Global Strichartz estimates for approximations of the Schrödinger equation

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Abstract

We consider a semidiscrete scheme for the linear Schrödinger equation with high order dissipative term. We obtain maximum norm estimates for its solutions and we prove global Strichartz estimates for the considered model, estimates that are uniform with respect to the mesh size. The methods we employ are based on classical arguments of harmonic analysis.

Keywords: Finite differences, Schrödinger equations, Strichartz estimates.

1 Introduction

Let us consider the linear Schrödinger equation in the whole space:

$$iu_t + \Delta u = 0. \tag{1}$$

This equation has two important properties, the conservation of energy

$$||u(t)||_{L^2(\mathbb{R}^d)} = ||u(0)||_{L^2(\mathbb{R}^d)}$$

and a dispersive property:

$$\|u(t)\|_{L^{\infty}(\mathbb{R}^d)} \le \frac{c(d)}{|t|^{d/2}} \|u(0)\|_{L^1(\mathbb{R}^d)}, \ t \ne 0.$$
(2)

These properties have been employed to develop well-posedness results for homogenous and nonlinear Schrödinger equations [16, 5, 18]. The main idea of these works is to obtain space-time estimates for the solutions of the linear Schrödinger equation, called Strichartz estimates after the pioneering work of Strichartz [16]:

$$\|u\|_{L^{q}(\mathbb{R}, L^{r}(\mathbb{R}^{d}))} \leq c(d, q, r)\|u(0)\|_{L^{2}(\mathbb{R}^{d})},$$
(3)

where (q, r) are the so-called d/2-admisible pairs:

$$\frac{1}{q} = \frac{d}{2} \left(\frac{1}{2} - \frac{1}{r} \right).$$

In [8] trying to introduce a numerical scheme for the nonlinear Schrödinger equation with low regular initial data, the authors prove the lack of uniform dispersive properties of type (2) or (3) for the solutions of the simplest approximation of the linear Schrödinger equation:

$$iu_t + \Delta_h u = 0, \tag{4}$$

where uniformity is with respect to the mesh size. Above, Δ_h is the second order approximation by finite differences of the Laplace operator Δ :

$$(\Delta_h u)_j = \frac{1}{h^2} \sum_{k=1}^d (u_{j+e_k} + u_{j-e_k} - 2u_j), \ j \in \mathbb{Z}^d,$$

 $\{e_k\}_{k=1}^d$ being the canonical basis in \mathbb{R}^d .

To be more precise, along this paper we will consider the spaces $l^p(h\mathbb{Z}^d)$ of sequences $\{\varphi_j\}_{j\in\mathbb{Z}^d}$ endowed with the norms

$$\|\varphi\|_{l^p(h\mathbb{Z}^d)} = \begin{cases} \left(h^d \sum_{j \in \mathbb{Z}^d} |\varphi_j|^p\right)^{1/p}, & 1 \le p < \infty, \\ \sup_{j \in \mathbb{Z}^d} |\varphi_j|, & p = \infty. \end{cases}$$

In dimension one, the lack of a uniform estimate of type (2) is due to the fact that the symbol $p_h(\xi) = 4/h^2 \sin(\xi h/2)$ of the operator $-\Delta_h$ changes the convexity at the points $\pm \pi/2h$, a property that the continuous one ξ^2 , does not satisfy. Observing this pathology, in [8] the following estimate for the solutions of scheme (4) is proved:

$$\|u(t)\|_{l^{\infty}(h\mathbb{Z}^d)} \le c(d) \Big(\frac{1}{|t|^{1/2}} + \frac{1}{|th|^{1/3}}\Big) \|u(0)\|_{l^1(h\mathbb{Z}^d)},$$

estimate that is not uniform on the mesh parameter h. This does not allow to prove uniform Strichartz-like estimates for the above semi-discretization. A similar result can be stated in dimension d in terms of the rank of the Hessian matrix $H_{p_h}(\xi)$, where p_h is the symbol of the discrete operator $-\Delta_h$:

$$p_h(\xi) = \frac{4}{h^2} \sum_{k=1}^d \sin^2\left(\frac{\xi_k h}{2}\right), \, \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d.$$

We mention that the Schrödinger equation on the lattice $h\mathbb{Z}^d$, without concern for the uniformity of the estimates with respect to the size of the lattice, has been also studied in [13]. The analysis of dispersive properties for fully discrete models is analyzed in [11] for the KdV equation and in [6] for the Schrödinger equation.

For numerical purposes, to avoid the lack of uniformness of the dispersive properties, in [7] the following viscous scheme is introduced:

$$iu_t + \Delta_h u = i\operatorname{sgn}(t)a(h)\Delta_h u,\tag{5}$$

where a(h) goes to zero as h goes to zero such that $\inf_{h>0} a(h)/h^{2-d/\alpha} > 0$ for some parameter $\alpha > d/2$. The authors have thus obtained that the solutions of (5) satisfy

$$\|u(t)\|_{l^{\infty}(h\mathbb{Z}^{d})} \leq c(d)(|t|^{-d/2} + |t|^{-\alpha})\|u(0)\|_{l^{1}(h\mathbb{Z}^{d})}$$

