The prime and maximal spectra and the reticulation of BL-algebras

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Abstract

In this paper we study the prime and maximal spectra of a BL-algebra, proving that the prime spectrum is a compact T_0 topological space and that the maximal spectrum is a compact Hausdorff topological space. We also define and study the reticulation of a BL-algebra.

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Introduction

BL-algebras are the algebraic structures for Hájek's Basic Logic [10], arising from the continuous triangular norms (t-norms), familiar in the frameworks of fuzzy set theory. The main example of a BL-algebra is the interval [0,1] endowed with the structure induced by a continuous t-norm.

The paper is divided in three sections. In the first section we recall some facts concerning BL-algebras.

In the second section we study the prime spectrum $Spec(A)$ and the maximal spectrum $Max(A)$ of a BL-algebra, following a standard method [1]. It turns out that $Spec(A)$ is a compact T_0 topological space and $Max(A)$ is a compact Hausdorff topological space.

The *reticulation* of a ring was defined by Simmons [14] for commutative rings and it was extended by Belluce to non-commutative rings [3]. The reticulation of a ring R is a bounded distributive lattice $L(R)$ such that the prime spectrum of R, endowed with the Zariski topology, is homeomorphic to the prime spectrum of $L(R)$, endowed with the Stone topology. By this connection, many properties can be transfered from R to $L(R)$ and vice versa. A similar construction was done by Belluce for MV-algebras [2]. Hence, a natural problem is to define a reticulation for some classes of universal algebras. This was done by Georgescu [8] for quantales [13], which constitute a good abstraction of the lattice of congruence for many types of algebraic structures.

In Section 3 we define the reticulation $\beta(A)$ of a BL-algebra A. We get that $\beta(A)$ is a normal and completely normal lattice such that the lattices of filters of A and $\beta(A)$ are isomorphic and that the prime (maximal) spectra of A and $\beta(A)$ are homeomorphic topological spaces.

1 Definitions and first properties

A *BL-algebra* [10] is an algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, \odot, \rightarrow$ and two constants $0, 1$ such that $(A, \wedge, \vee, 0, 1)$ is a bounded lattice, $(A, \odot, 1)$ is a commutative monoid, and for all $a, b, c \in A$,

$$c \leq a \rightarrow b \quad \text{iff} \quad a \odot c \leq b \tag{1.1}$$

$$a \wedge b = a \odot (a \rightarrow b) \tag{1.2}$$

$$(a \rightarrow b) \vee (b \rightarrow a) = 1. \tag{1.3}$$

Example 1.1. A *continuous t-norm* is a continuous map

$$\star : [0,1] \times [0,1] \rightarrow [0,1]$$

such that $([0,1], \star)$ is a commutative partially ordered monoid. There are three fundamental t-norms:

$$\begin{array}{ll} \text{Łukasiewicz } t\text{-norm:} & x \star_L y = max(x + y - 1, 0), \\ \text{Gödel } t\text{-norm:} & x \star_G y = min\{x, y\}, \\ \text{Product } t\text{-norm:} & x \star_P y = xy. \end{array}$$

Since the natural ordering on $[0,1]$ is a complete lattice ordering, each continuous t-norm induces naturally a *residuum*, or an implication in more logical terms, by

$$x \rightarrow y = max\{z \mid z \star x \leq y\}.$$

The implications associated to the three fundamental norms are:

$$x \rightarrow_L y = min(y - x + 1, 1),$$

$$x \rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

$$x \rightarrow_P y = \begin{cases} 1 & \text{if } x \leq y, \\ y/x & \text{otherwise.} \end{cases}$$

If $\star$ is a continuous t-norm and $\rightarrow$ is its residuum, then

$$([0,1], min, max, \star, \rightarrow, 0, 1)$$

is a BL-algebra. Taking the three fundamental norms and their residua, we get three particular BL-algebras:

$$\begin{array}{ll} \text{Łukasiewicz structure:} & ([0,1], min, max, \star_L, \rightarrow_L, 0, 1), \\ \text{Gödel structure:} & ([0,1], min, max, \star_G, \rightarrow_G, 0, 1), \\ \text{Product structure:} & ([0,1], min, max, \star_P, \rightarrow_P, 0, 1). \end{array}$$

A BL-algebra A is nontrivial iff $0 \neq 1$. For any BL-algebra A, the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. For any $a \in A$, we define $a^- = a \rightarrow 0$. We shall denote $(a^-)^-$ by a^{--}.

