} \le 2\|\Upsilon^h_{1/4}u_h^0\|_{1,h} \qquad \text{and} \qquad \|u_h^1\|_{0,h} \le 2\|\Upsilon^h_{1/4}u_h^1\|_{0,h}.$$

This proves that

$$\mathcal{E}_h(u_h) \le 4\mathcal{E}_h(\Upsilon^h_{1/4}u_h)$$

and finishes the proof.

□

4. Construction of the Control

In this section we introduce a numerical approximation for the HUM control v of the continuous wave equation (1) based on the two-grid method.

The idea of approximating the HUM control v for the continuous problem (1) is to consider the following discrete problem

$$\begin{cases} u_h'' - \Delta_h u_h = 0 & \text{in } \Omega_h \times (0,T), \\ u_h = v_h \mathbf{1}_{\Gamma_{0h}} & \text{on } \Gamma_h \times (0,T), \\ u_h(0) = u_h^0,\ \partial_t u_h(0) = u_h^1 & \text{in } \Omega_h. \end{cases}, \tag{23}$$

where the initial data $(u_h^0, u_h^1) \in V^h \times V^h$ are approximations of (u^0, u^1). In this way we will obtain a function v_h that approximate the continuous control function v and the projection of the solution u_h on the coarse grid G^{4h} vanishes at the time T. To be more precisely let us consider the spaces $\mathcal{G}_h$, $\mathcal{G}_{4h}$ of all the functions defined on the fine grid G^h, respectively the coarse one G^{4h}. We also introduce the extension operator Π which associate to any function $\psi \in \mathcal{G}_{4h}$ a new function $\Pi\psi \in \mathcal{G}_h$ obtained by an interpolation process:

$$(\Pi\psi)_{\mathbf{j}} = (\mathbf{P}_1\psi)(\mathbf{j}),\ \mathbf{j} \in \mathbb{Z}^2,$$

where $\mathbf{P}_1\psi$ is the piecewise linear interpolator of $\psi \in \mathcal{G}_{4h}$. It is easy to see that the space V^h represents the image of the operator Π. Also it is possible to define a restriction operator which carries any function of $\mathcal{G}_h$ to $\mathcal{G}_{4h}$. The most natural way is to define it as the formal adjoint of the operator Π:

$$(\psi, \Pi\phi)_h = (\Pi^*\psi, \phi)_{4h},\ \forall\ \phi \in \mathcal{G}_{4h},$$

where $(\cdot,\cdot)_h$ denote the $\mathcal{H}_h^0(\Omega_h)$ inner product:

$$(u,v)_h = h^2 \sum_{\mathbf{j}h \in \Omega_h} u_{\mathbf{j}} v_{\mathbf{j}}.$$

Let us introduce the adjoint discrete problem:

$$\begin{cases} \phi_h'' - \Delta_h \phi_h = 0 & \text{in } \Omega_h \times (0,T), \\ \phi_h(t) = 0 & \text{on } \Gamma_h \times (0,T), \\ \phi_h(T) = \phi_h^0,\ \partial_t \phi_h(T) = \phi_h^1 & \text{in } \Omega_h. \end{cases} \tag{24}$$

We define the duality product between $\mathcal{H}_h^0(\Omega_h) \times \mathcal{H}_h^{-1}(\Omega_h)$ and $\mathcal{H}_h^1(\Omega_h) \times \mathcal{H}_h^0(\Omega_h)$ by

$$\langle(\varphi^0, \varphi^1), (u^0, u^1)\rangle_h = (\varphi^1, u^0)_h - (\varphi^0, u^1)_h.$$

Concerning the HUM control of the discrete control problem (23) the following theorem gives us the existence of a sequence $\{v_h\}_{h>0}$ which will constitute a convergent approximation of the continuous one v.

Theorem 4.1. *Let be $T > 4$. There exists a constant $C(T)$ such that for any $h > 0$ and (u_h^0, u_h^1), there exists a function v_h with*

$$\|v_h\|^2_{L^2((0,T)\times\Gamma_{0h})} \leq C(T)(\|u_h^0\|^2_{0,h} + \|u_h^1\|^2_{-1,h}) \tag{25}$$

such that the solution u_h of system (23) with (u_h^0, u_h^1) as initial data and v^h acting as control satisfies:

$$\Pi^* u_h(T) = \Pi^* u_h'(T) = 0. \tag{26}$$

Following the same steps as in the continuous case, i.e. multiplying the control problem (23) by solutions of the adjoint problem (24) and integrating (summing) by parts we obtain the following description of the solutions of system (23):

Lemma 4.1. *Let u_h be a solution of system (23). Then the following*

$$\int_0^T \int_{\Gamma_{0h}} v_h(t)\partial_n^h \phi_h(t) d\Gamma_{1h} dt + \langle (u_h, u_h'), (\phi_h, \phi_h')\rangle_h \Big|_0^T = 0 \tag{27}$$

holds for all solutions ϕ_h of the adjoint problem (24).

In the sequel we introduce the following notations

$$\int_{\Omega_h} u d\Omega_h = h^2 \sum_{\mathbf{j}h\in\Omega_h} u_{\mathbf{j}} \text{ and } \int_{\Gamma_h} u d\Gamma_h := h \sum_{\mathbf{j}h\in\Gamma_h} u_{\mathbf{j}}.$$

Proof of Lemma 4.1. Multiplying (23) and (24) by ϕ_h, respectively u_h, and integrating on $[0,T]$ and summing on Ω_h yields to

$$\int_0^T \int_{\Omega_h} (u_h''\phi_h - \phi_h'' u_h) dt d\Omega_h = \int_0^T \int_{\Omega_h} [(\Delta_h u_h)\phi_h - (\Delta_h \phi_h) u_h] dt d\Omega_h.$$

Integration by parts in the left hand term gives us

$$\int_0^T \int_{\Omega_h} (u_h''\phi_h - \phi_h'' u_h) dt d\Omega_h \quad = \quad \int_{\Omega_h} \Big(u_h'\phi_h \Big|_0^T - \phi_h' u_h \Big|_0^T \Big) d\Omega_h = \langle (u_h, u_h'), (\phi_h, \phi_h')\rangle_h \Big|_0^T.$$

For the second term, replacing the parameter h in the notation, we have the following explicit computations:

$$\begin{aligned}
\int_0^T \int_{\Omega_h} [(\Delta_h u_h)\phi_h - (\Delta_h \phi_h) u_h] dt d\Omega_h &= h \sum_{i,j=1}^N \big[(u_{i-1,j} + u_{i+1,j})\phi_{ij} - (\phi_{i-1,j} + \phi_{i+1,j}) u_{ij}\big] \\
&\quad + h \sum_{i,j=1}^N \big[(u_{i,j-1} + u_{i,j+1})\phi_{ij} - (\phi_{i,j-1} + \phi_{i,j+1}) u_{ij}\big] \\
= h \sum_{j=1}^N (u_{0j}\phi_{1j} + u_{N+1,j}\phi_{N,j}) + \sum_{i=1}^N (u_{i0}\phi_{i1} + u_{i,N+1}\phi_{i,N}) &= h \sum_{j=1}^N u_{N+1,j}\phi_{N,j} + \sum_{i=1}^N u_{i,N+1}\phi_{i,N} \\
= - \int_0^T \int_{\Gamma_{0h}} v_h(t)\partial_n^h \phi_h(t) dt d\Gamma_{1h}.
\end{aligned}$$

□

Proof of Theorem 4.1. First, using variational methods we will prove the existence of a function v^h such that

$$\int_0^T \int_{\Gamma_{0h}} v_h(t)\partial_n^h \phi_h(t) d\Gamma_{0h} dt = \langle (u_h^0, u_h^1), (\phi_h(0), \phi_h'(0))\rangle_h \tag{28}$$

for all solutions ϕ_h of the adjoint problem (24) with final state $(\phi_h^0, \phi_h^1) \in V^h \times V^h$.