Observe that the behavior at $t \sim 0$ and $t \sim \infty$ is different. Thus, the estimates of the type (3) obtained in [7] for the solutions of scheme (5) are not global. More precisely, for any T > 0, the authors prove that the solutions of (5) satisfy for any α -admissible pair (q, r):

$$\frac{1}{q} = \alpha \left(\frac{1}{2} - \frac{1}{r}\right),$$

the following estimate:

$$\|u\|_{L^{q}([-T,T], l^{r}(h\mathbb{Z}^{d}))} \leq C(d, T, q, r) \|u(0)\|_{l^{2}(h\mathbb{Z}^{d})}.$$

The global estimates are useful in obtaining the global existence of solutions for the critical nonlinear Schrödinger equation:

$$iu_t + \Delta u = |u|^{4/d}u,$$

with large L^2 -initial data. If one assumes that for such initial datum φ , the norm $\|\exp(it\Delta)\varphi\|_{L^q(\mathbb{R},L^r(\mathbb{R}^d))}$ is small enough, then global existence of solutions is guaranteed by the global Strichartz estimates (see for example [3], Ch. 4.7, p. 119). Examples of $\varphi \in L^2(\mathbb{R}^d)$ satisfying the above condition are given in [10] (Ch. 5, p. 108).

In this paper we introduce a numerical scheme with a high order dissipative term as follows:

$$iu_t + \Delta_h u = -ia(h)(-\Delta_h)^m u, \tag{6}$$

with $m \ge 2$ an integer and $a(h) \to 0$ as $h \to 0$, such that

$$\inf_{h>0} \frac{a(h)}{h^{m-1}} > 0.$$

Observe that the solutions of (6) at time t satisfy

$$u(t) = \exp(it\Delta_h)\exp(-t(-\Delta_h)^m)u(0).$$

In order to derive $l^p(h\mathbb{Z}^d)$ -estimates for the solution u of (6) we need to analyze the action of the operator $\exp(-t(-\Delta_h)^m)$ on the spaces $l^p(h\mathbb{Z}^d)$. Using the results obtained in Section 2 for the operator $\exp(-t(-\Delta_h)^m)$ we prove that the solutions of (6) have uniform decay rates similar to those of the continuous equation (1). As a consequence we obtain Strichartz like estimates for our model similar to those of the continuous one. For further applications of these results for approximations of nonlinear Schrödinger equation we refer to [8].

The article is organized as follows. In Section 2 we obtain $l^p(h\mathbb{Z}^d) - l^q(h\mathbb{Z}^d)$ estimates on the operator $\exp(-t(-\Delta_h)^m)$. Section 3 is devoted to the $l^{p'}(h\mathbb{Z}^d) - l^p(h\mathbb{Z}^d)$, $p \geq 2$, estimates on the solutions of equation (6). Finally in Section 4 we prove global Strichartz estimates for the considered dissipative scheme.

2 Decay rates for the operator $\exp(-t(-\Delta_h)^m)$

Let us consider the following equation:

$$\begin{cases} u_t = -(-\Delta)^m u & \text{in } (0,\infty) \times \mathbb{R}^d, \\ u(0) = \varphi & \text{in } \mathbb{R}^d, \end{cases}$$
(7)

where m > 0. It is well known that, as long as the Fourier transform makes sense, the solution of equation (7) is given in the Fourier variable by

$$\widehat{u}(t,\xi) = \exp(-t|\xi|^m)\widehat{\varphi}(\xi), \, \xi \in \mathbb{R}^d, \, t \ge 0.$$

Classical properties of the Fourier transform guarantee that $u(t) = G_m(t) * \varphi$, where $\widehat{G}_m(t,\xi) = \exp(-t|\xi|^m)$. A scaling argument gives us that for any t > 0 and $x \in \mathbb{R}^d$, the following holds:

$$G_m(t,x) = t^{-d/ms} G_m(1,xt^{-1/m}).$$

Thus for any $p \ge 1$, the $L^p(\mathbb{R}^d)$ -norm of $G_m(t)$ satisfies

$$||G_m(t)||_{L^p(\mathbb{R}^d)} \le c(m, p, d)t^{-\frac{d}{m}(1-\frac{1}{p})}.$$

Using Young's inequality we get for any $1 \le p \le q \le \infty$ and t > 0 that the following holds

$$\|\exp(-t(-\Delta)^m)\varphi\|_{L^q(\mathbb{R}^d)} \le c(m,p,q,d)t^{-\frac{d}{m}(\frac{1}{p}-\frac{1}{q})}\|\varphi\|_{L^p(\mathbb{R}^d)}.$$

We point out that in the case m = 1 the above estimates can be obtained by energy methods.

We consider the following approximation of equation (7):

$$\begin{cases} \frac{du^h}{dt} = -(-\Delta_h)^m u^h, \ t > 0, \\ u^h(0) = \varphi^h, \end{cases}$$
(8)

where we have replaced the Laplace operator Δ by Δ_h . We will prove that the solutions of (8) have similar decay properties as the continuous counterpart and moreover the estimates are uniform with respect to the mesh size h.

The main result concerning the long time behavior of the semidiscrete solution u^h is given by the following theorem. **Theorem 2.1.** Let m be a positive integer and $1 \le p \le q \le \infty$. There exists a positive constant c = c(m, p, q, d) such that

$$\|u^{h}(t)\|_{l^{q}(h\mathbb{Z}^{d})} \leq c\|\varphi^{h}\|_{l^{p}(h\mathbb{Z}^{d})}t^{-\frac{d}{m}(\frac{1}{p}-\frac{1}{q})}$$
(9)

holds for all positive time t, uniformly on h > 0.

