Sheaf representations of BL-algebras

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Abstract In this paper we present a survey on sheaf representations of BL-algebras, based on our PhD thesis [36]. We define sheaf spaces and sheaf representations of BL-algebras, we study completely regular and compact sheaf spaces of BL-algebras, compact representations of BL-algebras and we develop a Gelfand theory for BL-algebras. Thus, we prove that the category of nontrivial BL-algebras is equivalent to the category of compact local sheaf spaces of BL-algebras. The last section of our paper is a contribution to the representation theory of BL-algebras by (weak) Boolean products. We characterize the (weak) Boolean products of BL-chains, the weak Boolean products of local BL-algebras and the weak Boolean products of perfect BL-algebras.

Introduction

In 1998, Hájek [31] introduced a very general many-valued logic, called *Basic Logic* (or *BL*), with the idea to formalize the many-valued semantics induced by a continuous t-norm on the unit real interval $[0,1]$. This Basic Logic turns to be a fragment common to three important many-valued logics: $\aleph_0$-valued Łukasiewicz logic, Gödel logic and Product logic. The Lindenbaum-Tarski algebras for Basic Logic are called *BL-algebras*. Apart from their logical interest, BL-algebras have important algebraic properties and they have been intensively studied from an algebraic point of view.

Sheaf representations turned out to be very useful for different classes of algebras, since they reduce the study of algebras to the study of the stalks, which usually have a better known structure. From the sheaf representations of rings of Grothendieck [30], Pierce [44], and Dauns and Hofmann [16], there evolved general constructions of sheaf representations for universal algebra [13,17]. A very important sheaf representation is the *Boolean product* representation. The Boolean product construction was introduced by Burris and Werner [8] as a reformulation of the Boolean sheaf construction [16,44] and it is very useful for describing a good structure theory.

The first section is dedicated to basic definitions and results on BL-algebras. We present some known facts about BL-algebras, but we also present new results. We emphasize the following constructions, which will have a very important role for our paper: the prime and maximal spectra of a BL-algebra, the reticulation, and the MV-center of a BL-algebra.

In Section 2 we study some classes of filters very important for our sheaf representations and Boolean products of BL-algebras. Thus, we study Stone filters, co-annihilators of a BL-algebra, and o-filters.

Stone (maximal) filters are the filters generated by (maximal) filters of the center of a BL-algebra. They will be the key ingredient for the study of weak Boolean products of BL-algebras in Section 5.

Given a BL-algebra A, the set $Co-An(A)$ of co-annihilator filters can be endowed with a structure of complete Boolean algebra, a fact of the most importance for the construction of a Baer extension of a BL-algebra.

The notion of o-filter is inspired by the dual notion of o-ideal from lattice theory [14]. Co-annulets (or principal co-annihilator filters) are o-filters. For our paper, the most important class of o-filters is the class of filters $\{O(P)\}_{P\in Spec(A)}$, where

$$O(P)=\{a\in A\mid a\vee x=1 \text{ for some } x\notin P\}.$$

The family $\{O(P)\}_{P\in Spec(A)}$ will be used in the sequel to obtain a general sheaf representation for BL-algebras, and its subfamily $\{O(M)\}_{M\in Max(A)}$ for obtaining a compact representation of a BL-algebra.

In Section 3, after defining sheaf spaces of BL-algebras, called also BL-sheaf spaces, and sheaf representations of BL-algebras, we give, following Davey's construction for universal algebra [17], a general theorem which associates with any family of filters of a BL-algebra, under some conditions, a sheaf representation of the BL-algebra. We use this theorem for two sheaf constructions.

The first one represents a nontrivial BL-algebra A as a BL-algebra of global sections of a BL-sheaf space whose base space is $Spec(A)$ and whose stalks are the quotients $A/O(P)$. This representation will induce the compact representation from Section 4. The second one embeds any BL-algebra A in a Baer algebra, called a Baer extension of A. Baer BL-algebras are BL-algebras with the property that co-annihilator filters are generated by central elements. The Baer extension of A is the BL-algebra of global sections of a Hausdorff BL-sheaf space whose base space is the complete Boolean algebra $Co-An(A)$. Moreover, if A is itself Baer, the embedding turns out to be an isomorphism of BL-algebras.

Section 4 contains the main result of our paper: *the equivalence between the category $\mathcal{BL}$ of nontrivial BL-algebras and the category $CL-BL-ShSp$ of compact local BL-sheaf spaces.* This result is similar to the one obtained for ringed spaces and Gelfand rings by Mulvey [43], generalizing Gelfand duality. We define and study compact BL-sheaf spaces and compact representations of a BL-algebra, and we associate with each nontrivial BL-algebra A a compact local BL-sheaf space $(F_A, p_A, Max(A))$, with base space $Max(A)$ and stalks $A/O(M)$. In this way we get a compact representation for any nontrivial BL-algebra, which is even an isomorphism. The functors which establish the equivalence of categories are

$$\mathcal{S} : CL - BL - ShSp \to \mathcal{BL}, \quad \mathcal{T} : \mathcal{BL} \to CL - BL - ShSp,$$

where $\mathcal{S}$ assigns to to each BL-sheaf space (F, p, X) the BL-algebra $\Gamma(X, F)$ of global sections, and $\mathcal{T}$ assigns to each nontrivial BL-algebra A the compact local BL-sheaf space $(F_A, p_A, Max(A))$.

The last section of our paper is a contribution to the representation theory of BL-algebras by (weak) Boolean products. After proving a general theorem of representation by weak Boolean products of BL-algebras, obtained using the Stone maximal filters, we study the following standard problem: *given a class $\mathcal{K}$ of BL-algebras, characterize (weak) Boolean products of members of $\mathcal{K}$*. We give an answer to this problem for three classes of BL-algebras: BL-chains, local BL-algebras, and perfect BL-algebras. Weak Boolean products of BL-chains are exactly the BL-algebras with the property that Stone maximal filters coincide with minimal prime filters, and Boolean products of BL-chains are BL-algebras characterized by the fact that the associated lattice is dual Stone. In order to characterize weak Boolean products of local BL-algebras, we define quasi-local BL-algebras. We obtain a theorem which gives more equivalent characterizations of this class of BL-algebras. We think that a very interesting property of quasi-local BL-algebras is the fact that the maximal spectrum is a Boolean space. Finally, we define quasi-perfect BL-algebras, and it turns out that they are the weak Boolean products of perfect BL-algebras.

1 BL-algebras. Definitions and first properties

A *BL-algebra* [31] is an algebra $(A, \wedge, \vee, \odot, \to, 0, 1)$ with four binary operations $\wedge, \vee, \odot, \to$ and two constants $0, 1$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a commutative monoid, and for all $a, b, c \in A$,

$$c \leq a \to b \text{ iff } a \odot c \leq b \tag{1.1}$$
$$a \wedge b = a \odot (a \to b) \tag{1.2}$$
$$(a \to b) \vee (b \to a) = 1. \tag{1.3}$$

In order to simplify the notation, a BL-algebra $(A, \wedge, \vee, \odot, \to, 0, 1)$ will be referred by its support set A.

A BL-algebra A is nontrivial iff $0 \neq 1$. For any BL-algebra A, the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. A *BL-chain* is a totally ordered BL-algebra, i.e. a BL-algebra such that its lattice order is total.

For any $a \in A$, we define $a^- = a \to 0$. We denote the set of natural numbers by ω. We define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for $n \in \omega - \{0\}$. The *order* of $a \in A$, in symbols $ord(a)$, is the smallest $n \in \omega$ such that $a^n = 0$. If no such n exists, then $ord(a) = \infty$.

We shall denote by $\mathcal{BL}$ the category whose objects are nontrivial BL-algebras and whose morphisms are BL-morphisms.

Let A be a BL-algebra. A *filter* of A is a nonempty set $F \subseteq A$ such that for all $a, b \in A$,
(i) $a, b \in F$ implies $a \odot b \in F$;
(ii) $a \in F$ and $a \leq b$ imply $b \in F$.

Trivial examples of filters are $\{1\}$ and A. A filter F of A is *proper* iff $F \neq A$. Any filter of A is also a filter of the lattice $L(A)$.

A proper filter M of A is called *maximal* (or *ultrafilter*) if it is not contained in any other proper filter.

A proper filter P of A is called *prime* provided that it is prime as a filter of $L(A)$, that is

$$a \vee b \in P \quad \text{implies} \quad a \in P \text{ or } \ b \in P.$$

A *minimal prime filter* of A is a minimal element of the poset $(Spec(A), \subseteq)$.