We consider the space $\mathcal{F}_h = V^h \times V^h$ endowed with the norm

$$\|(\phi_h^0, \phi_h^1)\|_{\mathcal{F}_h} = \|\phi_h^0\|_{1,h} + \|\phi_h^1\|_{0,h}$$

and the functional $\mathcal{J}_h : \mathcal{F}_h \to \mathbb{R}$ defined by

$$\mathcal{J}_h((\phi_h^0, \phi_h^1)) = \frac{1}{2}\int_0^T \int_{\Gamma_{0h}} |\partial_n^h \phi_h|^2 d\Gamma_{0h} dt + \langle (u_h^0, u_h^1), (\phi_h(0), \phi_h'(0))\rangle_h \tag{29}$$

where ϕ_h is the solutions of the adjoint problem (24) with final state (ϕ_h^0, ϕ_h^1). The existence of the function v^h will be obtained by minimizing $\mathcal{J}_h$ on the space $\mathcal{F}_h$. In the following in order to apply the fundamental theorem of calculus of variations we prove that the functional $\mathcal{F}_h$ is continuous and uniformly coercive (with respect to the parameter h) on $\mathcal{F}_h$.

The linear term in the right side of (29) satisfies

$$\begin{aligned} |\langle (u_h^0, u_h^1), (\phi_h(0), \phi_h'(0))\rangle_h| &\le \|u_h^1\|_{-1,h}\|\phi_h(0)\|_{1,h} + \|u_h^0\|_{0,h}\|\phi_h'(0)\|_{0,h} \\ &\le (\|u_h^1\|_{-1,h} + \|u_h^0\|_{0,h})\|(\phi_h(0), \phi_h'(0))\|_{\mathcal{F}_h}. \end{aligned}$$

Using the *direct inequality* (17) and the conservation of the energy $\mathcal{E}_h(\phi_h)$ we get

$$|\mathcal{J}_h((\phi_h^0, \phi_h^1))| \le \|(\phi_h^0, \phi_h^1)\|_{\mathcal{F}_h}\left(C(T)\|(\phi_h^0, \phi_h^1)\|_{\mathcal{F}_h} + \|u_h^1\|_{-1,h} + \|u_h^0\|_{0,h}\right)$$

which proves the continuity of the functional J_h.

In view of the observability inequality (20), for any $T > 4$, the functional $\mathcal{J}_h$ is uniformly (with respect to h) coercive on $\mathcal{F}_h$:

$$|\mathcal{J}_h((\phi_h^0, \phi_h^1))| \ge \|(\phi_h^0, \phi_h^1)\|_{\mathcal{F}_h}\left(C(T)\|(\phi_h^0, \phi_h^1)\|_{\mathcal{F}_h} - \|u_h^1\|_{-1,h} - \|u_h^0\|_{0,h}\right),$$

where $C(T)$ is a constant obtained in (20).

Applying the fundamental theorem of calculus of variations we obtain the existence of a minimizer $(\phi_h^{0,*}, \phi_h^{1,*}) \in \mathcal{F}_h$ such that

$$\mathcal{J}_h((\phi_h^{0,*}, \phi_h^{1,*})) = \min_{((\phi_h^0, \phi_h^1)) \in \mathcal{F}_h} \mathcal{J}_h((\phi_h^0, \phi_h^1)).$$

This implies that $\mathcal{J}_h'$, the Gateaux derivative of $\mathcal{J}_h$, satisfies

$$\mathcal{J}_h'((\phi_h^{0,*}, \phi_h^{1,*}))(\phi_h^0, \phi_h^1) = 0$$

for all $(\phi_h^0, \phi_h^1) \in \mathcal{F}_h$. This implies that ϕ_h^* solution of (24) with final state $(\phi_h^{0,*}, \phi_h^{1,*})$ satisfies

$$\int_0^T \int_{\Gamma_{0h}} (\partial_n^h \phi_h^*) \partial_n^h \phi(t) d\Gamma_{0h} dt + \langle (u_h^0, u_h^1), (\phi_h(0), \phi_h'(0))\rangle_h = 0$$

for all ϕ solution of the adjoint problem (24) with final state belonging to $\mathcal{F}_h$.

We set

$$v_h(t) = \partial_n^h \phi_h^*(t), \; t \in [0, T]$$

and then (28). In view of Lemma 4.1, the solution u_h of system (23) with the above function v_h acting as control on Γ_{1h} satisfies

$$(u_h'(T), \phi_h^0)_h - (u_h(T), \phi_h^1)_h = 0$$

for all function $(\phi_h^0, \phi_h^1) \in V^h \times V^h$. Using that $V^h = \Pi(\mathcal{G}^{2h})$ we obtain

$$(u_h(T), \Pi\psi)_h = (u_h'(T), \Pi\psi)_h = 0$$

for all functions $\psi \in \mathcal{G}^{2h}$. Then

$$(\Pi^* u_h(T), \psi)_{2h} = (\Pi^* u_h'(T), \psi)_{2h} = 0$$

for all $\psi \in \mathcal{G}^{2h}$ and obviously

$$\Pi^* u_h(T) = \Pi^* u_h'(T) = 0.$$

It remains to prove estimate (25). Using that $(\phi_h^{0,*}, \phi_h^{1,*})$ is a minimizer of $\mathcal{J}_h$ we have $\mathcal{J}_h((\phi_h^{0,*}, \phi_h^{1,*})) \leq \mathcal{J}_h((0_h, 0_h))$, where 0_h is the function that vanishes identically on the mesh G_h. Consequently

$$\int_0^T \int_{\Gamma_{0h}} |\partial_n^h \phi_h^{0,*}|^2 d\Gamma_{0h} dt \leq (\|u_h^1\|_{-1,h} + \|u_h^0\|_{0,h})(\|\phi_h^{0,*}\|_{1,h} + \|\phi_h^{1,*}\|_{0,h}).$$

Applying the *direct inequality* (17) to the solution ϕ_h^* we obtain that

$$\|v_h\|^2_{L^2((0,T)\times\Gamma_{1h})} = \int_0^T \int_{\Gamma_{1h}} |\partial_n^h \phi_h^*|^2 d\Gamma_1 dt \leq C(T)(\|u_h^1\|_{-1,h} + \|u_h^0\|_{0,h})^2$$

where the constant $C(T)$ is independent of h.

The proof is now complete. □

5. Convergence of the method

Let us consider the family $\{u_h\}_{h>0}$ of solutions of (4) and let us denote by $\mathbf{P}_1 u_h$ their piecewise linear interpolator. Given the fact that the solutions of the continous problem belong to $H_0^1(0,1)$ is then natural to consider a interpolator with enough regularity.

The following propositions describes how a uniformly bounded family of solutions of (4) weakly converges as $h \to 0$ to a solution of finite energy of the continuous wave equation (2).