As in the continuous case, in the semidiscrete case the $l^p(h\mathbb{Z}^d) - l^q(h\mathbb{Z}^d)$ estimates are reduced to estimates on the fundamental solutions $G_m^h(t)$ of (8). The main difficulty is given by the fact that the new operator $-\Delta_h$ introduces a symbol $p_h(\xi)$ that is not homogenous. In the continuous case this was the key point to establish that the fundamental solution of (7) can be written at any time t in terms of itself at time t = 1, and then the $L^p(\mathbb{R}^d)$ -estimates of the solutions u. Thus one cannot apply the above scaling arguments to obtain $l^p(h\mathbb{Z}^d)$ -estimates on the fundamental solution $G_m^h(t)$.

In the case $2 \leq p \leq \infty$, the $l^p(h\mathbb{Z}^d)$ -norm of $G_m^h(t)$ is easily estimated by interpolating between the cases p = 2 and $p = \infty$. The case p = 2 follows by Plancherel's identity. Also the case $p = \infty$ follows by rough estimates. The main difficulty is to estimate the $l^1(h\mathbb{Z}^d)$ -norm of the discrete kernel $G_m^h(t)$. In the case m = 1 this follows by using the fact that $\exp(t\Delta_h)$ satisfies the maximum principle (see for instance [4]) and the fact that the mass of solutions does not increases as t increase. To estimate the $l^1(h\mathbb{Z}^d)$ norm of G_m^h we will proceed as in [2] (Ch. 3, p. 71), using Carlson-Beurling's inequality (see for instance [2], Ch. 1, Th. 3.1, p. 18):

$$\|f\|_{L^{1}(\mathbb{R}^{d})} \lesssim \|\widehat{f}\|_{L^{2}(\mathbb{R}^{d})}^{1-\frac{d}{2n}} \|\widehat{f}\|_{\dot{H}^{n}(\mathbb{R}^{d})}^{\frac{d}{2n}},\tag{10}$$

inequality that holds for any n > d/2 and for all $\widehat{f} \in H^n(\mathbb{R}^d)$. Observe that both right hand side terms contains the Fourier transform of f and then the $L^1(\mathbb{R}^d)$ -norm of the function f is easily estimated if its Fourier transform is known explicitly.

In what follows, to avoid the presence of constants, we will use the notation $A \leq B$ to report the inequality $A \leq constant \times B$, where the constant is independent of h. The statement $A \simeq B$ is equivalent to $A \leq B$ and $B \leq A$. *Proof of Theorem 2.1.* Using the semidiscrete Fourier transform at scale h(see [17] for the main properties of this transform):

$$\widehat{u}(\xi) = \mathcal{F}_h(u)(\xi) = h^d \sum_{j \in \mathbb{Z}^d} u_j \exp(ij\xi h), \ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d,$$

the solutions of equation (8) are given in the Fourier variable by

$$\widehat{u}^{h}(t,\xi) = \exp(-tp_{h}^{m}(\xi))\widehat{\varphi}^{h}(\xi), \ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^{d}, \ t > 0.$$
(11)

Observe that $u^{h}(t)$ can be written in the convolution form

$$u^{h}(t) = G^{h}_{m}(t) * \varphi^{h}, \ t > 0,$$
 (12)

where * is the discrete convolution on $h\mathbb{Z}^d$:

$$(u * v)_n = h^d \sum_{j \in \mathbb{Z}^d} u_{n-j} v_j, n \in \mathbb{Z}^d$$

and

$$G_m^h(j,t) = \int_{[-\pi/h,\pi/h]^d} \exp(-tp_h^m(\xi)) \exp(ij\xi h) d\xi, \ j \in \mathbb{Z}^d.$$
(13)

In view of Young's inequality it is easy to see that (9) holds if for any h > 0 and $p \ge 1$ the fundamental solution G_m^h satisfies:

$$\|G_m^h(t)\|_{l^p(h\mathbb{Z}^d)} \le c(p,d,m)t^{-\frac{d}{m}(1-\frac{1}{p})}, \ t > 0$$
(14)

for some positive constant c(p, d, m), independent of h.

A scaling argument allows us to deal with the case h = 1:

$$G_m^h(t) = \frac{1}{h^d} G_m^1\left(\frac{t}{h^{2m}}\right)$$

and

$$\|G_m^h(t)\|_{l^p(h\mathbb{Z}^d)} = \frac{1}{h^d} \left\|G_m^1\left(\frac{t}{h^{2m}}\right)\right\|_{l^p(h\mathbb{Z}^d)} = h^{d(\frac{1}{p}-1)} \left\|G_m^1\left(\frac{t}{h^{2m}}\right)\right\|_{l^p(\mathbb{Z}^d)}.$$

The case $p = \infty$ follows by the rough estimate:

$$\begin{split} \|G_m^1(\cdot,t)\|_{l^{\infty}(h\mathbb{Z}^d)} &\leq \int_{[-\pi,\pi]^d} \exp(-tp_1^m(\xi))d\xi \\ &\leq \int_{[-\pi,\pi]^d} \exp(-tc(m)|\xi|^{2m})d\xi \lesssim t^{-\frac{d}{m}}, \end{split}$$

once we observe that

$$p_1(\xi) \ge \frac{4}{\pi^2} |\xi|^2, \forall \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d.$$

In the following we consider the case p = 1:

$$\|G_m^1(t)\|_{l^1(\mathbb{Z}^d)} \le c(d,m), \ t > 0,$$

the other cases 1 , coming by Hölder's inequality.