We shall denote by $\mathcal{F}(A)$ the set of filters of A, by $Spec(A)$ the set of prime filters of A, by $Max(A)$ the set of maximal filters of A, and by $MinSpec(A)$ the set of minimal prime filters of A.
If $X \subseteq A$, then the filter of A generated by X will be denoted by $< X >$. For any $a \in A$, $< a >$ denotes the principal filter of A generated by $\{a\}$.

With any filter F of A we can associate a congruence relation $\equiv (mod\, F)$ on A by defining

$$a \equiv b(mod\, F) \text{ iff } a \to b \in F \text{ and } b \to a \in F \text{ iff } (a \to b) \odot (b \to a) \in F.$$

For any $a \in A$, let a/F be the equivalence class $a/_{\equiv(mod\, F)}$. If we denote by A/F the quotient set $A/_{\equiv(mod\, F)}$, then A/F becomes a BL-algebra with the natural operations induced from those of A.

Let $B(A)$ be the Boolean algebra of all complemented elements in the distributive lattice $L(A)$. We shall refer to $B(A)$ as the *center* of A and to elements of $B(A)$ as *central elements* of A.

A BL-algebra A is called *directly indecomposable* iff A is nontrivial and whenever $A \cong A_1 \times A_2$ then either A_1 or A_2 is trivial. In a similar manner with [11, Chapter 6.4], we can prove that

Proposition 1 *A BL-algebra A is directly indecomposable iff* $B(A) = \{0, 1\}$.

It follows immediately that any BL-chain is directly indecomposable.

Let A be a nontrivial BL-algebra. The *prime spectrum* of A is the set $Spec(A)$ of prime filters of A, endowed with the Zariski topology, of which the subsets of the form

$$D(a) = \{P \in Spec(A) \mid a \notin P\}, \quad a \in A$$

form a basis of open sets.

The *maximal spectrum* of A is the subspace $Max(A)$ of $Spec(A)$ consisting of the maximal filters of A with the induced topology. The subsets

$$D_{Max}(a) = D(a) \cap Max(A) = \{M \in Max(A) \mid a \notin M\}, \quad a \in A$$

form a basis for the topology of the maximal spectrum.

The prime and maximal spectra will be the base spaces for the BL-sheaf spaces defined in Section 3.1, respectively Section 4.3. Following standard methods, we get

Proposition 2 [37, Theorems 2.7, 2.9]
$Spec(A)$ is a compact T_0 topological space and $Max(A)$ is a compact Hausdorff topological space.

The *reticulation* of a ring was defined by Simmons [45] for commutative rings and it was extended by Belluce to non-commutative rings [3]. The reticulation of a ring R is a bounded distributive lattice $L(R)$ such that the prime spectrum of R, endowed with the Zariski topology, is homeomorphic to the prime spectrum of $L(R)$, endowed with the Stone topology. By this connection, many properties can be transfered from R to $L(R)$ and vice versa. A similar construction was done by Belluce for MV-algebras [2].

In the sequel, we present properties of the reticulation $\beta(A)$ of a BL-algebra A; all the results can be found in [37].

Let A be a nontrivial BL-algebra. For any $a, b \in A$ define

$$a \equiv b \text{ iff } D(a) = D(b).$$

Then $\equiv$ is an equivalence relation on A compatible with the operations $\odot, \wedge$ and $\vee$. For $a \in A$ let us denote by $[a]$ the class of $a \in A$ with respect to $\equiv$. The bounded distributive lattice $\beta(A) = (A/\equiv, \vee, \wedge, [0], [1])$ is called the *reticulation* of the BL-algebra A.

For any $F \in \mathcal{F}(A)$, let

$$\beta(F) = \{[a] \mid a \in F\}.$$

Proposition 3 [37]
Let A be a nontrivial BL-algebra. Then
(i) the mapping $F \mapsto \beta(F)$ is an isomorphism between the lattices $\mathcal{F}(A)$ and $\mathcal{F}(\beta(A))$;
(ii) the mapping $P \mapsto \beta(P)$ is a homeomorphism between the prime spectrum of A and the prime spectrum of $\beta(A)$;
(iii) the mapping $M \mapsto \beta(M)$ is a homeomorphism between the maximal spectrum of A and the maximal spectrum of $\beta(A)$.

Proposition 4 [37, Proposition 3.14]
$\beta(A)$ is a normal and completely normal lattice.

Local BL-algebras are introduced and studied in [53,54] in the same way as local MV-algebras are analyzed in [4]. Thus, a BL-algebra is called *local* iff it has a unique maximal filter.

Define [53]

$$M(A) = \{a \in A \mid a^n \neq 0 \text{ for all } n \in \omega\} = \{a \in A \mid ord(a) = \infty\}.$$

A proper filter P of A is called *primary* if, for all $a, b \in A$,

$$(a \odot b)^- \in P \text{ implies } (a^n)^- \in P \text{ or } (b^n)^- \in P \text{ for some } n \in \omega.$$

Proposition 5 [54]
Let A be a BL-algebra. The following are equivalent:
(i) A is local;
(ii) $M(A)$ is the unique maximal filter of A;
(iii) for all $a \in A$,

$$ord(a) < \infty \text{ or } ord(a^-) < \infty.$$

(iv) any proper filter of A is primary.

Proposition 6 [19, Proposition 4.4]
Any local BL-algebra is directly indecomposable.

We remind that an MV-algebra [9] is an algebra $(A, \oplus, -, 0)$ with one binary operation $\oplus$, one unary operation - and one constant 0 such that:
(i) $(A, \oplus, 0)$ is a commutative monoid;
(ii) $a^{=} = a$;
(iii) $a \oplus 0^- = 0^-$;
(iv) $(a^- \oplus b)^- \oplus b = (b^- \oplus a)^- \oplus a$.
If A is a MV-algebra, then the binary operations $\odot$, $\wedge$, $\vee$ and the constant 1 are defined by the following relations:

$$a \odot b = (a^- \oplus b^-)^-, \quad a \wedge b = (a \oplus b^-) \odot b, \quad a \vee b = (a \odot b^-) \oplus b, \quad 1 = 0^-.$$

For a detailed exposition of MV-algebras see [11].

MV-algebras and BL-algebras are closely related. Indeed, a BL-algebra A is a MV-algebra iff $a^{=} = a$ for all $a \in A$. If A is a BL-algebra consider, following [54], the subset

$$MV(A) = \{a \in A \mid a^{=} = a\} = \{a^- \mid a \in A\}.$$

If one defines $a^- \oplus b^- = (a \odot b)^-$, then $(MV(A), \oplus, -, 0)$ becomes a MV-algebra (see [54]) such that $B(A) = B(MV(A))$.
Since $a^- \in MV(A)$ for any $a \in A$, we can define the map

$$\varphi : A \to MV(A), \quad \varphi(a) = a^-.$$

The map φ determines a bijection between the set of maximal filters of A and the set of maximal ideals of $MV(A)$. As a consequence, a BL-algebra A is local iff $MV(A)$ is a local MV-algebra.

2 Classes of filters of BL-algebras

The aim of this section is to study classes of filters used in the subsequent sections for obtaining (compact) sheaf or (weak) Boolean product representations of BL-algebras, namely Stone (maximal) filters, co-annihilators and o-filters.

2.1 Stone filters

Stone filters are very important for Boolean product representations of BL-algebras from Section 5, and we shall use them extensively in that section.

Let $(L, \wedge, \vee, 0, 1)$ be a bounded distributive lattice. A *Stone filter* of L is a filter of L generated by a filter of the Boolean algebra $B(L)$ and a *Stone maximal filter* of L is a Stone filter generated by a prime filter of $B(L)$. For a more detailed analysis of these notions see [10].

A *Stone (maximal) filter* of a BL-algebra A is a Stone (maximal) filter of the lattice $L(A)$, that is a filter of $L(A)$ generated by a (prime) filter of $B(A)$. We shall denote by $\mathcal{F}_{Stone}(A)$ the set of Stone filters of A and by $Max_{Stone}(A)$ the set of Stone maximal filters of A.
It follows that $\mathcal{F}_{Stone}(A) = \{< H >| \ H \text{ is a filter of } B(A)\}$ and $Max_{Stone}(A) = \{< R >| \ R \text{ is a prime filter of } B(A)\}$.