Proposition 5.1. *(Proposition 4.1,* [20]*) Let $\{\phi_h\}_{h>0}$ be a family of solutions of (24) depending on the parameter $h \to 0$, whose energies are uniformly bound, i.e.*

$$E_h(0) \leq C, \forall\, h > 0. \tag{30}$$

Then there exists a solution $\phi \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega))$ of problem (2) such that by extracting a suitable subsequence $h \to 0$ we may guarantee that

$$\mathbf{P}_1\phi_h \rightharpoonup \phi \textit{ in } H^1([0,T], L^2(\Omega)) \cap L^2([0,T], H_0^1(\Omega)), \tag{31}$$

$$E(0) \leq \liminf_{h\to 0} E_h(0) \tag{32}$$

and

$$\|\mathbf{P}_1\phi_h(t)\|_{L^2(\Omega)} \to \|\phi(t)\|^2_{L^2(\Omega)} \textit{ in } L^\infty(0,T). \tag{33}$$

Also the normal derivatives of $\mathbf{P}_1\phi_h$ converges to the continuous one.

Proposition 5.2. *(Proposition ,* [20]*) Let $\{\phi_h(t)\}_h$ be a family of solutions of (24) satisfying (30). Let ϕ be any solution of (4) obtained as limit when $h \to 0$ as in the statement of Proposition 5.1. Then*

$$\int_0^T \int_{\Gamma_1} \left|\frac{\partial \phi}{\partial n}\right|^2 d\sigma dt \leq \liminf_{h\to 0} \int_0^T \int_{\Gamma_{1h}} |\partial_n^h \phi_h(t)|^2 d\Gamma_{1h} dt. \tag{34}$$

and

$$\frac{\partial(\mathbf{P}_1\phi_h)}{\partial n} \rightharpoonup \frac{\partial \phi}{\partial n} \textit{ on } L^2((0,T),\Gamma_0). \tag{35}$$

Concerning the convergence of the discrete control problem we have the following result.

Theorem 5.1. *Let* $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ *and* (u_h^0, u_h^1) *such that*

$$\mathbf{P}_1 u_h^0 \rightharpoonup u^0 \text{ in } L^2(\Omega), \quad \mathbf{P}_1 u_h^1 \rightharpoonup u^1 \text{ in } H^{-1}(\Omega).$$

Then for any $T > 4$ *the solution* $(u_h(t), \partial_t u_h(t))$ *and its partial controls* $v_h \in L^2(0,T)$ *given by Theorem 4.1 satisfy*

$$\mathbf{P}_1 u_h \overset{*}{\rightharpoonup} u \text{ in } L^\infty([0,T], L^2(\Omega)),\ (\mathbf{P}_1 u_h)_t \overset{*}{\rightharpoonup} u_t \text{ in } L^\infty([0,T], H^{-1}(\Omega)),$$
$$\mathbf{P}_1 v_h \rightharpoonup v \text{ in } L^2([0,T], L^2(\Gamma_0)),$$

where (u, u_t) *solves (1), with the control* $v(t)$, *and* $u(T) = u_t(T) = 0$. *The limit control* v *is given by*

$$v(t) = \partial_n \phi^* \text{ on } \Gamma_0,$$

where ϕ^* *is solution of (2), with data* $(\phi^{0,*}, \phi^{1,*}) \in H_0^1(\Omega) \times L^2(\Omega)$ *minimizing the functional*

$$J((\phi^0, \phi^1)) = \frac{1}{2} \int_0^T \int_{\Gamma_0} |\partial_n \phi|^2 dt + \langle (u^0, u^1), (\phi^0, \phi^1) \rangle \tag{36}$$

in $H_0^1(\Omega) \times L^2(\Omega)$.

Proof. For each $h > 0$ we consider the minimization problem associated with (u_h^0, u_h^1), i.e. (29) on $\mathcal{F}_h$. In view of Theorem 4.1 there exists a function $v_h = \partial_n^h \phi_h^*(t)$, that depends on (u_h^0, u_h^1) and satisfies (25). Recall that ϕ_h^* solves (24) with final state $(\phi_h^{0,*}, \phi_h^{1,*}) \in V^h \times V^h$ minimizing the function J_h.

Moreover, as a consequence of the observability inequality (20) we have

$$\|\phi_h^{0,*}\|_{1,h} + \|\phi_h^{1,*}\|_{0,h} \leq C(T) \|v_h\|_{L^2((0,T)\times\Gamma_{1h})} \leq C(T)(\|u_h^1\|_{0,h} + \|u_h^0\|_{-1,h}) \leq C(T).$$

In these conditions, Proposition 5.1 guarantees the existence of a function ϕ^* that solves (2) and in addition

$$\mathbf{P}_1 v_h(t) = \mathbf{P}_1 \partial_n^h \phi_h^*(t) = \partial_n(\mathbf{P}_1 \phi_h^*) \rightharpoonup \partial_n \phi^* \text{ in } L^2(0,T) \text{ as } h \to 0.$$

Step II. Let us now consider equation (23) with initial data (u_h^0, u_h^1) and v_h as above. Then for any solution ϕ_h of the adjoint problem (24) with final state at $T = s$ the following

$$\int_0^s \int_{\Gamma_{1h}} v_h(t) \partial_n^h \phi_h(t) d\Gamma_{1h} dt + \langle (u_h, u_h'), (\phi_h, \phi_h') \rangle_h \Big|_0^s = 0 \tag{37}$$

holds for all $0 < s < T$. Thus in view of *direct inequality* (17) and the conservation of the energy applied to ϕ_h we get for any $s < T$ that

$$\begin{aligned} |\langle (u_h(s), u_h'(s)), (\phi_h^0, \phi_h^1) \rangle_h| &\leq |\langle (u_h^0, u_h^1), (\phi_h(0), \phi_h'(0)) \rangle_h| + \|v_h\|_{L^2((0,T)\times\Gamma_{1h})} \|\partial_n^h \phi_h\|_{L^2((0,T)\times\Gamma_{1h})} \\ &\leq C(T)(\|u_h^0\|_{0,h} + \|u_h^1\|_{-1,h})(\|\phi_h^0\|_{1,h} + \|\phi_h^1\|_{0,h}). \end{aligned}$$

This means that

$$\|u_h(s)\|_{0,s} + \|u_h'(s)\|_{-1,s} \leq C.$$

We deduce the existence of a subsequence of indexes $\{h\}$ such that

$$\mathbf{P}_1 u_h \overset{*}{\rightharpoonup} u \text{ in } L^\infty([0,T], L^2(\Omega)) \tag{38}$$

and

$$\mathbf{P}_1 u_h' \overset{*}{\rightharpoonup} u' \text{ in } L^\infty([0,T], H^{-1}(\Omega)). \tag{39}$$

Also using that u_h verifies equation (23) we also obtain

$$\|\mathbf{P}_1 u_h''(s)\|_{L^2([0,T],\,H^{-2}(\Omega))} \leq C. \tag{40}$$

Using (38), (39) and (40) we get

$$u \in C([0,T], L^2(\Omega)) \text{ and } u' \in C([0,T], H^{-1}(\Omega)).$$

Moreover, from (38,39) and (39,40) and Aubin-Lions compactness lemma, we deduce that

$$\mathbf{P}_1 u_h \to u \text{ in } C([0,T], L^2(\Omega)) \cap C^1([0,T], H^{-1}(\Omega)).$$

Thus

$$\mathbf{P}_1 u_h(0) \rightharpoonup u(0) \,\text{in}\, L^2(\Omega)$$

and

$$\mathbf{P}_1 u_h'(0) \rightharpoonup u'(0) \,\text{in}\, H^{-1}(\Omega).$$

Observe that, by hypothesis of the theorem $u(0) = u^0$ and $u'(0) = u^1$. Using (37) we have

$$\int_0^s \int_{\Gamma_1} \mathbf{P}_0 v_h(t) \partial_n (\mathbf{P}_1 \phi_h)(t) d\sigma dt + \langle (u_h, u_h'), (\phi_h, \phi_h')\rangle_h \Big|_0^s = 0.$$

Getting $h \to 0$ we obtain

$$\int_0^s \int_{\Gamma_0} \partial_n \phi^* \partial_n \phi d\sigma dt + \langle (u, u'), (\phi, \phi')\rangle_h \Big|_0^s = 0, \,\forall\, s < T,$$

where ϕ is solution of problem (2) with final state (ϕ^0, ϕ^1). Thus u is a solution by transposition of (1) with control $v = \partial_n \phi^*$.