Let us consider the new function G_m defined by

$$G_m(x,t) = \int_{[-\pi,\pi]^d} \exp(-tp_1^m(\xi)) \exp(ix\xi) d\xi, \ x \in \mathbb{R}^d.$$

In fact this function represents the band-limited interpolator of the sequence $G_m^1(t)$ (cf. [19], Ch. II). The results of Plancherel and Pólya on band-limited function [12] (see also [20], Ch. 2, p. 82, Th. 17) show that the discrete norms of G_m^1 can be controlled by the continuous one of G_m :

$$\|G_m^1(\cdot,t)\|_{l^1(\mathbb{Z}^d)} \lesssim \|G_m(\cdot,t)\|_{L^1(\mathbb{R}^d)}$$

Now, we choose n > d/2 and apply inequality (10) to the function G_m :

$$\|G_m(t)\|_{L^1(\mathbb{R}^d)} \le \|\widehat{G}_m(t)\|_{L^2(\mathbb{R}^d)}^{1-d/2n} \|\widehat{G}_m(t)\|_{\dot{H}^n(\mathbb{R}^d)}^{d/2n}.$$
 (15)

Taking into account that $p_1(\xi) \simeq |\xi|^2$ on $[-\pi, \pi]^d$, by Plancherel's identity we easily estimate the L^2 -norm of $\widehat{G}_m(t)$:

$$\|\widehat{G}_m(t)\|_{L^2(\mathbb{R}^d)}^2 \le \int_{[-\pi,\pi]^d} \exp(-c(m)t|\xi|^{2m})d\xi \lesssim \langle t \rangle^{-\frac{d}{2m}}, \ t > 0,$$

where $\langle \cdot \rangle$ is the Japanese bracket $\langle \cdot \rangle = t + 1$. In view of inequality (15) it remains to prove that

$$\|\widehat{G}_m(t)\|_{\dot{H}^n(\mathbb{R}^d)} \le \langle t \rangle^{\frac{n}{2m} - \frac{d}{4m}}$$
(16)

holds for all positive time t.

By symmetry it is sufficient to prove that

$$\|\partial_{\xi_1}^n \exp(-tp_1^m)\|_{L^2([-\pi,\pi]^d)} \lesssim \langle t \rangle^{\frac{n}{2m} - \frac{d}{4m}}.$$
 (17)

For an integer $n \ge 1$, we recall the following identity

$$\partial_{\xi_1}^n(\exp(g)) = \exp(g) \sum_{\alpha_1 + 2\alpha_2 + \dots + n\alpha_n = n} a_{\alpha_1,\dots,\alpha_n} (\partial_{\xi_1}^1 g)^{\alpha_1} (\partial_{\xi_1}^2 g)^{\alpha_2} \dots (\partial_{\xi_1}^n g)^{\alpha_n}$$

where $a_{\alpha_1,\ldots,\alpha_n}$ are constants independent of g.

Applying the above identity to the function $g = -tp_1^m(\xi)$ we obtain

$$\begin{aligned} \partial_{\xi_1}^n(\exp(-tp_1^m))(\xi) &| \lesssim \\ &\lesssim \exp(-tp_1^m(\xi)) \sum_{\alpha_1+2\alpha_2+\ldots+n\alpha_n=n} t^{\alpha_1+\ldots+\alpha_n} \prod_{j=1}^n |\partial_{\xi_1}^j(p_1^m)(\xi)|^{\alpha_j}. \end{aligned}$$

Using that for any $\xi \in [-\pi,\pi]^d$ the function p_1 satisfies

$$|\partial_{\xi_1}^j p_1(\xi)| \lesssim \min\{1, |\xi|^{2m-j}\},$$

we obtain by Cauchy's inequality that the following holds for all $\xi \in [-\pi, \pi]^d$:

$$\begin{aligned} |\partial_{\xi_1}^n(\exp(-tp_1^m))(\xi)|^2 &\lesssim \\ &\lesssim \exp(-2tp_1^m(\xi)) \sum_{\alpha_1+2\alpha_2+\ldots+n\alpha_n=n} t^{2(\alpha_1+\ldots+\alpha_n)} \prod_{j=1}^{\min\{2m,n\}} |\xi|^{2(2m-j)\alpha_j}. \end{aligned}$$
(18)

For any 0 < t < 1 we obviously have

$$\int_{[-\pi,\pi]^d} |\partial_{\xi_1}^n(\exp(-tp_1^m))(\xi)|^2 d\xi \lesssim 1.$$

It remains to prove that for all $t \ge 1$ the following holds

$$\int_{[-\pi,\pi]^d} |\partial_{\xi_1}^n(\exp(-tp_1^m))(\xi)|^2 d\xi \lesssim t^{\frac{n}{m} - \frac{d}{2m}}.$$
(19)

Integrating inequality (18) on $[-\pi,\pi]^d$ and using that $p_1(\xi) \simeq |\xi|^2$ on this interval we get

$$\int_{[-\pi,\pi]^d} |\partial_{\xi_1}^n(\exp(-tp_1^m))(\xi)|^2 d\xi \lesssim \sum_{\alpha_1+2\alpha_2+\ldots+n\alpha_n=n} t^{2(\alpha_1+\ldots+\alpha_n)} \times \\
\times \int_{[-\pi,\pi]^d} e^{-c(m)t|\xi|^{2m}} \prod_{j=1}^{\min\{2m,n\}} |\xi|^{2(2m-j)\alpha_j} d\xi.$$