We point out two very useful properties of maximal Stone filters.

Proposition 7 [19, Proposition 1.15]
If U is a Stone maximal filter of A, then A/U is a directly indecomposable BL-algebra.

Proposition 8 [19, Proposition 1.19]
If C is a Boolean subalgebra of $B(A)$, then Then $\bigcap_{R\in Spec(C)} < R >= \{1\}$.

It follows that $\bigcap Max_{Stone}(A) = \{1\}$, so A is isomorphic to a subdirect product of the family $\{A/ < R >\}_{R\in Spec(B(A))}$.

2.2 Co-annihilators

For any non-empty subset X of A, the *co-annihilator of* X is the set

$$^{\perp}X = \{a \in A \mid a \vee x = 1 \text{ for any } x \in X\}.$$

It is easy to see that $^{\perp}A = \{1\}$ and $^{\perp}\emptyset = {}^{\perp}\{1\} = A$.
Let us recall some facts from lattice theory (see [29]). Let $(L, \vee, \wedge, 0)$ be a lattice with 0. An element $a^* \in L$ is a *pseudocomplement* of $a \in L$ if

$$a \wedge a^* = 0 \text{ and } (a \wedge x = 0 \quad \text{implies} \quad x \leq a^*).$$

A bounded lattice L is called *pseudocomplemented* iff every element has a pseudocomplement.

Let $(L, \vee, \wedge, 0, 1)$ be a bounded pseudocomplemented distributive lattice. L is called a *Stone lattice* iff it satisfies the *Stone identity*:

$$a^* \vee a^{**} = 1.$$

Proposition 9 [29, Theorem 3.14,3]
Let L be a bounded pseudocomplemented distributive lattice. Then L is a Stone lattice iff $(a \wedge b)^* = a^* \vee b^*$ for all $a, b \in L$.

It turns out that the lattice $\mathcal{F}(A)$ is pseudocomplemented, for any filter F, its pseudocomplement being $^{\perp}F$.

We define

$$Co - An(A) = \{^{\perp}F \mid F \in \mathcal{F}(A)\} = \{^{\perp}X \mid X \subseteq A\}.$$

The elements of $Co - An(A)$ will be called *co-annihilator filters* of A .

Proposition 10 [39]
Let A be a BL-algebra. Then

$$(Co - An(A), \cap, \vee_{Co-An(A)}, ^{\perp}, \{1\}, A)$$

is a Boolean algebra, where $F \vee_{Co-An(A)} G = {}^{\perp}({}^{\perp}F \cap {}^{\perp}G)$ for any $F, G \in Co - An(A)$.
Moreover, $Co - An(A)$ is a complete Boolean algebra such that for any family $(F_i)_{i \in I} \subseteq Co - An(A)$,

$$^{\perp\perp}(\bigcap_{i \in I} F_i) = \bigcap_{i \in I} {}^{\perp\perp}F_i.$$

We shall denote by

$$^{\perp}a = {}^{\perp}\{a\}.$$

2.3 o-filters

Let I be an ideal of $L(A)$. We define

$$o(I) = \{a \in A \mid a \vee x = 1 \text{ for some } x \in I\}.$$

A filter F of A is called an *o-filter* if $F = o(I)$ for some ideal I of $L(A)$.

The notion of o-filter of a BL-algebra is the dual of the notion of o-ideal introduced by Cornish [14] in the case of bounded distributive lattices. We remind that an ideal I of a bounded distributive lattice L is called an o-ideal if there is a filter F of L such that $I = \{a \in A \mid a \wedge x = 0 \text{ for some } x \in F\}$.

Our study of o-filters is very much inspired by the study of o-ideals from [14]. It is easy to see that $o(\{0\}) = \{1\}$, and $o(A) = A$, so $A, \{1\}$ are o-filters. A first example of o-filters are co-annihilators $^{\perp}a$, with $a \in A$. We give in the sequel another example.

To any prime filter P of a bounded distributive lattice or a BL-algebra A we associate the set

$$O(P) = \{a \in A \mid a \vee x = 1 \text{ for some } x \in A - P\}.$$

It is easy to see that $O(P)$ is a proper filter of A such that $O(P) \subseteq P$ and also that $O(P)$ is an o-filter.

We obtain characterizations of o-filters (in particular of co-annulets) as intersections of families of $O(P)$-filters

Proposition 11 [39]
Let I be an ideal of $L(A)$. Then

$$o(I) = \bigcap\{O(P) \mid P \text{is a prime filter, } I \subseteq A - P\}.$$

A set $\mathcal{K}$ of prime filters of A is called *full* if any proper filter of A is contained in some member of $\mathcal{K}$. $Max(A)$, and $Spec(A)$ are examples of full sets of prime filters.

Proposition 12 [39]
Let $\mathcal{K}$ be a full set of prime filters of A. Then for every filter F of A,

$$F = \bigcap\{F \vee O(P) \mid P \in \mathcal{K}\}$$

Corollary 1 *Let $\mathcal{K}$ be a full set of prime filters of A. Then*

$$\bigcap\{O(P) \mid P \in \mathcal{K}\} = \{1\}.$$

In particular, $\bigcap_{P \in Spec(A)} O(P) = \bigcap_{M \in Max(A)} O(M) = \{1\}$.

The family $\{O(P)\}_{P \in Spec(A)}$ will be used in Subsection 3.1 to obtain a general sheaf representation for BL-algebras. Its subfamily $\{O(M)\}_{M \in Max(A)}$ is the key ingredient in Section 4.3 for obtaining a compact representation of a BL-algebra.

Proposition 13 [20, Proposition 15]
Let A be a nontrivial BL-algebra. Then
(i) for any maximal filter M of A, M is the unique maximal filter that contains $O(M)$;
(ii) for any distinct maximal filters M, N of A, $O(M) \vee O(N) = A$;
(iii) $A/O(M)$ is local for any $M \in Max(A)$.

We want to point out (ii) from the above proposition. This property, obtained using the reticulation of a BL-algebra, is essential for compact representations (see Proposition 25).

3 BL-sheaf spaces and sheaf representations

The notion of a BL-sheaf space (or sheaf space of BL-algebras) is defined following the general line of sheaf spaces of universal algebras [17]. For details about sheaf theory, see also [50,49].

A *sheaf space of BL-algebras* (or a *BL-sheaf space*) is a triple (F, p, X) such that the following properties are satisfied:
(i) F and X are topological spaces;
(ii) $p : F \to X$ is a local homeomorphism from F onto X;
(iii) for each $x \in X$, $p^{-1}(\{x\}) = F_x$ is a nontrivial BL-algebra with operations denoted by $\vee_x, \wedge_x, \odot_x, \to_x, 0_x, 1_x$;
(iv) the functions $(a, b) \mapsto a \vee_x b$, $(a, b) \mapsto a \wedge_x b$, $(a, b) \mapsto a \odot_x b, (a, b) \mapsto a \to_x b$ from the set $\{(a, b) \in F \times F \mid p(a) = p(b)\}$ into F are continuous, where $x = p(a) = p(b)$;
(v) the functions $\underline{0}, \underline{1} : X \to F$, which assign to each x in X the zero 0_x and the unit 1_x of F_x respectively, are continuous.

X is known as the *base space*, F as the *total space* and F_x is called the *stalk* of F at $x \in X$.

If $Y \subseteq X$, then a *section* σ over Y is a continuous map $\sigma : Y \to F$ satisfying $(p \circ \sigma)(y) = y$ for all $y \in Y$. The set of all sections over Y form a nontrivial BL-algebra with the operations defined pointwise, denoted by $\Gamma(Y, F)$. The elements of $\Gamma(X, F)$ are called *global sections.*

For every $\sigma, \tau \in \Gamma(Y, F)$, we shall use the following notation:

$$[\sigma = \tau] = \{y \in Y \mid \sigma(y) = \tau(y)\}.$$

We shall use the expression *a BL-algebra of global sections* to refer to any BL-subalgebra of $\Gamma(X, F)$. If A is a BL-algebra of global sections, then for each $x \in X$, we define $p_x^A : A \to F_x$ by $p_x^A(\sigma) = \sigma(x)$ for all $\sigma \in A$. If $A = \Gamma(X, F)$, then we shall denote p_x^A by p_x.

If A is a BL-algebra of global sections, U is an open subset of X and σ is a section over U, we say that σ is *locally in the BL-algebra of global sections* A if

(*) there are an open covering $\{U_i\}_{i \in I}$ of U and a family $\{\sigma_i\}_{i \in I} \subseteq A$ such that $\sigma|_{U_i} = \sigma_i|_{U_i}$ for all $i \in I$.