Let us prove that $u(T) = u'(T) = 0$. We prove that $u(T) = 0$ the other case being similar. Since $(u_h(T), v_h)_h = 0$ for all functions $v_h \in V^h$ we obtain that

$$\int_\Omega \mathbf{P}_1 u_h(T) \mathbf{P}_1 v_h dx \to 0 \text{ as } h \to 0.$$

Using that $\mathbf{P}_1(V^h)$ is dense in $L^2(\Omega)$ we get

$$\int_\Omega \mathbf{P}_1 u_h(T) v dx \to 0 \text{ as } h \to 0$$

for any function $v \in L^2(\Omega)$ that implies $u(T) \equiv 0$.

Finally, using the uniqueness results for problem (1) we obtain that control v obtained before satisfies $v = \partial_n \psi^*$ where ψ^* is the solution of problem (2) with final state minimizing functional (36).

□

6. Proof of Theorem 2.1

Proof of Theorem 2.1. In view (13), for any $T > T(\gamma)$ there exists two positive constants δ and ϵ such that

$$\mathcal{E}_h(v_h) \leq C(T, \gamma, \epsilon, \delta) \int_{2\delta}^{T-2\delta} \int_{\Gamma_{0h}} |\partial_n^h v_h|^2 d\Gamma_{0h} dt \tag{41}$$

for all $v_h \in I_h(\gamma + \epsilon)$. More precisely, using the continuity of the map $\gamma \to T(\gamma)$ we obtain the existence of a small constant ϵ such that $T > T(\gamma + \epsilon)$. Sequently we choose a positive δ such that $T - 4\delta > T(\gamma + \epsilon)$. Then, a time translation guarantees that (41) holds.

With ϵ verifying (41) let us choose positive constants a, b, c and μ satisfying

$$1 < c < \frac{b-\mu}{a+\mu} \text{ and } \frac{b}{a+\mu} < \frac{\gamma+\epsilon}{\gamma}. \tag{42}$$

In the following we make precise the time projectors P_k which give us the a time-spectral decomposition of u. These are essentially the ones introduced in [12].

Let $F \in C_c^\infty(\mathbb{R})$ be supported in (a,b), $0 \le F \le 1$ such that $F \equiv 1$ in $[a+\mu, b-\mu]$. Set $P(\tau) = F(\tau) + F(-\tau)$. For any function $f \in L^1(\mathbb{R})$ and $k \ge 0$ we consider the projector $P_k f$ defined by

$$(P_k f)(t) = \int_{\mathbb{R}_\tau} \int_{\mathbb{R}_s} P(c^{-k}\tau) f(s) e^{i(t-s)\tau} ds d\tau. \tag{43}$$

In view of (10) the Fourier transform of u_h, in the t variable, reads

$$\widehat{u_h}(\tau) = \sum_{\mathbf{j}\in\mathbb{Z}^2} \left[\delta(\tau - \omega_{\mathbf{j}}(h)) \widehat{u}^h_+(\mathbf{j}) + \delta(\tau + \omega_{\mathbf{j}}(h)) \widehat{u}^h_-(\mathbf{j}) \right] \varphi^{\mathbf{j}}_h.$$

Therefore, the projector $P_k u_h$ satisfies

$$P_k u_h(t) = \sum_{\mathbf{j}\in\mathbb{Z}^2} F(c^{-k}\omega_{\mathbf{j}}(h)) \left[e^{it\omega_{\mathbf{j}}(h)} \widehat{u}_+(\mathbf{j}) + e^{-it\omega_{\mathbf{j}}(h)} \widehat{u}_-(\mathbf{j}) \right] \varphi^{\mathbf{j}} \tag{44}$$

and its energy is given by

$$\mathcal{E}_h(P_k u_h) = \sum_{\mathbf{j}\in\mathbb{Z}^2} F^2(c^{-k}\omega_{\mathbf{j}}(h)) \omega^2_{\mathbf{j}}(h) (|\widehat{u}_{\mathbf{j}+}|^2 + |\widehat{u}_{\mathbf{j}-}|^2). \tag{45}$$

Conditions (42) guarantee that $(P_k u_h)_{k\ge 0}$ covers all the frequencies occurring in the representation of u_h. Also $P_k f = f$ for all functions f that contains only frequencies in the range $[-(b-\mu)c^k, -(a+\mu)c^k] \cup [(a+\mu)c^k, (b-\mu)c^k]$.

We first give the main ideas of the proof. Let us choose two positive integers k_0 and k_h, $k_0 \le k_h$, k_0 independent of h, such that $\{P_k u_h\}_{k=k_0}^{k_h}$ covers, except possibly for a finite number, all the frequencies occurring in $\Pi^h_\gamma u_h$. The precise value of k_0 and k_h will be specified later.

Firstly we will prove that

$$\mathcal{E}_h(\Pi^h_\gamma u_h) \le \sum_{k=k_0}^{k_h} \mathcal{E}_h(P_k u_h) + LOT \tag{46}$$

where LOT is a lower order term, involving all the frequencies smaller than $c^{k_0}(a+\mu)$, in particular this LOT will be compact when passing to the limit $h \to 0$.

Next we use that each projection $P_k u_h$, $k_0 \le k \le k_h$ belongs to the class $I_h(\gamma+\epsilon)$ and, consequently, according to (41), satisfies the observability inequality:

$$\mathcal{E}_h(P_k u_h) \le C(T, \gamma, \delta, \epsilon) \int_{2\delta}^{T-2\delta} \int_{\Gamma_h} |\partial^h_n P_k u_h|^2 d\Gamma_h dt. \tag{47}$$

Thus, combining (46) and (47) we obtain the following estimate:

$$\mathcal{E}_h(\Pi^h_\gamma u_h) \le C(T, \gamma, \delta, \epsilon) \sum_{k=k_0}^{k_h} \int_{2\delta}^{T-2\delta} \int_{\Gamma_h} |\partial^h_n P_k u_h|^2 d\Gamma_h dt + LOT.$$

Using the previous ideas of [12] and [2] the right hand side sum can be estimated in terms of the energy of u_h measured on Γ_{0h}. More precisely, we will prove the existence of a constant $C(\epsilon,\delta,T)$ such that

$$\sum_{k\geq k_0}\int_{2\delta}^{T-2\delta}\int_{\Gamma_h}|\partial_n^h P_k u_h|^2 d\Gamma_h dt \leq 2\int_0^T\int_{\Gamma_{0h}}|\partial_n^h u_h|^2 d\Gamma_h dt + \frac{C(\epsilon,\delta,T)}{c^{2k_0}}\mathcal{E}_h(u_h) \tag{48}$$

holds for any $k_0\geq 0$ and u_h solution of (4) uniformly on $h>0$. Then the following holds:

$$\mathcal{E}_h(u_h)\leq C\mathcal{E}_h(\Pi_\gamma^h u_h)\leq C(T,\gamma,\delta,\epsilon)\int_0^T\int_{\Gamma_{0h}}|\partial_n^h u_h|^2 d\Gamma_{0h}dt + \frac{C(\epsilon,\delta,T)}{c^{2k_0}}\mathcal{E}_h(u_h)+LOT. \tag{49}$$

Choosing h small and k_0 sufficiently large, but still independent of h, the energy term from the right side may be absorbed and then we obtain

$$\mathcal{E}_h(u_h)\leq C(T,\gamma,\delta,\epsilon)\int_0^T\int_{\Gamma_{0h}}|\partial_n^h u_h|^2 d\Gamma_{0h}dt + LOT. \tag{50}$$

Finally classical arguments of compactness-uniqueness allow us to get rid of the lower order term.