The following properties hold in any BL-algebra A and will be used in the sequel:

$$a \odot b \leq a \wedge b \leq a, b \tag{1.4}$$

$$a \odot a^- = 0 \tag{1.5}$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c). \tag{1.6}$$

Let A be a BL-algebra. A *filter* of A is a nonempty set $F \subseteq A$ such that for all $a, b \in A$,
(i) $a, b \in F$ implies $a \odot b \in F$;
(ii) $a \in F$ and $a \leq b$ imply $b \in F$.

Trivial examples of filters are $\{1\}$ and A. By (1.4) it is obvious that any filter of A is also a filter of the lattice $L(A)$.

A *deductive system* [15] of A is a set $D \subseteq A$ such that
(i) $1 \in D$
(ii) for all $a, b \in A$,
$a, a \rightarrow b \in D$ imply $b \in D$.

Proposition 1.2. [16, Proposition 2]
Let $F \subseteq A$. The following are equivalent:
(i) F is a filter of A;
(ii) F is a deductive system of A.

The following remark is obvious and it will be very used in the sequel.

Remark 1.3. *Let F be a filter of A and $a, b \in A$. Then*

$$a \odot b \in F \textit{ iff } a \wedge b \in F \textit{ iff } a \in F, b \in F.$$

A filter F of A is *proper* iff $F \neq A$. It is easy to see that a filter F is proper iff $0 \notin F$. A proper filter P of A is called *prime* provided that it is prime as a filter of $L(A)$, that is

$$a \vee b \in P \quad \text{implies} \quad a \in P \text{ or } \ b \in P.$$

A proper filter M of A is called *maximal* (or *ultrafilter*) if it is not contained in any other proper filter.

We shall denote the set of prime filters of A by $Spec(A)$ and the set of maximal filters of A by $Max(A)$.

We remind some properties of filters that will be used in the sequel.

Proposition 1.4. [5, Corollary 4.26]
If P is a prime filter of A and F is a proper filter of A such that $P \subseteq F$, then F is also prime.

Proposition 1.5. Prime filter theorem[6, Theorem 4.28]
Let F be a filter of the BL-algebra A and let $S \neq \emptyset$ be a $\vee$-closed subset of A (that is, $a, b \in S$ implies $a \vee b \in S$) such that $F \cap S = \emptyset$. Then there exists a prime filter P of A such that $F \subseteq P$ and $P \cap S = \emptyset$.

Proposition 1.6. [10, Lemma 2.3.15]
Let $a \in A$, $a \neq 1$. Then there is a prime filter P of A such that $a \notin P$.

Proposition 1.7. [15, Theorem 3]
If A is a nontrivial BL-algebra, then any proper filter of A can be extended to a prime, maximal filter.

Proposition 1.8. [16, Proposition 7]
Any maximal filter of A is a prime filter of A.

Proposition 1.9. [7, Proposition 1.4]
If A is a nontrivial BL-algebra, then any proper filter F of A is the intersection of all prime filters containing F.

Proposition 1.10. [7, Proposition 1.6]
If A is a nontrivial BL-algebra, then any prime filter of A is contained in a unique maximal filter.

Let $X \subseteq A$. The filter of A generated by X will be denoted by $< X >$. We have that $< \emptyset >= \{1\}$ and, if $X \neq \emptyset$,

$$< X > \ = \ \{y \in A \mid x_1 \odot \ldots \odot x_n \leq y \text{ for some } n \in \omega - \{0\} \text{ and some } x_1, \ldots, x_n \in X\}.$$

For any $a \in A$, $< a >$ denotes the principal filter of A generated by $\{a\}$. Then,

$$< a >= \{b \in A \mid a^n \leq b \text{ for some } n \in \omega - \{0\}\}.$$

It follows immediately that $< 1 >= \{1\}$ and $< 0 >= A$.