Proposition 14 [20, Proposition 18]
Let (F, p, X) be a BL-sheaf space and A a BL-algebra of global sections. The following are equivalent:
(i) every section over an open subset of X is locally in the BL-algebra A;
(ii) for each $x \in X$, the BL-morphism p_x^A is onto.

Let X and Y be topological spaces and $f : Y \to X$ a continuous function. Let (F, p, X) and (G, q, Y) be two BL-sheaf spaces. A *morphism* $\alpha : F \to G$ *over* f is a family $(\alpha_y : F_{f(y)} \to G_y)_{y \in Y}$ of BL-morphisms satisfying the following condition:

If U is open in X and $\sigma \in \Gamma(U,F)$, define $\gamma : f^{-1}(U) \to G$ by

$$\gamma(y) = \alpha_y(\sigma(f(y))).$$

Then γ is continuous, and therefore $\gamma \in \Gamma(f^{-1}(U), G)$.

We shall write $\gamma = \alpha^U_{\#}(\sigma)$.

It follows that a morphism $\alpha : F \to G$ over f induces a BL-morphism $\alpha^U_{\#} : \Gamma(U,F) \to \Gamma(f^{-1}(U), G)$ for all open U in X. We shall denote $\alpha^X_{\#}$ by $\alpha_{\#}$. Since $f^{-1}(X) = Y$, $\alpha_{\#}$ is a BL-morphism between the BL-algebras of global sections $\Gamma(X,F)$ and $\Gamma(Y,G)$.

An example of a morphism over f is given by the canonical mapping from a BL-sheaf space (F,p,X) to the BL-sheaf space $(f^{-1}(F), q, Y)$, *induced* by f and (F,p,X), defined as follows.

Define $f^{-1}(F) = \{(y,a) \in Y \times F \mid f(y) = p(a)\} = \bigcup_{y\in Y}\{y\} \times F_{f(y)}$ and $q : f^{-1}(F) \to Y$ by $q(y,a) = y$. Then for all $y \in Y$, $f^{-1}(F)_y = \{y\} \times F_{f(y)}$. For each $y \in Y$, define $i_y : F_{f(y)} \to f^{-1}(F)_y$ by $i_y(a) = (y,a)$. We get easily that i_y is a bijection. We make $f^{-1}(F)_y$ a BL-algebra by transporting the BL-structure of $F_{f(y)}$ to $f^{-1}(F)_y$ by means of i_y.

Thus, we have got a BL-sheaf space $(f^{-1}(F), q, Y)$ and a morphism $i : F \to f^{-1}(F)$ over f, where i is the family $(i_y)_{y\in Y}$.

A *morphism of BL-sheaf spaces* $(f,\alpha) : (F,p,X) \to (G,q,Y)$ consists of a continuous function $f : Y \to X$ and a morphism $\alpha : F \to G$ over f. An *isomorphism of BL-sheaf spaces* is a morphism (f,α) such that f is a homeomorphism and α_y is an isomorphism of BL-algebras for all $y \in Y$.

If $(f,\alpha) : (F,p,X) \to (G,q,Y)$ and $(g,\beta) : (G,q,Y) \to (H,r,Z)$ are two morphisms of BL-sheaf spaces, then their composition is the morphism $(f \circ g, \beta \circ \alpha)$, where $(\beta \circ \alpha)_z = \beta_z \circ \alpha_{g(z)}$ for all $z \in Z$.

Let (F,p,X) and (G,q,X) be BL-sheaf spaces over the same topological space X. If $(\alpha_x : F_x \to G_x)_{x\in X}$ is a family of functions, then we can define a function $\alpha : F \to G$ by $\alpha(a) = \alpha_x(a)$, where $x \in X$ is unique such that $a \in F_x$. Conversely, a function $\alpha : F \to G$ can be seen as a family $(\alpha_x : F_x \to G_x)_{x\in X}$, where $\alpha_x = \alpha \mid F_x$ for all $x \in X$.

Proposition 15 $(1_X, \alpha) : (F,p,X) \to (G,q,X)$ *is a morphism of BL-sheaf spaces iff* $\alpha : F \to G$ *is a continuous function such that* $q \circ \alpha = p$ *and* $\alpha_x : F_x \to G_x$ *is a BL-morphism for all* $x \in X$.

We shall denote by $BL - ShSp$ the category of BL-sheaf spaces and morphisms of BL-sheaf spaces.

Define $\mathcal{S}(F,p,X) = \Gamma(X,F)$ for any BL-sheaf space (F,p,X) and $\mathcal{S}(f,\alpha) = \alpha_{\#}$ for every morphism $(f,\alpha) : (F,p,X) \to (G,q,Y)$. Then it is easy to see that

Proposition 16 $\mathcal{S} : BL - ShSp \to \mathcal{BL}$ *is a functor.*

Thus, we have defined a functor from the category of BL-sheaf spaces to the category of nontrivial BL-algebras, the *section* functor $\mathcal{S}$, which associates with every BL-sheaf space (F,p,X) the BL-algebra $\Gamma(X,F)$ of global sections. This functor will be used in Section 4 to obtain the equivalence between the category of nontrivial BL-algebras and the category of compact local BL-sheaf spaces.

3.1 Sheaf representations of BL-algebras

Following Mulvey [42], by a *sheaf representation* (or simply *representation*) of a nontrivial BL-algebra A will be meant a BL-morphism

$$\varphi : A \to \Gamma(X, F)$$

from A to the BL-algebra $\Gamma(X, F)$ of global sections of a BL-sheaf space (F, p, X).

Hence, $\varphi(A)$ is a BL-algebra of global sections of (F, p, X). In a representation φ, each $a \in A$ determines a global section $\varphi(a)$; in particular, for every $x \in X$, $\varphi(a)(x)$ is an element of the stalk F_x.

For each $x \in X$, we define

$$\varphi_x : A \to F_x, \quad \varphi_x(a) = \varphi(a)(x) \text{ for all } a \in A,$$
$$K_x = Ker(\varphi_x) = \{a \in A \mid \varphi(a)(x) = 1_x\}.$$

Since $\varphi_x = p_x \circ \varphi$, we have that φ_x is a BL-morphism, so K_x is a proper filter of A for every $x \in X$.

It is easy to see that $Ker(\varphi) = \bigcap_{x \in X} K_x$, hence φ is a monomorphism iff $\bigcap_{x \in X} K_x = \{1\}$.

For every $a \in A$, we shall use the following notation:

$$V(a) = [\varphi(a) = \underline{1}] = \{x \in X \mid \varphi(a)(x) = 1_x\} = \{x \in X \mid a \in K_x\}.$$

Then $V(a)$ is open in X for all $a \in A$.

Proposition 17 *Any sheaf representation $\varphi : A \to \Gamma(X, F)$ such that φ is a monomorphism determines a subdirect representation of A*

A *filter space* of a BL-algebra A is a family $\{T_x\}_{x \in X}$ of proper filters of A, indexed by a topological space X.

Let $\varphi : A \to \Gamma(X, F)$ be a representation of A. The filter space $\{K_x\}_{x \in X}$ will be called the *representation space* of the representation, and the filters the *representation filters*. The topology generated by the family $\{V(a)\}_{a \in A}$ of subsets of X is called the *representation topology* on the space X. Then, any topology on X contains the representation topology.

We say that a filter space $\{T_x\}_{x \in X}$ *canonically determines* a representation of A if there is a representation $\varphi : A \to \Gamma(X, F)$ such that $T_x = K_x$ for all $x \in X$.

The use of sheaf spaces to obtain representation theorems and (or) embedding theorems for various algebras is very popular (see [32,17] for an extensive list of references). After a study of common patterns which appear in the various representations, Davey [17] gave a general procedure to construct, under some hypothesis, a sheaf representation of a universal algebra.

We follow Davey's construction to get an existence theorem for representations of BL-algebras, which associates with any family of filters of a BL-algebra, under some conditions, a sheaf representation of the BL-algebra A.