In the following we give the details of the proofs of the above steps.

Step I. Upper bounds of $\mathcal{E}_h(\Pi_\gamma^h u_h)$ in terms of $\{\mathcal{E}_h(P_k u_h)\}_{k\geq 0}$.

The condition $1<c<b/a$ imposed in (42) shows that $\bigcup_{k\geq 0}(ac^k, bc^k)=(a,\infty)$. This means that any frequency $\omega_{\mathbf{j}}(h)\geq a$ occurs in at least one of the projections $P_k u_h$, $k\geq 0$.

Let us choose a positive integer k_h such that

$$c^{k_h}(a+\mu)\leq \gamma/h < c^{k_h+1}(a+\mu). \tag{51}$$

Also let us fix a positive integer $k_0\leq k_h$ independent of h. Its precise value will be chosen later on in the proof. The choice of k_h is always possible for small parameter h. Using that $c<(b-\mu)/(a+\mu)$ (see (42)) we obtain that the following inequality holds

$$c^{k_h}(a+\mu)\leq\gamma/h\leq c^{k_h+1}(a+\mu)\leq c^{k_h}(b-\mu).$$

Then any frequency $\omega_{\mathbf{j}}(h)$ belonging to $[(a+\mu)c^{k_0},\gamma/h]$ is contained in at least one interval of the form $[c^k(a+\mu), c^k(b-\mu)]$ with $k_0\leq k\leq k_h$ where the function $F(c^{-k}\cdot)$ is identically one. Then any frequency $\omega_{\mathbf{j}}(h)\in[(a+\mu)c^{k_0},\gamma/h]$ we get

$$1\leq\sum_{k=k_0}^{k_h}F(c^{-k}\omega_{\mathbf{j}}(h))^2. \tag{52}$$

In view of (45) and (52) the energy of $\Pi_\gamma^h u_h$ excepting a low order term, can be bounded from above by the energy of all the projections $(P_k u_h)_{k=k_0}^{k_h}$:

$$\begin{aligned}\mathcal{E}_h(\Pi_\gamma^h u_h) &\leq c^{2k_0}(a+\mu)^2\sum_{\omega_{\mathbf{j}}(h)<(a+\mu)c^{k_0}}\left(|\widehat{u}_{\mathbf{j}+}^h|^2+|\widehat{u}_{\mathbf{j}-}^h|^2\right)\\ &\quad+\sum_{k=k_0}^{k_h}\sum_{\mathbf{j}\in\mathbb{Z}^2}F^2(c^{-k}\omega_{\mathbf{j}}(h))\omega_{\mathbf{j}}^2(h)\left(|\widehat{u}_{\mathbf{j}+}^h|^2+|\widehat{u}_{\mathbf{j}-}^h|^2\right)\\ &= C(a,k_0,\mu)\sum_{\omega_{\mathbf{j}}(h)<(a+\mu)c^{k_0}}\left(|\widehat{u}_{\mathbf{j}+}^h|^2+|\widehat{u}_{\mathbf{j}-}^h|^2\right)+\sum_{k=k_0}^{k_h}\mathcal{E}_h(P_k u_h).\end{aligned} \tag{53}$$

Step II. Observability inequalities for the projections $(P_k u_h)_{k\geq k_0}^{k_h}$**.**
The next step is to apply the observability inequality (41) to each projection $P_k u_h$, $k_0 \leq k \leq k_h$. To do that we have to prove that each of them belongs to the class $I_h(\gamma+\epsilon)$ where (41) holds. We remark that the projector $P_k u_h(t)$ contains only the frequencies $\omega_{\mathbf{j}}(h) \in (c^k a, c^k b)$. For any given $k \leq k_h$ any frequency $\omega_{\mathbf{j}}(h)$ involved in the decomposition (44) of $P_k u_h$, in view of (51), satisfies

$$\omega_{\mathbf{j}}(h) \leq c^{k_h} b < \frac{\gamma b}{h(a+\mu)} < \frac{\gamma+\epsilon}{h},$$

which shows that $P_k u_h(t) \in I_h(\gamma+\epsilon)$.

Step III. The final step. Now we apply inequality (41) to each projection $P_k u_h$:

$$\mathcal{E}_h(P_k u_h) \leq C(T,\delta,\epsilon,\gamma) \int_{2\delta}^{T-2\delta} \int_{\Gamma_{0h}} |\partial_n^h (P_k u_h)|^2 d\Gamma_{0h} dt, \ \ k_0 \leq k \leq k_h. \tag{54}$$

Using (53) and the above inequalities we obtain that

$$\begin{aligned} \mathcal{E}_h(\Pi_\gamma^h u_h) \ &\leq \ C(T,\gamma,\delta,\epsilon) \sum_{k=k_0}^{k_h} \int_{2\delta}^{T-2\delta} \int_{\Gamma_{0h}} |\partial_n^h (P_k u_h)|^2 d\Gamma_{0h} dt \\ &\quad + C(k_0,a,\mu) \sum_{\omega_{\mathbf{j}}(h)<(a+\mu)c^{k_0}} \left[|\widehat{u}_{\mathbf{j}+}^h|^2 + |\widehat{u}_{\mathbf{j}-}^h|^2 \right]. \end{aligned} \tag{55}$$

It remains to prove (48). Once this inequality holds then (49) and (50) hold as well, which finishes the proof.

The key point is the following Lemma which will be proved in Appendix A.

Lemma 6.1. *Let μ be a Borel measure and Ω a μ-measurable set such that $\mu(\Omega) < \infty$. We set $X = L^p(\Omega, d\mu)$, $1 \leq p \leq \infty$. Then for any positive δ and T there is a constant $C(\delta,T)$ such that the following holds*

$$\sum_{k\geq k_0} \int_{2\delta}^{T-2\delta} \|P_k a\|_X^2 dt \leq 2\int_0^T \|a\|_X^2 dt + \frac{C(\delta,T)}{c^{2k_0}} \sup_{l\in\mathbb{Z}} \|a\|_{L^2((lT,(l+1)T),X)} \tag{56}$$

for all positive integer k_0 and $a \in L^2_{loc}(\mathbb{R}, X)$.