We now use that for any s and m positive the following holds:

$$\int_{\mathbb{R}^d} \exp(-t|\xi|^{2m}) |\xi|^s d\xi \lesssim t^{-\frac{d}{2m} - \frac{s}{2m}}$$

This implies that

$$\int_{[-\pi,\pi]^d} |\partial_{\xi_1}^n(\exp(-tp_1^m))(\xi)|^2 d\xi \lesssim t^{-\frac{d}{2m}} \sum_{\alpha_1+2\alpha_2+\ldots+n\alpha_n=n} t^{2p(\alpha_1,\ldots,\alpha_n)}$$

where

$$p(\alpha_1,\ldots,\alpha_n) = \alpha_1 + \cdots + \alpha_n - \frac{1}{2m} \sum_{j=1}^{\min\{2m,n\}} (2m-j)\alpha_j.$$

In order to prove (19) it is sufficient to show that

$$p(\alpha_1,\ldots,\alpha_n) \le \frac{n}{2m}$$

for all indexes $(\alpha_1, \ldots, \alpha_n)$ such that $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$. If $2m \ge n$ the last inequality is in fact an equality. If not, explicit calculations show that

$$p(\alpha_1, \dots, \alpha_n) = \sum_{j=2m+1}^n \alpha_j + \frac{1}{2m} \sum_{j=1}^{2m} j\alpha_j \le \frac{1}{2m} \sum_{j=1}^n j\alpha_j = \frac{n}{2m}.$$

3 A Higher Order Dissipative Scheme for the Schrödinger equation

In the following we will consider a numerical scheme with a high order dissipative term. We will replace in the right hand side of (5) the operator Δ_h by $-(-\Delta_h)^m$. The scheme we will analyze is the following

$$\begin{cases} i\frac{du^h}{dt} + \Delta_h u^h = -i\operatorname{sgn}(t)a(h)(-\Delta_h)^m u^h, \quad t \neq 0, \\ u^h(0) = \varphi^h. \end{cases}$$
(20)

The term $-(-\Delta_h)^m$ will introduce more dissipation than Δ_h . Observe that for high frequencies $|\xi| \sim 1/h$ the contribution of the term $-(-\Delta_h)^m$ is of order $1/h^{2m}$, which is greater than $1/h^2$, introduced by Δ_h in scheme (5).

The following theorem shows that for any integer $m \ge 2$ we can recover the same behavior of the solutions as in the continuous case, uniform on the mesh size h. In contrast with the scheme (5) in this case the behavior of the solutions will be the same for $t \sim 0$ and $t \sim \infty$.

Theorem 3.1. Let be $m \ge 2$ an integer and a(h) a positive function such that

$$\inf_{h>0} \frac{a(h)}{h^{2(m-1)}} = a > 0.$$
(21)

For any $p \in [2, \infty]$ there exist positive constants c = c(d, p, m, a) such that

$$\|u^{h}(t)\|_{l^{p}(h\mathbb{Z}^{d})} \leq \frac{c}{|t|^{\frac{d}{2}(1-\frac{2}{p})}} \|\varphi^{h}\|_{l^{p'}(h\mathbb{Z}^{d})}$$
(22)

holds for all $t \neq 0$, $\varphi^h \in l^{p'}(h\mathbb{Z}^d)$ and h > 0.

Proof. Taking the semidiscrete Fourier transform in (20) we obtain that \hat{u}^h satisfies the following ODE:

$$\begin{cases} i\widehat{u}_{t}^{h}(t,\xi) - p_{h}(\xi)\widehat{u}^{h}(t,\xi) = -i\operatorname{sgn}(t)a(h)p_{h}^{m}(\xi)\widehat{u}^{h}(t,\xi), & t \neq 0, \, \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^{d} \\ \\ \widehat{u}(0,\xi) = \widehat{\varphi}^{h}(\xi), & \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^{d}. \end{cases}$$

Solving this ODE we find that for all time t, u^h satisfies:

$$\widehat{u}^{h}(t,\xi) = \exp(-itp_{h}(\xi))\exp(-|t|a(h)p_{h}^{m}(\xi))\widehat{\varphi}^{h}(\xi).$$
(23)

We will consider the cases p = 2 and $p = \infty$, the other come by interpolation.

In the case p = 2, Plancherel's identity gives us that

$$\begin{aligned} \|u^{h}(t)\|_{l^{2}(h\mathbb{Z}^{d})} &= \frac{1}{(2\pi)^{d}} \int_{[-\pi/h,\pi/h]^{d}} \exp(-2|t|a(h)p_{h}^{m}(\xi)) |\widehat{\varphi}^{h}(\xi)|^{2} d\xi \\ &\leq \|\varphi\|_{l^{2}(h\mathbb{Z}^{d})}^{2}. \end{aligned}$$

In the following we analyze the case $p = \infty$:

$$\|u^{h}(t)\|_{l^{\infty}(h\mathbb{Z}^{d})} \lesssim |t|^{-\frac{d}{2}} \|\varphi^{h}\|_{l^{1}(h\mathbb{Z}^{d})}.$$
(24)

In view of (23) we write the solution $u^{h}(t)$ in the convolution form

$$u^h(t) = K^h(t) * \varphi^h,$$

where the kernel $K^{h}(t)$ is given by

$$K^{h}(t,j) = \int_{[-\frac{\pi}{h},\frac{\pi}{h}]^{d}} e^{-itp_{h}(\xi)} e^{-|t|a(h)p_{h}^{m}(\xi)} e^{ij\cdot\xi h} d\xi.$$

In order to prove (24) it is sufficient to prove that $K^{h}(t)$ satisfies:

$$\|K^h(t)\|_{l^{\infty}(h\mathbb{Z}^d)} \lesssim |t|^{-\frac{d}{2}}$$

for all $t \neq 0$.