Let A be a nontrivial BL-algebra and $\{T_x\}_{x\in X}$ a filter space of A such that the subset $V(a) = \{x \in X \mid a \in T_x\}$ is open in X for all $a \in A$. Then a BL-sheaf space (F_A, p_A, X) and a representation $\varphi : A \to \Gamma(X, F_A)$ can be constructed in the following way, given in [17] for universal algebra. Let F_A be the disjoint union of the sets $\{A/T_x\}_{x\in X}$ and $p_A : F_A \to X$ the canonical projection, so $p_A^{-1}(\{x\}) = A/T_x$ for all $x \in X$. For all $x \in X$, T_x is a proper filter of A, so A/T_x is a nontrivial BL-algebra. For each $a \in A$, define the map $[a] : X \to F_A$ by $[a](x) = a/T_x$. Endow F_A with the topology generated by the family $\{[a](U) \mid a \in A$ and U is open in $X\}$. Applying [17, Corollary 2], we get that (F_A, p_A, X) is a sheaf space of BL-algebras and the function $\varphi : A \to \Gamma(X, F_A)$, defined by $\varphi(a) = [a]$ for all $a \in A$, is a representation of A. It is easy to see that $K_x = T_x$ for all $x \in X$.

Hence, we get the following theorem.

Theorem 1 *Let A be a nontrivial BL-algebra and $\{T_x\}_{x\in X}$ a filter space of A such that the subset $V(a) = \{x \in X \mid a \in T_x\}$ is open in X for all $a \in A$. Then $\{T_x\}_{x\in X}$ canonically determines a representation of A.*

As in the universal algebra case, our theorem allows us to convert any subdirect representation into a sheaf representation of a BL-algebra A.

Corollary 2 *Any subdirect representation of A determines a sheaf representation of A.*

In [7], Brezuleanu and Diaconescu proved a lattice-theoretic analogue of Grothendieck's well-known sheaf representation of a commutative ring R with 1, using the family of local rings $\{R_P\}_{P\in Spec(R)}$. Lambek [35] has given a sheaf representation of non-commutative rings R with 1 which was inspired by and related to Grothendieck's representation, and which found many uses in ring theory. $Spec(R)$ is the base space, and the stalks are the quotients $R/O(P)$, where P is a prime ideal of R and $O(P) = \{a \in R \mid aRs = 0$ for some $s \in R - P\}$.

In [14], Cornish gave a lattice-theoretic analogue of Lambek's representation. For any bounded distributive lattice L, Cornish defined a sheaf representation whose base space is $Spec(L)$ and whose stalks are the lattices $L/O(P)$, P a prime ideal in L. Georgescu and Voiculescu [26] obtained a sheaf representation which can be defined on any subframe of ideals of a bounded distributive lattice, which for the case of normal lattices and the subframe of σ-ideals is isomorphic to both sheaves induced on the space of maximal ideals by the Brezuleanu-Diaconescu and Cornish representations.

Applying Theorem 1, we obtain the similar sheaf representation for BL-algebras.

Proposition 18 *Let A be a nontrivial BL-algebra. Then the nonempty family $\{O(P)\}_{P\in Spec(A)}$ canonically determines a sheaf representation of A.*

Let $(F_A, p_A, Spec(A))$ be the BL-sheaf space and $\varphi : A \to \Gamma(Spec(A), F_A)$ the sheaf representation determined by the family $(O(P))_{P\in Spec(A)}$. Then $(F_A)_P = A/O(P)$ for all $P \in Spec(A)$, $p_A : F_A \to Spec(A)$ is the canonical projection and $\varphi(a) = [a]$ for all $a \in A$, where $[a] \in \Gamma(Spec(A), F_A)$ is defined by $[a](P) = a/O(P)$ for all $P \in Spec(A)$.

This representation will induce the compact representation from Section 4. It turns out that φ is a monomorphism of BL-algebras, but it is still an open problem if it is an isomorphism of BL-algebras.

3.2 Baer extensions of BL-algebras

The aim of this subsection is to construct a Baer extension of any BL-algebra, that is to embed any nontrivial BL-algebra A in a Baer BL-algebra.

A BL-algebra A is called *Baer* if every co-annihilator filter of A is a principal filter of A generated by an element from the center $B(A)$.

The definition is similar to the one of Baer rings, extensively studied (see [33,34,47,6]), or Baer MV-algebras, defined in [21] (also studied under the name of strongly stonian MV-algebras in [1]).

It is easy to see that a BL-algebra A is Baer iff for all $X \subseteq A$, there is a unique $e \in B(A)$ such that ${}^{\perp}X =< e >= e \vee A$.

Let A be a BL-algebra. A Baer BL-algebra A^* is called a *Baer extension of A* if A is isomorphic to a BL-subalgebra of A^*.

The sheaf-theoretic technique we use for obtaining a Baer extension of a BL-algebra is inspired by Keimel's construction for rings and semigroups [33], which is similar to the methods used in [34].

By Proposition 10, $Co - An(A)$ is a complete Boolean algebra. Then $Spec(Co - An(A))$, the set of its prime filters, is a Boolean space and the clopen sets of the basis are all the sets of the form

$$N_H = \{\underline{m} \in Spec(Co - An(A)) \mid H \in \underline{m}\},$$

where $H \in Co - An(A)$.

Let us recall that a topological space X is called *extremally disconnected* if the closure $\overline{U}$ of any open subset U of X is also an open subset of X. Since $Co - An(A)$ is a complete Boolean algebra, it follows that $Spec(Co - An(A))$ is extremally disconnected (see, e.g., [5, P.10.3.6, p.209]).

For any $\underline{m} \in Spec(Co - An(A))$, we define

$$P_{\underline{m}} = \cup\{H \in Co - An(A) \mid H \notin \underline{m}\} = \cup\{H \in Co - An(A) \mid \underline{m} \notin N_H\}.$$

Then $P_{\underline{m}}$ is a prime filter of A, and $\bigcap_{\underline{m}\in Spec(Co-An(A))} A/P_{\underline{m}} = \{1\}$, so A is isomorphic to a subdirect product of the family $\{A/P_{\underline{m}}\}_{\underline{m}\in Spec(Co-An(A))}$.

Proposition 19 *The family $\{P_{\underline{m}}\}_{\underline{m}\in Spec(Co-An(A))}$ canonically determines a sheaf representation of A.*

Let $(F_A, p_A, Spec(Co - An(A)))$ be the sheaf space of BL-algebras and

$$\varphi : A \to \Gamma(Spec(Co - An(A)), F_A)$$

the sheaf representation determined by the family $\{P_{\underline{m}}\}_{\underline{m} \in Spec(Co-An(A))}$. Then $(F_A)_{\underline{m}} = A/P_{\underline{m}}$ for all $\underline{m} \in Spec(Co-An(A))$, p_A is the canonical projection and $\varphi(a) = [a]$ for all $a \in A$, where $[a] \in \Gamma(Spec(Co - An(A)), F_A)$ is defined by $[a](\underline{m}) = a/P_{\underline{m}}$.

The total space F_A is Hausdorff and φ is a monomorphism of BL-algebras, embedding A into the BL-algebra of global sections of the BL-sheaf space $(F_A, p_A, Spec(Co - An(A)))$.

Let us denote by A^* the BL-algebra $\Gamma(Spec(Co - An(A)), F_A)$.

Theorem 2 [40]
A^* is a Baer extension of the BL-algebra A. Moreover, if A is a Baer BL-algebra, then $\varphi : A \cong A^*$.

4 Gelfand theory for BL-algebras

The Gelfand duality theorem states that the functor from the category of compact Hausdorff spaces to the category of commutative C^*-algebras, obtained by assigning to each compact Hausdorff space X the commutative C^*-algebra $\mathbb{C}(X)$ of continuous complex functions on X, determines a duality between these categories. The dual functor is obtained by assigning to each commutative C^*-algebra A the compact Hausdorff space $Max(A)$ of maximal ideals of A endowed with the Zariski topology.

In [41], Mulvey extended the concepts of complete regularity and compactness from topological spaces to ringed spaces and proved a compactness theorem for completely regular ringed spaces generalizing the Gelfand-Kolmogoroff criterion concerning maximal ideals in the ring $\mathbb{R}(X)$ of continuous real functions on a completely regular space X [23] (see also [28]). In [42], Mulvey introduced compact representations of rings, showing that they are exactly those representations of rings that establish an equivalence of categories of modules. To extend the Gelfand duality to (not necessarily) commutative rings, Mulvey introduced the notion of Gelfand ring [43], and, using compact representations, he proved that the functor from the category of compact local ringed spaces to the category of Gelfand rings obtained by assigning to each ringed space $(X, \mathcal{O}_X)$ the ring of sections $\mathcal{O}_X(X)$ determines a duality between these categories. Sun [48] proved a Gelfand-Mulvey duality theorem for a class of rings which includes Gelfand rings.