We now apply Lemma 6.1 with $a = \partial_n^h u$ and $X = l^2(\Gamma_{0h})$. Using that $P_k(\partial_n^h u) = \partial_n^h (P_k u)$, we obtain the existence of a constant $C(\delta,T)$ such that

$$\begin{aligned} \sum_{k\geq k_0} \int_{2\delta}^{T-2\delta} \int_{\Gamma_{0h}} |\partial_n^h P_k u(t)|^2 d\Gamma_{0h} dt \ &\leq \ 2\int_0^T \int_{\Gamma_{0h}} |\partial_n^h u(t)|^2 dt \\ &\quad + C(\delta,T) \sup_{l\in\mathbb{Z}} \int_{lT}^{(l+1)T} \int_{\Gamma_{0h}} |\partial_n^h u(t)|^2 d\Gamma_{0h} dt. \end{aligned}$$

At this point we apply the so-called *"direct inequality"* (17), which holds for all solutions u of (4). Thus, a translation in time in (17) together with the conservation of energy shows that

$$\sup_{l\in\mathbb{Z}} \int_{lT}^{(l+1)T} \int_{\Gamma_{0h}} |\partial_n^h u(t)|^2 d\Gamma_{0h} dt \leq C(T)\mathcal{E}_h(u). \tag{57}$$

and then (48).

$\square$

Appendix A. Proof of Lemma 6.1

In this Appendix we prove Lemma 6.1. The main ingredient is the following lemma inspired in ideas of [12], [2] and adapted to our context. In the sequel X denote the space $L^p(\Omega, d\mu)$, where μ is a Borel measure and $\mu(\Omega) < \infty$. In our problem $X = l^2(\Gamma_{0h})$. The following lemmas can be applied also for internal observability, for instance by choosing $X = l^2(O_h)$, O_h being the internal observability zone.

Lemma A.1. *Let $\varphi \in C_0^\infty(0,T)$, and $\psi \in L^\infty(\mathbb{R})$ be such that $\psi \equiv 1$ on $(0,T)$ and $(P_k)_{k\geq 0}$ be defined as above. There exists a constant $C = C(T, \varphi, \psi, F)$ such that*

$$\int_{\mathbb{R}} \|\varphi(t)P_k(a)(t)\|_X^2 dt \leq 2\int_{\mathbb{R}} \|\varphi(t)P_k(\psi a)(t)\|_X^2 dt + Cc^{-2k} \sup_{l\in\mathbb{Z}} \|a\|^2_{L^2((lT,(l+1)T),\,X)} \tag{58}$$

holds for all $a \in L^2_{loc}(\mathbb{R},\, X)$ and for all $k \geq 0$.

Proof of Lemma A.1. We denote $I_l = [lT, (l+1)T)$ and $a_l = 1_{I_l}a$. We claim the existence of a positive constant $C = C(P)$ such that for all $\varphi \in C_0^\infty(\mathbb{R})$ and $l \in \mathbb{Z}$ with $\mathrm{dist}(\mathrm{I_l}, \mathrm{supp}(\varphi)) \geq \delta > 0$ the following holds:

$$\sup_{t\in\mathbb{R}} \|\varphi(t)P_k(a_l)\|_X \leq Cc^{-k}\delta^{-2}T^{1/2}\|\varphi\|_{L^\infty(\mathbb{R})} \sup_{l\in\mathbb{Z}} \|a_l\|_{L^2(\mathbb{R},X)}, \tag{59}$$

uniformly in $h > 0$.

Using estimate (59) we will prove the existence of a positive constant $C = C(T, \varphi, \psi, P)$ such that

$$\sup_{t\in\mathbb{R}} \|\varphi(t)(P_k(a) - P_k(\psi a))(t)\|_X \leq Cc^{-k} \sup_{l\in\mathbb{Z}} \|a_l\|_{L^2(\mathbb{R},\,X)}. \tag{60}$$

Then, (58) will be a consequence of Cauchy's inequality:

$$\begin{aligned}\int_{\mathbb{R}} \|\varphi(t)P_k(a)(t)\|_X^2 dt &\leq 2\int_{\mathbb{R}} \|\varphi(t)P_k(\psi a)(t)\|_X^2 dt + 2\int_{\mathbb{R}} \|\varphi(t)P_k(a-\psi a)(t)\|_X^2 dt\\ &\leq 2\int_{\mathbb{R}} \|\varphi(t)P_k(\psi a)(t)\|_X^2 dt + 2T\sup_{t\in\mathbb{R}} \|\varphi(t)(P_k(a-\psi a))(t)\|_X^2\\ &\leq 2\int_{\mathbb{R}} \|\varphi(t)P_k(\psi a)(t)\|_X^2 dt + Cc^{-k}\sup_{l\in\mathbb{Z}} \|a_l\|^2_{L^2(\mathbb{R},\,X)}.\end{aligned}$$

In the following we prove (60). Observe that on I_0, $a \equiv a\psi$. This yields to the following decomposition of the difference $P_k(a) - P_k(\psi a)$:

$$P_k(a) - P_k(\psi a) = \sum_{|l|\geq 1} P_k(a_l - (\psi a)_l) = \sum_{|l|\geq 1} P_k(b_l), \tag{61}$$

with $b_l = a_l - (\psi a)_l$. Let us choose an $\delta > 0$ such that φ is supported on $(\delta, T-\delta)$. Thus for all $|l| \geq 2$, the function b_l satisfies $\mathrm{dist}(\mathrm{supp}(\varphi), I_l) \geq T(|l|-1)$. Also, for $|l| = 1$: $\mathrm{dist}(\mathrm{supp}(\varphi), I_l) \geq \delta$. By (59) we obtain the existence of a constant $C = C(T, \varphi, \psi, P)$ such that

$$\sup_{t\in\mathbb{R}} \|\varphi(t)P_k(b_l)(t)\|_X \leq Cc^{-k}\sup_{l\in\mathbb{Z}} \|b_l\|_{L^2(\mathbb{R},\,X)} \begin{cases} \frac{1}{(|l|-1)^2}, & |l| \geq 2,\\[2ex] \frac{1}{\delta^2}, & |l| = 1.\end{cases} \tag{62}$$

Finally, (61) and (62) give for any $t \in \mathbb{R}$

$$\|\varphi(t)[P_k(a) - P_k(\psi a)]\|_X \le \sum_{|l|\ge 1} \|\varphi(t) P_k(b_l)\|_X \le C(\delta) c^{-k} \sup_{l\in\mathbb{Z}} \|b_l\|_{L^2(\mathbb{R},X)}$$
$$\le C(\delta) c^{-k} \sup_{l\in\mathbb{Z}} \|a\|_{L^2(\mathbb{R},X)}.$$

We now prove estimate (59). The definition of the projector P_k and integration by parts give us

$$\begin{aligned} \varphi(t) P_k(a_l)(t) &= \int_{\mathbb{R}_\tau}\int_{\mathbb{R}_s} e^{i\tau(t-s)} P(c^{-k}\tau)\varphi(t) a_l(s) ds d\tau \\ &= \int_{\mathbb{R}_\tau}\int_{\mathbb{R}_s} e^{i\tau(t-s)} i^2 \partial_\tau^2 [P(c^{-k})\tau] \frac{\varphi(t) a_l(s)}{(t-s)^2} ds d\tau. \end{aligned}$$

Thus, for any t in the support of φ, Minkowsky's inequality yelds

$$\begin{aligned} \|\varphi(t) P_k(a_l)(t)\|_X &\le c^{-2k} \|\varphi\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}_\tau} |(\partial_\tau^2 P)(c^{-k}\tau)| d\tau \int_{I_l} \frac{\|a_l(s)\|_X}{(t-s)^2} ds \\ &\le c^{-k}\delta^{-2} \|\varphi\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}_\tau} |(\partial_\tau^2 P)(\tau)| d\tau \int_{I_l} \|a_l(s)\|_X ds. \end{aligned}$$

Applying Cauchy's inequality in time we get

$$\|\varphi(t) P_k(a_l)(t)\|_X^2 \le \left(c^{-k}\delta^{-2}\|\varphi\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}_\tau} |(\partial_\tau^2 P)(\tau)| d\tau\right)^2 T \int_{I_l} \|a_l(s)\|_X^2 ds \tag{63}$$

which finishes the proof.