We decompose the kernel $K^h(t)$ in two components: a low frequency component, respectively a high frequency one. To illustrate this fact let us denote $\Omega_h = [-\pi/h, \pi/h]^d \setminus [-\pi/4h, \pi/4h]^d$. We define

$$K_{low}^{h}(t,j) = \int_{[-\pi/4h,\pi/4h]^{d}} e^{-itp_{h}(\xi)} e^{-|t|a(h)p_{h}^{m}(\xi)} e^{ij\cdot\xi h} d\xi$$

and

$$K_{high}^{h}(t,j) = \int_{\Omega_{h}} e^{-itp_{h}(\xi)} e^{-|t|a(h)p_{h}^{m}(\xi)} e^{ij\cdot\xi h} d\xi$$

Then the high component $K^h_{high}(t)$ satisfies the rough estimate

$$\begin{split} \|K_{high}^{h}(t)\|_{l^{\infty}(h\mathbb{Z}^{d})} &\leq \int_{\Omega_{h}} e^{-|t|a(h)p_{h}^{m}(\xi)}d\xi \\ &\leq \frac{c(d)}{h^{d}}\exp\Big(-\frac{|t|a(h)}{h^{2m}}\big(d\sin^{2}\frac{\pi}{8}\big)^{m}\Big) \leq \frac{c(m,d)}{h^{d}}\left(\frac{h^{2m}}{|t|a(h)}\right)^{d/2} \\ &\leq \frac{c(m,d)}{|t|^{d/2}}\left(\inf_{h>0}\frac{a(h)}{h^{2(m-1)}}\right)^{-d/2} \leq \frac{c(m,d,a)}{|t|^{d/2}}. \end{split}$$

It remains to estimate $K_{low}^{h}(t)$, the restriction of the kernel $K^{h}(t)$ on the low frequencies. Observe that $K_{low}^{h}(t)$ satisfies

$$K_{low}^h(t) = K_3^h(t) * G_m^h(|t|a(h)),$$

where G_m^h is defined in (13) and $K_3^h(t)$ is given by:

$$K_{3,j}^{h}(t) = \int_{[-\pi/4h,\pi/4h]^d} \exp(-itp_h(\xi)) \exp(ij\xi h) d\xi.$$

Applying estimate (14) with p = 1 and Young's inequality we get

$$\|K_{low}^{h}(t)\|_{l^{\infty}(h\mathbb{Z}^{d})} \leq \|K_{3}^{h}(t)\|_{l^{\infty}(h\mathbb{Z}^{d})} \|G_{m}^{h}(|t|a(h))\|_{l^{1}(h\mathbb{Z}^{d})} \lesssim \|K_{3}^{h}(t)\|_{l^{\infty}(h\mathbb{Z}^{d})}.$$

Thus, it is sufficient to prove that

$$||K_3^h(t)||_{l^{\infty}(h\mathbb{Z}^d)} \lesssim |t|^{-\frac{d}{2}}.$$

Using the separation and change of variables, it is sufficient to prove that

$$\sup_{j\in\mathbb{Z}} \left| \int_{-\pi/4}^{\pi/4} \exp(-it\sin^2(\xi/2)) \exp(ij\xi) d\xi \right| \le |t|^{-\frac{1}{2}}$$
(25)

for all $t \neq 0$. Using that the function $\xi \to \sin^2(\xi/2)$ does not change the convexity on the interval $[-\pi/4, \pi/4]$ we apply Van der Corput's Lemma (Prop. 2, Ch. 8, p. 332, [15]) and then we obtain (25).

The proof is now complete.

4 Strichartz estimates

In this section we obtain space-time estimates for the solutions of (20), similar to those given in (3) for the continuous case. We denote by $S^{h}(t)$, the solution of (20) at time t:

$$S^{h}\varphi(t) = \exp(it\Delta_{h})\exp(-|t|(-\Delta_{h})^{m})\varphi.$$
(26)

Observe that $S^{h}(t)$ satisfies the semigroup condition $S^{h}(t+s) = S^{h}(t)S^{h}(s)$ restricted on $[0, \infty)$ and $(-\infty, 0]$ but not on the whole interval $(-\infty, \infty)$, $S^{h}(t)S^{h}(s)$ being more dissipative than $S^{h}(t+s)$ in the case ts < 0.

The main result of this section is given by the following theorem.