Gelfand rings have a property that can be formulated in terms of universal algebra, namely that each prime ideal is contained in a unique maximal ideal. Universal algebras with this property and their Gelfand representations were studied in [27] and in a lattice-theoretical setting in [46]. MV-algebras, lattice-ordered groups [5], and BL-algebras are classes of algebras that also satisfy this property. Hence, the problem of obtaining similar results for these structures is natural. Filipoiu and Georgescu [22] proved that the category of MV-algebras is equivalent to the category of compact sheaf spaces of MV-algebras with local stalks.

In this section, we give an answer for this problem in the case of BL-algebras.

4.1 Compact BL-sheaf spaces

Following techniques used by Mulvey for ringed spaces, we define completely regular and compact BL-sheaf spaces, and, more generally, the notions of BL-algebra of global sections completely regular (compact) in a BL-sheaf space.

Throughout, BL-algebras are nontrivial and X will be assumed to denote a Hausdorff topological space.

A BL-sheaf space (F, p, X) is called *completely regular* if it satisfies the following:

(CR) for each $x \in X$ and closed set $C \subseteq X$ not containing x, there is $\sigma \in \Gamma(X, F)$ such that $\sigma(x) = 0_x$ and $\sigma|_C = \underline{1}|C$.

A completely regular BL-sheaf space (F, p, X) is called *compact* if X is compact.

It is easy to see that **(CR)** is equivalent with the following condition:

(CR$'$) for each $x \in X$ and every open neighborhood U of x, there is $\sigma \in \Gamma(X, F)$ such that $\sigma(x) = 0_x$ and $\sigma(y) = 1_y$ for all $y \notin U$.

The following proposition gives some properties of completely regular BL-sheaf spaces.

Proposition 20 [20, Proposition 21]
Let (F, p, X) be a completely regular BL-sheaf space. Then
(i) X is a regular topological space;
(ii) every section over an open subset of X is locally in the BL-algebra $\Gamma(X, F)$ of global sections of the BL-sheaf space;
(iii) the family $[\sigma = \underline{1}]_{\sigma \in \Gamma(X,F)}$ form a basis for the topology of X;
(iv) $F_x \cong A/Ker(p_x)$ for all $x \in X$.

Let A be a BL-algebra of global sections of the BL-sheaf space (F, p, X). We say that A is *completely regular in the BL-sheaf space* (F, p, X) if for each $x \in X$ and closed set $C \subseteq X$ not containing x, there is $\sigma \in A$ such that $\sigma(x) = 0_x$ and $\sigma|_C = \underline{1}|_C$.

If A is completely regular in (F, p, X) and X is compact, then A is said to be *compact in the BL-sheaf space* (F, p, X).

We point out the following result, which gives a sufficient condition for a BL-algebra A of global sections compact in a BL-sheaf space to coincide with $\Gamma(X, F)$. This result allows us to prove that the compact representation associated with every nontrivial BL-algebra in Subsection 4.3 is an isomorphism (see Proposition 27).

Proposition 21 [20, Proposition 22]
Let A be a BL-algebra of global sections that is compact in (F, p, X) and suppose that every global section is locally in A. Then A is necessarily the BL-algebra $\Gamma(X, F)$.

4.2 The compactness theorem

The compactness theorem is similar to the Gelfand-Kolmogoroff criterion concerning maximal ideals in the ring $\mathbb{R}(X)$ of continuous real functions on a completely regular space X [23] (see also [28]). This theorem gives equivalent conditions for a BL-algebra of global sections completely regular in a BL-sheaf space (F, p, X) to be compact in the BL-sheaf space; all these equivalent conditions make use of the notion of fixed filter of $\Gamma(X, F)$, inspired by the notion of fixed ideal of the ring $\mathbb{R}(X)$ [28].

In the sequel, A will be a BL-algebra of global sections of the BL-sheaf space (F, p, X).

For each $x \in X$, let us denote $K_x = Ker(p_x^A) = \{\sigma \in A \mid \sigma(x) = 1_x\}$. Since A is nontrivial, it follows that K_x is a proper filter of A.

A filter T of A is called *fixed* if there is $x \in X$ such that $T \vee K_x$ is a proper filter of A. Otherwise, T is said to be a *free* filter of A.
In the following, we shall denote by $Spec_X(A)$ the set of prime filters of A that are fixed and by $Max_X(A)$ the set of maximal filters of A that are fixed.

If A is completely regular in (F, p, X), then for any $P \in Spec_X(A)$ there is a unique $x \in X$ such that $K_x \subseteq M_P$, so for any $M \in Max_X(A)$, there is a unique $x \in X$ such that $K_x \subseteq M$.

Hence, we can define a function $\mathbf{s} : Spec_X(A) \to X$ that assigns to each $P \in Spec_X(A)$ the unique $x \in X$ such that $K_x \subseteq M_P$. We shall denote by $\mathbf{m}$ its restriction to $Max_X(A)$. Then $\mathbf{m}$ assigns to every fixed maximal filter M of A the unique $x \in X$ such that $K_x \subseteq M$.

Proposition 22 [20, Proposition 23]
Let A be completely regular in (F, p, X). Then $\mathbf{s}$ is onto and $\mathbf{m}$ is continuous and onto.

Suppose that A is compact in (F, p, X). Then $Spec_X(A) = Spec(A)$ and it is easy to see that $\mathbf{s} : Spec(A) \to X$ assigns to every prime filter P of A the unique $x \in X$ such that $K_x \subseteq P$. We obtain the following consequence.

Corollary 3 [20, Corollary 2]
Let A be compact in the BL-sheaf space (F, p, X). Then $\mathbf{s}$ and $\mathbf{m}$ are continuous, closed and onto.

Theorem 3 (The compactness theorem)[20, Theorem 1]
Suppose that A is completely regular in the BL-sheaf space (F, p, X). The following are equivalent
(i) the Hausdorff topological space X is compact;

(ii) every proper filter of A is fixed;
(iii) every prime filter of A is fixed;
(iv) every maximal filter of A is fixed;
(v) A is compact in the BL-sheaf space (F, p, X).

A BL-sheaf space (F, p, X) is called *local* if for each $x \in X$ the stalk F_x is a local BL-algebra.

Proposition 23 [20, Proposition 24]
If (F, p, X) is a compact BL-sheaf space and $A = \Gamma(X, F)$, then
$\mathbf{m}$ is a homeomorphism iff (F, p, X) is a local BL-sheaf space.

Let (F, p, X) be a compact local BL-sheaf space and $A = \Gamma(X, F)$. By the above proposition, we can define a function $\mathbf{n} : X \to Max(A)$, that associates with every $x \in X$ the unique maximal filter M of A such that $K_x \subseteq M$. It follows that

Proposition 24 *Let (F, p, X) be a compact local BL-sheaf space. Then $\mathbf{n}$ is the inverse of $\mathbf{m}$, hence $\mathbf{n} : X \to Max(A)$ is also a homeomorphism.*

4.3 Compact representations

We shall define completely regular and compact representations and, finally, we shall prove that any compact representation arises canonically from a filter space of the BL-algebra satisfying certain conditions.

Thus, a representation $\varphi : A \to \Gamma(X, F)$ of a BL-algebra A in a BL-sheaf space (F, p, X) will be said to be a *completely regular representation of* A if φ is a monomorphism and $\varphi(A)$ is completely regular in (F, p, X).

A *compact representation* of A is a monomorphism $\varphi : A \to \Gamma(X, F)$ such that $\varphi(A)$ is compact in (F, p, X). Hence, a compact representation is a completely regular representation $\varphi : A \to \Gamma(X, F)$ with the property that X is compact.

For any BL-algebra A, a family $\{T_x\}_{x \in X}$ of proper filters of A will be said to be *coprime* if $\bigcap_{x \in X} T_x = \{1\}$ and for any distinct $x, y \in X$ we have

$$T_x \vee T_y = A.$$

The family $\{T_x\}_{x \in X}$ is called *strongly coprime* if $\bigcap_{x \in X} T_x = \{1\}$ and for any $x \in X$ and $a \in T_x$, we have

$$T_x \vee \bigcap \{T_y \mid y \in X \text{ and } a \notin T_y\} = A.$$

In the sequel, let us consider a filter space $\{T_x\}_{x \in X}$ of A such that the subset $V(a) = \{x \in X \mid a \in T_x\}$ is open in X for all $a \in A$. By Theorem 1, there is a representation $\varphi : A \to \Gamma(X, F)$ of A such that $T_x = K_x = \{a \in A \mid \varphi(a)(x) = 1_x\}$ for all $x \in X$.