□

Proof of Lemma 6.1. Let us choose a function $\varphi \in C_0^\infty(0,T)$ such that $\varphi \equiv 1$ on $[2\delta, T-2\delta]$. Applying Lemma A.1 to the function u and $\psi = \mathbf{1}_{(0,T)}$, we obtain

$$\int_{2\delta}^{T-2\delta} \|P_k u\|_X^2 dt \le \int_{\mathbb{R}} \varphi^2 \|P_k(u)\|_X^2 dt$$
$$\le 2\int_{\mathbb{R}} \varphi^2 \|P_k(\psi u)\|_X^2 dt + C c^{-2k} \sup_{l\in\mathbb{Z}} \|u\|_{L^2((lT,(l+1)T,X))}.$$

Summing all these inequalities we get

$$\sum_{k\ge k_0} \int_{2\delta}^{T-2\delta} \|P_k u\|_X^2 dt \le 2\sum_{k\ge k_0} \int_{\mathbb{R}} \varphi^2 \|P_k(\psi u)\|_X^2 dt + \frac{C(\delta,T)}{c^{2k_0}} \sup_{l\in\mathbb{Z}} \|u\|_{L^2((lT,(l+1)T),X)}. \tag{64}$$

In the following we prove that

$$\sum_{k\ge 0} \int_{\mathbb{R}} \varphi^2 \|P_k(\psi u)\|_X^2 dt \le \int_0^T \|u(t)\|_X dt.$$

Observe that any real number τ belongs either to a finite number of intervals of the form $(\pm ac^k, \pm bc^k)$ or to none of them. Then there is a positive constant C such that

$$\sup_{\tau\in\mathbb{R}} \sum_{k\ge 0} P^2(c^{-k}\tau) \le C. \tag{65}$$

Applying Plancherel's identity in time variable we obtain

$$\begin{aligned}\sum_{k\geq 0}\int_{\mathbb{R}}\varphi^2(t)\|P_k(\psi u)(t)\|_X^2 dt &\leq \|\varphi\|^2_{L^\infty(\mathbb{R})}\sum_{k\geq 0}\int_{\mathbb{R}}\|P_k(\psi u)(t)\|_X^2 dt\\
&= \|\varphi\|^2_{L^\infty(\mathbb{R})}\sum_{k\geq 0}\int_{\mathbb{R}}P^2(c^{-k}\tau)\|\widehat{\psi u}(\tau)\|_X^2 d\tau\\
&\leq \|\varphi\|^2_{L^\infty(\mathbb{R})}\sup_{\tau\in\mathbb{R}}\sum_{k\geq 0}P^2(c^{-k}\tau)\int_{\mathbb{R}}\|\widehat{\psi u}(\tau)\|_X^2 d\tau\\
&\leq C\|\varphi\|^2_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\|(\psi u)(t)\|_X^2 dt = C\|\varphi\|^2_{L^\infty(\mathbb{R})}\int_0^T\|u(t)\|_X^2 dt.\end{aligned}$$

□

Appendix B. Spectral analysis of V^h-functions

Let M be a positive integer, $N = 4M-1$ and $h = 1/(N+1)$. Let us consider a function $v \in V^h$ and its projection $\Upsilon^h_{1/4}u$. For each positive s the norm of the projection satisfies $\|\Upsilon^h_{1/4}v\|_{s,h} \leq \|v\|_{s,h}$. Lemma 3.1 shows that the converse inequality

$$\|v\|_{s,h} \leq C\|\Upsilon^h_{1/4}v\|_{s,h} \tag{66}$$

also holds for all $0 \leq s \leq 2$ and v in the space V^h.

We first obtain in the following Lemma a description of the Fourier coefficients $\widehat{v}(\mathbf{k})$ of a function $v \in V^h$ and then prove Lemma 3.1.

Lemma B.1. *Let $v \in V^h$. Then for any $\mathbf{k} = (k_1, k_2) \in \Lambda_{4M-1}$ the $\mathbf{k}$-th Fourier coefficient satisfies*

$$\widehat{v}(\mathbf{k}) = \frac{64}{(4M-1)^2}\prod_{l=1}^{2}\cos^2\left(\frac{k_l\pi h}{2}\right)\cos^2(k_l\pi h)\sum_{\mathbf{j}\in\Lambda_{M-1}} v_{4\mathbf{j}}\varphi^{\mathbf{k}}_{4\mathbf{j}}. \tag{67}$$

Proof. Firstly we analyze the one-dimensional case. The result extends to the 2d-case by iterating the same argument in each direction.

For each $1 \leq k \leq 4M-1$ the coefficient $\widehat{v}(k)$ is given by

$$\widehat{v}(k) = \frac{2}{(4M-1)}\sum_{j=1}^{4M-1} v_j \sin(kj\pi h).$$

Using that the function v satisfies $v_{2j+1} = (v_{2j} + v_{2j+2})/2$, $j = 0, ..., 2M-1$, we obtain

$$\begin{aligned}\sum_{j=1}^{4M-1} v_j\sin(kj\pi h) &= \sum_{j=1}^{2M-1} v_{2j}\sin(2kj\pi h) + \sum_{j=0}^{2M-1} v_{2j+1}\sin((2j+1)k\pi h)\\
&= 2\cos^2\left(\frac{k\pi h}{2}\right)\sum_{j=1}^{2M-1} v_{2j}\sin(2kj\pi h).\end{aligned}$$

In a similar way, taking into account that $v_{4j+2} = (v_{4j} + v_{4j+4})/2$ we also obtain:

$$\sum_{j=1}^{2M-1} v_{2j} \sin(2kj\pi h) \quad = \quad 2\cos^2(k\pi h) \sum_{j=1}^{M-1} v_{4j} \sin(4kj\pi h).$$

These identities prove (67) in the one-dimensional case. Applying the same argument in each space direction we obtain (67) in the two-dimensional case. □

Proof of Lemma 3.1. Let us choose an integer M such that $N+1=4M$. We first analyze the one dimensional case and apply it to the 2d case.

Step I. The 1-d case. In the sequel we denote $\lambda_k(h) = 4/h^2 \sin^2(k\pi h/2)$. By Lemma B.1 for any $k = 1, ..., 4M-1$, the k-th Fourier coefficient $\widehat{v}(k)$ is given by $\widehat{v}(k) = a(k)g(k)$ where

$$a(k) = 4\cos^2\left(\frac{k\pi h}{2}\right)\cos^2(k\pi h)\,,\; g(k) = \frac{2}{4M-1}\sum_{j=0}^{M-1} v_{4j}\sin(4kj\pi h).$$

A explicit computation shows that for any $k = 1, ..., 4M-1$ the following holds

$$\begin{aligned} a(k)\lambda_k(h) &= a(2M+k)\lambda_{2M+k}(h) = a(2M-k)\lambda_{2M-k}(h) = a(4M-k)\lambda_{4M-k}(h) \\ &= \frac{1}{h^2}\sin^2(2k\pi h) \end{aligned}$$

and $g(k) = g(2M+k) = -g(2M-k) = -g(4M-k)$.