Theorem 4.1. Let be a(h) satisfying (21) and (q, r), (\tilde{q}, \tilde{r}) two d/2-admissible pairs. Then

i) There exists a positive constant C = C(d, r, m, a) such that

$$\|S^{h}(\cdot)\varphi\|_{L^{q}(\mathbb{R}, l^{r}(h\mathbb{Z}^{d}))} \leq C\|\varphi\|_{l^{2}(h\mathbb{Z}^{d})}$$

$$(27)$$

holds for all $\varphi^h \in l^2(h\mathbb{Z}^d)$ uniformly on h > 0.

ii) There exists a positive constant C(d, r, m, a) such that

$$\left\| \int_{\mathbb{R}} S^{h}(s)^{*} f(s) ds \right\|_{l^{2}(h\mathbb{Z}^{d})} \leq C(d, r, m, a) \|f\|_{L^{q'}(\mathbb{R}, l^{r'}(h\mathbb{Z}^{d}))}$$
(28)

holds for all $f \in L^{\tilde{q}'}(\mathbb{R}, l^{\tilde{r}'}(h\mathbb{Z}^d))$, uniformly on h > 0.

iii) There exists a positive constant $C = C(d, \alpha, r, m, a)$ such that

$$\left\| \int_{s < t} S^h(t-s) f(s) ds \right\|_{L^q(\mathbb{R}, l^r(h\mathbb{Z}^d))} \le \|f\|_{L^{\tilde{q}'}(\mathbb{R}, l^{\tilde{r}'}(h\mathbb{Z}^d))}.$$
 (29)

holds for all $f \in L^{\tilde{q}'}(\mathbb{R}, l^{\tilde{r}'}(h\mathbb{Z}^d))$, uniformly on h > 0.

The first part of the theorem will be obtained as a consequence of the following result of Keel and Tao, [9]. We state here the original result.

Proposition 4.1. ([9], Theorem 1.2) Let H be a Hilbert space, (X, dx)be a measure space and $U(t) : H \to L^2(X)$ be a one parameter family of mappings, which obey the energy estimate

$$\|U(t)f\|_{L^2(X)} \le C \|f\|_H \tag{30}$$

and the decay estimate

$$\|U(t)U(s)^*g\|_{L^{\infty}(X)} \le C|t-s|^{-\sigma}\|g\|_{L^1(X)}$$
(31)

for some $\sigma > 0$. Then

$$\begin{split} \|U(t)f\|_{L^{q}(\mathbb{R},L^{r}(X))} &\leq C\|f\|_{L^{2}(X)}, \\ \left\|\int_{\mathbb{R}} U(s)^{*}F(s)ds\right\|_{L^{2}(X)} &\leq C\|F\|_{L^{q'}(\mathbb{R},L^{r'}(X))}, \\ \left\|\int_{0}^{t} U(t)U(s)^{*}F(s)ds\right\|_{L^{q}(\mathbb{R},L^{r}(X))} &\leq C\|F\|_{L^{\tilde{q}'}(\mathbb{R},L^{\tilde{r}'}(X))} \end{split}$$

for all (q, r) and (\tilde{q}, \tilde{r}) , σ -admissible pairs.

Proof of Theorem 4.1. We will apply Proposition 4.1 to the operator $S^{h}(t)$ defined in (26). In view of (31), it is sufficient to prove that

$$\|S^{h}(t)S^{h}(s)^{*}\varphi\|_{l^{\infty}(h\mathbb{Z}^{d})} \lesssim |t-s|^{-\frac{a}{2}}\|\varphi\|_{l^{1}(h\mathbb{Z}^{d})}$$

holds for all $\varphi \in l^1(h\mathbb{Z}^d)$ and $t \neq s$.

Observe that this property is not an immediate consequence of (22). This is due to the fact that

$$S^{h}(t)S^{h}(s)^{*}\varphi = S^{h}(t)S^{h}(-s)\varphi \neq S^{h}(t-s)\varphi$$

and thus we can not apply (22) directly. However the estimates obtained in Section 2 give us the right estimate, by pointing out that $S^h(t)S^h(s)^*$ is more dissipative than $S^h(t-s)$. Observe that the following

$$S^{h}(t)S^{h}(s)^{*} = \exp(i(t-s)\Delta_{h})\exp(-|t|a(h)(-\Delta_{h})^{m})\exp(-|s|a(h)(-\Delta_{h})^{m})$$
$$= S^{h}(t-s)\exp(-(|t|+|s|-|t-s|)a(h)(-\Delta_{h})^{m})$$

holds for all t and s. Thus

$$\begin{split} \|S^{h}(t)S^{h}(s)^{*}\varphi\|_{l^{\infty}(h\mathbb{Z}^{d})} &\leq \\ &\leq \|S^{h}(t-s)\exp(-(|t|+|s|-|t-s|)a(h)(-\Delta_{h})^{m})\varphi\|_{l^{1}(h\mathbb{Z}^{d})} \\ &\leq |t-s|^{-d/2}\|\exp(-(|t|+|s|-|t-s|)a(h)(-\Delta_{h})^{m})\varphi\|_{l^{1}(h\mathbb{Z}^{d})} \\ &\leq |t-s|^{-d/2}\|\varphi\|_{l^{1}(h\mathbb{Z}^{d})}. \end{split}$$

This guarantees that property (30) is satisfied. As a consequence we obtain (27) and (28). Unfortunately (29) does not follow from Proposition 4.1. We remark that Proposition 4.1 gives us that

$$\left\| \int_{s < t} S^h(t) S^h(s)^* f(s) ds \right\|_{L^q(\mathbb{R}, l^r(h\mathbb{Z}^d))} \le \|f\|_{L^{\tilde{q}'}(\mathbb{R}, l^{\tilde{r}'}(h\mathbb{Z}^d))},$$

which is weaker than (29), the operator $S^h(t)S^h(s)^*$ being more dissipative than $S^h(t-s)$. However, a slight modification of the proof of Proposition 4.1 gives the desired result. In the following we prove (29).