Theorem 4 [20, Theorem 3]
Let A be a nontrivial BL-algebra and $\{T_x\}_{x\in X}$ a filter space of A such that the subset $V(a) = \{x \in X \mid a \in T_x\}$ is open in X for all $a \in A$. The following are equivalent:
(i) the filter space canonically determines a compact representation of A;
(ii) X is compact and the family $\{T_x\}_{x\in X}$ is coprime;
(iii) the family $\{T_x\}_{x\in X}$ is strongly coprime, the topology on X is generated by the family $V(a)_{a\in A}$ and any maximal filter of A contains a filter of the filter space.

Applying Theorem 4, we prove the existence of a compact representation for any nontrivial BL-algebra A.

Proposition 25 [20, Proposition 29]
The family $\{O(M)\}_{M\in Max(A)}$ canonically determines a compact representation of A.

Let $(F_A, p_A, Max(A))$ be the BL-sheaf space and $\varphi : A \to \Gamma(Max(A), F_A)$ the compact representation determined by the family $(O(M))_{M\in Max(A)}$. Then $(F_A)_M = A/O(M)$ for all $M \in Max(A)$, $p_A : F_A \to Max(A)$ is the canonical projection and $\varphi(a) = [a]$ for all $a \in A$, where $[a] \in \Gamma(Max(A), F_A)$ is defined by $[a](M) = a/O(M)$ for all $M \in Max(A)$.

Let us remind that in Section 3.1 we defined a BL-sheaf space with base space $Spec(A)$ and with stalks $\{A/O(P)\}_{P\in Spec(A)}$. Let us denote this BL-sheaf space with $\mathcal{S}h_1$. Then

Proposition 26 *$(F_A, p_A, Max(A))$ is the BL-sheaf space induced by $\mathcal{S}h_1$ and the inclusion $Max(A) \hookrightarrow Spec(A)$.*

Proposition 27 [20, Proposition 30]
$\varphi : A \cong \Gamma(Max(A), F_A)$.

4.4 The equivalence between nontrivial BL-algebras and compact local BL-sheaf spaces

Let us denote by $CL - BL - ShSp$ the full subcategory of $BL - ShSp$ whose objects are compact local BL-sheaf spaces. By Proposition 16, there is a section functor $\mathcal{S} : \mathcal{BL} - ShSp \to \mathcal{BL}$. Then, by composing $\mathcal{S}$ with the inclusion functor, we get a functor from $CL - BL - ShSp$ to $\mathcal{BL}$, denoted by $\mathcal{S}$, too.

In the sequel, we shall define a functor $\mathcal{T} : \mathcal{BL} \to CL - BL - ShSp$ and we shall prove that the functors $\mathcal{S}, \mathcal{T}$ determine an equivalence between $CL - BL - ShSp$ and $\mathcal{BL}$.

For any nontrivial BL-algebra A, let us define $\mathcal{T}(A) = (F_A, p_A, Max(A))$. By the previous section, $(F_A, p_A, Max(A))$ is a compact BL-sheaf space. For any $M \in Max(A)$, we have that the stalk at M is $(F_A)_M = A/O(M)$. By

Proposition 13(iii), $A/O(M)$ is a local BL-algebra, so $(F_A, p_A, Max(A))$ is a compact local BL-sheaf space.

Let A and B be nontrivial BL-algebras and $h : A \to B$ a BL-morphism. If M is a maximal filter of B, then $h^{-1}(M)$ is a maximal filter of A. Let us define $\overline{h} : Max(B) \to Max(A)$ by $(\overline{h})(M) = h^{-1}(M)$ for any maximal filter M of B. We get that $\overline{h}$ is continuous.
Let $(\overline{h}^{-1}(F_A), q_A, Max(B))$ be the BL-sheaf space induced by the function $\overline{h} : Max(B) \to Max(A)$ and $(F_A, p_A, Max(A))$ and $i : F_A \to \overline{h}^{-1}(F_A)$ the canonical morphism over $\overline{h}$. Since $\overline{h} : Max(B) \to Max(A)$ is continuous, we get that $(\overline{h}, i) : (F_A, p_A, Max(A)) \to (\overline{h}^{-1}(F_A), q_A, Max(B))$ is a morphism of BL-sheaf spaces.

For any maximal filter $M \in Max(B)$, we define

$$\psi_M : (\overline{h}^{-1}(F_A))_M \to (F_B)_M, \quad \psi_M(M, a/O(\overline{h}(M))) = h(a)/O(M)$$

for any $a \in A$. Then

$$(1_{Max(B)}, \psi) : (\overline{h}^{-1}(F_A), q_A, Max(B)) \to (F_B, p_B, Max(B))$$

is a morphism of BL-sheaf spaces [20, Proposition 32].

Hence, for a BL-morphism $h : A \to B$, we have got the morphisms of BL-sheaf spaces

$$(\overline{h}, i) : (F_A, p_A, Max(A)) \to (\overline{h}^{-1}(F_A), q_A, Max(B)),$$

and

$$(1_{Max(B)}, \psi) : (\overline{h}^{-1}(F_A), q_A, Max(B)) \to (F_B, p_B, Max(B)).$$

We define

$$\mathcal{T}(h) = (1_{Max(B)}, \psi) \circ (\overline{h}, i) = (\overline{h}, \alpha_h) : \mathcal{T}(A) \to \mathcal{T}(B),$$

where $\alpha_h = \psi \circ i$.

We completed the definition of the functor $\mathcal{T} : \mathcal{BL} \to CL - BL - ShSp$.

Theorem 5 [20, Propositions 33, 34, Theorem 4]
The functors

$$\mathcal{S} : CL - BL - ShSp \to \mathcal{BL}, \quad \mathcal{T} : \mathcal{BL} \to CL - BL - ShSp$$

establish an equivalence between the category of nontrivial BL-algebras and the category of compact local BL-sheaf spaces.

As a consequence, we get the corresponding result for the most important classes of BL-algebras; in particular we obtain the result from [22].

Corollary 4 [36, Corollary 4.3.6]
The category of nontrivial MV-algebras (G-algebras, product algebras) is equivalent to the category of compact local sheaf spaces of MV-algebras (G-algebras, product algebras).

5 Boolean products of BL-algebras

This last section of our paper is a contribution to the representation theory of BL-algebras by Boolean products.

A *weak Boolean product* of a nonempty family $\{A_x\}_{x\in X}$ of BL-algebras is a subdirect product A of the given family, in such a way that X can be endowed with a Boolean space topology having the following two properties:
(i) if $a, b \in A$, then the set $\|a = b\| = \{x \in X \mid a(x) = b(x)\}$ is open in X;
(ii) if $a, b \in A$ and Z is a clopen subset of X, then $a|_Z \cup b|_{X-Z} \in A$.
By requiring in condition (i) that $\|a = b\|$ be clopen we obtain the notion of *Boolean product.*

A *(weak) Boolean product representation* of a BL-algebra A is an isomorphism from A onto a (weak) Boolean product of BL-algebras.

Remark 1 For any nontrivial BL-algebra A, the Baer extension A^* constructed in Subsection 3.2 is a Boolean product of the family

$$\{A/P_{\underline{m}}\}_{\underline{m}\in Spec(Co-An(A))}.$$

By Proposition 2, it follows that any Baer BL-algebra A is a Boolean product of the above family.

Using the Pierce-Comer construction, we can give a general theorem of (weak) Boolean product representation of a BL-algebra.

Firstly, let us remind that if B is a Boolean algebra, then $Spec(B)$ is a Boolean space and the clopen sets of the basis are all the sets of the form

$$N_a = \{P \in Spec(B) \mid a \in P\},$$

for $a \in B$.