We point out that for any $k = 1, ..., M-1$ and $j = M+1, ..., 4M-1$ the following holds:

$$\lambda_k(h) \leq \frac{4}{h^2}\sin^2\left(\frac{\pi}{8}\right) \leq \lambda_j(h).$$

Also for any $s \in [0,2]$, $k = 1, ..., M-1$ and $j \in \{2M-k, 2M+k, 4M-k\}$ we get

$$\begin{aligned} a^2(k)\lambda_k^s(h) &= \lambda_k^{s-2}(h)a^2(k)\lambda_k^2(h) = \lambda_k^{s-2}(h)a^2(j)\lambda_j^2(h) \\ &= (\lambda_j(h)/\lambda_k(h))^{2-s}a^2(j)\lambda_j^s(h) \geq a^2(j)\lambda_j^s(h). \end{aligned}$$

Using all these estimates, the $\hbar^s$-norm of v satisfies:

$$\begin{aligned} \|v\|_{s,h}^2 &= \sum_{k=1}^{M-1} g^2(k)\left[a^2(k)\lambda_k^s(h) + a^2(2M-k)\lambda_{2M-k}^s(h)\right. \\ &\qquad\qquad \left. + a^2(2M+k)\lambda_{2M+k}^s(h) + a^2(4M-k)\lambda_{4M-k}^s(h)\right] \\ &\leq 4\sum_{k=1}^{M-1} g^2(k)a^2(k)\lambda_k^s(h) = 4\|\Upsilon_{1/4}^h v\|_{s,h}^2. \end{aligned}$$

which finishes the proof of the 1-d case.

Step II. The 2-d case. We reduce this case to the previous one. The function v admits a representation in the Fourier space as:

$$v(x,y) = \sum_{j,k=1}^{4M-1} a_{jk}\varphi^j(x)\varphi^k(y),\;\; x = j_1 h,\;\; y = k_1 h,\;\; j_1, k_1 = 1, \ldots, 4M-1,$$

where $\varphi^j(x) = \sin(j\pi x)$, $j = 1, \ldots, 4M-1$.

Observe that for each x fixed, the function $v(x,\cdot)$ is obtained by a one-dimensional interpolation of the two-grid type. Then,

$$\|v(x,\cdot)\|_{s,h}^2 \leq 4\|\Upsilon_{1/4}^h v(x,\cdot)\|_{s,h}^2. \tag{68}$$

A similar argument guarantees that $\|v(\cdot,y)\|_{s,h}^2 \leq 4\|\Upsilon_{1/4}^h v(\cdot,y)\|_{s,h}^2$. Taking into account that

$$\lambda_{jk}^s(h) = (\lambda_j(h) + \lambda_k(h))^s \leq 2^{s-1}(\lambda_j^s(h) + \lambda_k^s(h)),$$

we obtain that

$$\|v\|_{s,h} \leq 2^{(s+1)/2}\|\Upsilon_{1/4}^h v\|_{s,h}.$$

□

References

1. M. Asch and G. Lebeau, *Geometrical aspects of exact boundary controllability for the wave equation–a numerical study*, ESAIM Control Optim. Calc. Var. **3** (1998), 163–212.
2. N. Burq, *Contrôlabilité exacte des ondes dans des ouverts peu réguliers*, Asymptot. Anal. **14** (1997), no. 2, 157–191.
3. N. Burq, P. Gérard, and N. Tzvetkov, *Strichartz inequalities and the nonlinear Schrödinger equation on compact manifolds*, Amer. J. Math. **126** (2004), no. 3, 569–605.
4. C. Castro and S. Micu, *Boundary controllability of a linear semi-discrete 1-D wave equation derived from a mixed finite element method*, Numer. Math. **102** (2006), no. 3, 413–462.
5. C. Castro, S. Micu, and A. Münch, *Boundary controllability of a semi-discrete 2-D wave equation with mixed finite elements*, Preprint.
6. R. Glowinski, *Ensuring well-posedness by analogy; Stokes problem and boundary control for the wave equation.*, J. Comput. Phys. **103** (1992), no. 2, 189–221.
7. R. Glowinski, W. Kinton, and M. F. Wheeler, *A mixed finite element formulation for the boundary controllability of the wave equation*, Internat. J. Numer. Methods Engrg. **27** (1989), no. 3, 623–635.
8. R. Glowinski, C. H. Li, and J.-L. Lions, *A numerical approach to the exact boundary controllability of the wave equation. I. Dirichlet controls: description of the numerical methods*, Japan J. Appl. Math. **7** (1990), no. 1, 1–76.
9. R. Glowinski and J.-L. Lions, *Exact and approximate controllability for distributed parameter systems*, Acta numerica, 1995, Cambridge Univ. Press, Cambridge, 1995, pp. 159–333.
10. L.I. Ignat and E. Zuazua, *A two-grid approximation scheme for nonlinear Schrödinger equations: dispersive properties and convergence.*, C. R. Acad. Sci. Paris, Ser. I **341** (2005), no. 6, 381–386.
11. J.A. Infante and E. Zuazua, *Boundary observability for the space semi-discretizations of the 1-D wave equation.*, M2AN Math. Model. Numer. Anal. **33** (1999), no. 2, 407–438.
12. G. Lebeau, *Contrôle de l'équation de Schrödinger*, J. Math. Pures Appl. **71** (1992), no. 3, 267–291.
13. ______, *The wave equation with oscillating density: observability at low frequency*, ESAIM Control Optim. Calc. Var. **5** (2000), 219–258.
14. J.-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, Recherches en Mathématiques Appliquées [Research in Applied Mathematics], vol. 8, Masson, Paris, 1988.
15. J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications. Vol. 1-2*, Travaux et Recherches Mathématiques, No. 17, Dunod, Paris, 1968.
16. P. Loreti and M. Mehrenberger, *An Ingham type proof for a two-grid observability theorem*, ESAIM:COCV, to appear (2005).
17. S. Micu, *Uniform boundary controllability of a semi-discrete 1-D wave equation*, Numer. Math. **91** (2002), no. 4, 723–768.
18. M. Negreanu and E. Zuazua, *Convergence of a multigrid method for the controllability of a 1-D wave equation.*, C. R. Math. Acad. Sci. Paris **338** (2004), no. 5, 413–418.
19. L.N. Trefethen, *Group velocity in finite difference schemes*, SIAM Rev. **24** (1982), no. 2, 113–136.
20. E. Zuazua, *Boundary observability for the finite-difference space semi-discretizations of the 2-D wave equation in the square*, J. Math. Pures Appl. **78** (1999), no. 5, 523–563.

21. ———, *Propagation, Observation, Control and Numerical Approximation of Waves approximated by finite difference methods.*, SIAM Review **2** (2005), no. 47, 197–243.

L. I. Ignat
Departamento de Matemáticas,
U. Autónoma de Madrid,
28049 Madrid, Spain
and
Institute of Mathematics of the Romanian Academy,
P.O.Box 1-764, RO-014700 Bucharest, Romania.
E-mail address: liviu.ignat@uam.es

E. Zuazua
Departamento de Matemáticas,
U. Autónoma de Madrid,
28049 Madrid, Spain.
E-mail address: enrique.zuazua@uam.es
Web page: http://www.uam.es/enrique.zuazua