Let us define the operator

$$Tf(t) = \int_{s < t} S^h(t - s)f(s)ds.$$

The operator T being linear, the proof of (29) is reduced to the cases $(\tilde{q}, \tilde{r}) = (\infty, 2), (q, r) = (\infty, 2)$ and $(q, r) = (\tilde{q}, \tilde{r})$. The other cases are a consequence of an interpolation between these cases (see [1]).

In the sequel we denote by $\langle \cdot, \cdot \rangle_h$ and $\langle \langle \cdot, \cdot \rangle \rangle_h$ the inner product on $l^2(h\mathbb{Z}^d)$:

$$\langle f,g\rangle_h = \Re\Big(h^d \sum_{j\in\mathbb{Z}^d} f_j \overline{g}_j\Big),$$

respectively on $L^2(\mathbb{R}, l^2(h\mathbb{Z}^d))$:

$$\langle\langle f,g\rangle\rangle_h = \int_{\mathbb{R}} \langle f(t),g(t)\rangle_h dt.$$

By duality

$$\|Tf\|_{L^{q}(\mathbb{R}, l^{r}(h\mathbb{Z}^{d}))} = \sup_{\|g\|_{L^{q'}(\mathbb{R}, l^{r'}(h\mathbb{Z}^{d}))} \leq 1} \langle \langle Tf, g \rangle \rangle,$$

so we will estimate the right hand side of the above identity.

In all the analyzed cases we will use the following property of the operator Tf:

$$\begin{split} \langle \langle Tf,g \rangle \rangle_h &= \int_{\mathbb{R}_t} \left\langle \int_{s < t} S^h(t-s)f(s)ds, g(t) \right\rangle_h dt \\ &= \int_{\mathbb{R}_s} \left\langle f(s), \int_{t > s} S^h(t-s)^*g(t)dt \right\rangle_h ds. \end{split}$$

Case I: $(\tilde{q}, \tilde{r}) = (\infty, 2)$. Applying Cauchy's inequality in the space variable we obtain:

$$\begin{aligned} \langle \langle Tf,g \rangle \rangle_h &\leq \int_{\mathbb{R}_s} \|f(s)\|_{l^2(h\mathbb{Z}^d)} \left\| \int_{t>s} S^h(t-s)^*g(t)dt \right\|_{l^2(h\mathbb{Z}^d)} ds \\ &\leq \|f\|_{L^1(\mathbb{R},l^2(h\mathbb{Z}^d))} \sup_{s\in\mathbb{R}} \left\| \int_{t>0} S^h(t)^*g(t+s)dt \right\|_{l^2(h\mathbb{Z}^d)}. \end{aligned}$$

Estimate (28) gives us

$$\left\| \int_{t>0} S^h(t)^* g(t+s) dt \right\|_{l^2(h\mathbb{Z}^d)} \le \|g(\cdot+s)\|_{L^{q'}((0,\infty), l^{r'}(h\mathbb{Z}^d))} \le 1.$$

and then

$$\langle \langle T_2 f, g \rangle \rangle_h \le \|f\|_{L^1(\mathbb{R}, l^2(h\mathbb{Z}^d))}.$$

This finishes the proof of the first case.

Case II: $(q,r) = (\infty, 2)$. With the same arguments as above

$$\langle\langle Tf,g\rangle\rangle_h \leq \sup_{t\in\mathbb{R}} \left\|\int_{s<0} S^h(s)^* f(t+s)ds\right\|_{l^2(h\mathbb{Z}^d)} \|g\|_{L^1(\mathbb{R},l^2(h\mathbb{Z}^d))}.$$

Applying again estimate (28) to the function $f(\cdot + t)$ we obtain:

$$\left\| \int_{s<0} S^h(s)^* f(t+s) ds \right\| \le \|f(\cdot+t)\|_{L^{\tilde{q}'}((-\infty,0), l^{\tilde{r}'}(h\mathbb{Z}^d))} \le 1$$

and finish the second case.

Case III: $(q, r) = (\tilde{q}, \tilde{r})$. Observe that Tf satisfies

$$\|Tf(t)\|_{l^{r}(h\mathbb{Z}^{d})} \leq \int_{\mathbb{R}} \|S^{h}(t-s)f(s)\|_{l^{r}(h\mathbb{Z}^{d})} ds \leq \int_{\mathbb{R}} \frac{\|f(s)\|_{l^{r'}(h\mathbb{Z}^{d})} ds}{|t-s|^{2/q}}$$

Applying Hardy-Littlewood-Sobolev's inequality (cf. [14], p. 119):

$$|||s|^{-2/q} * \varphi||_{L^q(\mathbb{R})} \le C(q,d) ||\varphi||_{L^{q'}(\mathbb{R})},$$

to the function $\varphi(s) = \|f(s)\|_{l^{r'}(h\mathbb{Z}^d)}$ we get

$$||Tf||_{L^{q}(\mathbb{R}, l^{r}(h\mathbb{Z}^{d}))} \leq ||f||_{L^{q'}(\mathbb{R}, l^{r'}(h\mathbb{Z}^{d}))}.$$

This ends the proof.

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