Theorem 6 [19, Theorem 2.1]
Let $\{A_x\}_{x\in X}$ be a nonempty family of nontrivial BL-algebras and A be a weak Boolean product of $\{A_x\}_{x\in X}$. Define $C \subseteq A$ by

$$C = \{a \in A \mid a(x) \in \{0_x, 1_x\} \text{ for all } x \in X\}.$$

Then
(i) C is a subalgebra of the Boolean algebra $B(A)$;
(ii) the correspondence $x \overset{\sigma_C}{\mapsto} P_x = \{a \in C \mid a(x) = 1_x\}$ is a homeomorphism from X onto $Spec(C)$;
(iii) for all $x \in X$, A_x is isomorphic to $A/ < P_x >$;
(iv) C coincides with $B(A)$ iff all algebras A_x are directly indecomposable.
Conversely, suppose that A is a nontrivial BL-algebra and C is a subalgebra of $B(A)$. Then A is representable as the weak Boolean product of the family $\{A/ < R >\}_{R\in Spec(C)}$.

If $\mathcal{K}$ is a class of universal algebras, then a standard problem is to characterize $\Gamma(\mathcal{K})$ or $\Gamma^a(\mathcal{K})$, that is (weak) Boolean products of members of $\mathcal{K}$.

Applying the above theorem and Proposition 7 we get a first result of this type.

Corollary 5 *Any nontrivial BL-algebra is representable as a weak Boolean product of directly indecomposable BL-algebras.*

In the sequel we give an answer to this problem for some proper classes of BL-algebras. Namely, we characterize the (weak) Boolean products of BL-chains, the weak Boolean products of local BL-algebras and the weak Boolean products of perfect BL-algebras. Some of our results extend some theorems related to the representation of MV-algebras by (weak) Boolean products [22,12,51,52].

5.1 Boolean products of BL-chains

A first characterization of weak Boolean products of BL-chains we get as an immediate application of Theorem 6.

Proposition 28 [19, Proposition 3.1]
A nontrivial BL-algebra A is a weak Boolean product of BL-chains iff the Stone maximal filters of A are prime filters of A.

We can prove more, namely that weak Boolean products of BL-chains are BL-algebras with the property that Stone maximal filters are exactly the minimal prime filters. Hence, we get the following theorem.

Theorem 7 [36, Proposition 5.2.2]
Let A be a BL-algebra. The following are equivalent:
(i) A is a weak Boolean product of BL-chains;
(ii) $Max_{Stone}(A) \subseteq Spec(A)$*;*
(iii) $Max_{Stone}(A) = MinSpec(A)$.

In the sequel, we shall give a characterization of BL-algebras representable by Boolean products of BL-chains.

Dualizing the notions of pseudocomplement, pseudocomplemented lattice and Stone lattice, we get the concepts of *dual pseudocomplement*, *dual pseudocomplemented lattice* and *dual Stone lattice*.

Theorem 8 *A nontrivial BL-algebra A is a Boolean product of BL-chains iff* $L(A)$ *is a dual Stone lattice.*

5.2 Weak Boolean products of local BL-algebras

In a similar manner with Proposition 28, we get from Theorem 6 a first characterization of weak Boolean products of local BL-algebras.

Proposition 29 *A nontrivial BL-algebra A is a weak Boolean product of local BL-algebras iff the Stone maximal filters of A are primary filters of A.*

In order to obtain an algebraic characterization, we define a new class of BL-algebras, namely quasi-local BL-algebras, inspired by the quasi-local MV-algebras defined in [22].

A nontrivial BL-algebra A is called *quasi-local* if, for any $a \in A$, there are $e \in B(A)$ and $n \in \omega - \{0\}$ such that

$$a^n \odot e = 0 \text{ and } (a^-)^n \odot e^- = 0.$$

It turns out that a BL-algebra A is quasi-local iff $MV(A)$ is a quasi-local MV-algebra [19, Proposition 4.8], and that local BL-algebras are exactly quasi-local directly indecomposable BL-algebras [19, Proposition 4.9]. Another very interesting property of this class of BL-algebras is the fact that the maximal spectrum is a Boolean space [36, Proposition 5.3.10].

The filters associated with quasi-local BL-algebras are quasi-primary filters, defined as follows. A proper filter F of a BL-algebra A is called *quasi-primary* if, for all $a, b \in A$,

$(a \odot b)^- \in F$ implies there are $n \in \omega - \{0\}$ and $u \in A$ such that
$u \vee u^- \in B(A)$, $(a^n \odot u)^- \in F$ and $(b^n \odot u^-)^- \in F$.

It is easy to see that any primary filter of a BL-algebra A is a quasi-primary filter of A.

Proposition 30 [19, Proposition 4.11]
Let F be a filter of A. The following are equivalent:
(i) A/F is a quasi-local BL-algebra;
(ii) F is a quasi-primary filter of A.

We end this subsection with a theorem which gives more equivalent characterizations of quasi-local BL-algebras.

Theorem 9 [36, Theorem 5.3.11]
Let A be a nontrivial BL-algebra A. The following are equivalent:
(i) A is quasi-local;
(ii) if $a, b \in A$ are such that $a \odot b = 0$, then there are $e \in B(A)$ and $n \in \omega - \{0\}$ such that $a^n \leq e^-$ and $b^n \leq e$ (or equivalently, $a^n \wedge e = a^n \odot e = 0$ and $b^n \wedge e^- = b^n \odot e^- = 0$);
(iii) if $a, b \in A$ are such that $a \wedge b = 0$, then there are $e \in B(A)$ and $n \in \omega - \{0\}$ such that $a^n \leq e^-$ and $b^n \leq e$ (or equivalently, $a^n \wedge e = a^n \odot e = 0$ and $b^n \wedge e^- = b^n \odot e^- = 0$);
(iv) if $a, b \in A$ are such that $a \vee b = 1$, then there are $e \in B(A)$ and $n \in \omega - \{0\}$ such that $(a^-)^n \leq e^-$ and $(b^-)^n \leq e$ (or equivalently, $(a^-)^n \wedge e = (a^-)^n \odot e = 0$ and $(b^-)^n \wedge e^- = (b^-)^n \odot e^- = 0$);

(v) $MV(A)$ is a quasi-local MV-algebra;
(vi) each Stone maximal filter of A is a primary filter of A;
(vii) each Stone maximal filter of A is contained in a unique maximal filter of A;
(viii) each prime filter of $B(A)$ is contained in a unique maximal filter of A;
(ix) any proper filter of A is quasi-primary;
(x) $\{1\}$ is quasi-primary;
(xi) A is a weak Boolean product of local BL-algebras.

As a consequence, it follows that the class of quasi-local BL-algebras is the class of weak Boolean products of local BL-algebras.

5.3 Weak Boolean products of perfect BL-algebras

Perfect MV-algebras were introduced in [4] and they are one of the most important and studied classes of MV-algebras. In a similar manner with the MV-case, Turunen and Sessa defined perfect BL-algebras in [54].

A BL-algebra A is called *perfect* if it is local and for any $a \in A$,

$$ord(a) < \infty \text{ implies } ord(a^-) = \infty.$$

The filters corresponding to perfect BL-algebras are perfect filters. A proper filter P of A is *perfect* if, for all $a \in A$,

$$(a^n)^- \in P \text{ for some } n \in \omega \text{ iff } ((a^-)^m)^- \notin P \text{ for all } m \in \omega.$$

Any perfect filter of a BL-algebra A is a primary filter of A and a filter P of A is perfect iff A/P is a perfect BL-algebra.

Again, from Theorem 6 we get a first characterization of weak Boolean products of perfect BL-algebras.

Proposition 31 [19, Proposition 5.3]
A nontrivial BL-algebra A is a weak Boolean product of perfect BL-algebras iff the Stone maximal filters of A are perfect filters of A.

In a similar manner with [22], we define quasi-perfect BL-algebras and quasi-perfect filters.

A BL-algebra A is *quasi-perfect* if it is quasi-local and satisfies

(*) for any $a \in A, e \in B(A) - \{0\}$,
$a^n \odot e = 0$ for some $n \in \omega$ implies $(a^-)^m \odot e \neq 0$ for all $m \in \omega$.

As in the quasi-local case, a BL-algebra A is quasi-perfect iff $MV(A)$ is a quasi-perfect MV-algebra [19, Proposition 5.5], and perfect BL-algebras are exactly the quasi-perfect directly indecomposable BL-algebras [19, Proposition 5.6].

Theorem 10 [36, Theorem 5.4.8]
Let A be a nontrivial BL-algebra A. The following are equivalent:
(i) A is quasi-perfect;
(ii) each Stone maximal filter of A is a perfect filter of A;
(iii) A is a weak Boolean product of perfect BL-algebras.

Thus, quasi-perfect BL-algebras are exactly the weak Boolean products of perfect BL-algebras.

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