Lemma 1.11. *Let $a \in A$. Then*
$< a >= \{1\}$ *iff* $a = 1$.

We shall denote by $\mathcal{F}(A)$ the set of filters of the BL-algebra A.

Proposition 1.12. *$(\mathcal{F}(A), \subseteq)$ is a complete lattice. For every family $\{F_i\}_{i\in I}$ of filters of A, we have that*

$$\bigwedge_{i\in I} F_i = \bigcap_{i\in I} F_i,$$
$$\bigvee_{i\in I} F_i = <\bigcup_{i\in I} F_i> .$$

Let A, B be two BL-algebras. A *BL-morphism* is a function $h : A \to B$ such that $h(a \wedge b) = h(a) \wedge h(b)$, $h(a \vee b) = h(a) \vee h(b)$, $h(a \odot b) = h(a) \odot h(b)$, $h(a \to b) = h(a) \to h(b)$, and $h(0) = 0$, $h(1) = 1$. A *BL-isomorphism* is a bijective BL-morphism.

Proposition 1.13. [7, Lemma 1.7]
Let $h : A \to B$ be a BL-morphism.
(i) if G is a (proper) filter of B, then $h^{-1}(G)$ is a (proper) filter of A;
(ii) if Q is a prime filter of B, then $h^{-1}(Q)$ is a prime filter of A.

2 The prime and maximal spectra

Let A be a nontrivial BL-algebra. For each subset X of A, we define

$$V(X) = \{P \in Spec(A) \mid X \subseteq P\}.$$

Proposition 2.1. *Let A be a nontrivial BL-algebra. Then*
(i) $X \subseteq Y \subseteq A$ implies $V(Y) \subseteq V(X) \subseteq Spec(A)$;
(ii) $V(\{0\}) = \emptyset$ and $V(\emptyset) = V(\{1\}) = Spec(A)$;
(iii) $V(X) = \emptyset$ iff $< X >= A$;
(iv) $V(X) = Spec(A)$ iff $X = \emptyset$ or $X = \{1\}$;
(v) if $\{X_i\}_{i\in I}$ is any family of subsets of A, then $V(\bigcup_{i\in I} X_i) = \bigcap_{i\in I} V(X_i)$;
(vi) $V(X) = V(< X >)$;
(vii) $V(X) \cup V(Y) = V(< X > \cap < Y >)$;
(viii) if X, $Y \subseteq A$, then $< X >=< Y >$ iff $V(X) = V(Y)$;
(ix) if F, G are filters of A, then $F = G$ iff $V(F) = V(G)$.

Proof. (i) Obviously.
(ii) For any $P \in Spec(A)$, P is a proper filter of A, so $0 \notin P$, that is $P \notin V(\{0\})$. Hence, $V(\{0\}) = \emptyset$. It is obvious that $V(\emptyset) = Spec(A)$. Since 1 is an element of any filter of A, it follows that 1 is an element of any prime filter of A, that is, $V(\{1\}) = Spec(A)$.
(iii)"$\Rightarrow$" Suppose that $< X >\neq A$, that is $< X >$ is a proper filter of A. Applying Proposition 1.7, there is a prime filter P of A that includes the proper filter $< X >$. Since $X \subseteq< X >$, it follows that $X \subseteq P$, so $P \in V(X)$. Thus, $V(X) \neq \emptyset$.
"$\Leftarrow$" If $V(X) \neq \emptyset$, then there is $P \in V(X)$. Since P is a filter including X and $< X >$ is the least filter of A with this property, it follows that $A =< X >\subseteq P$, i.e. $P = A$. We have got that P is not a proper filter. This is a contradiction, since P is prime.
(iv) "$\Leftarrow$" By (ii).
"$\Rightarrow$" Suppose that $X \neq \emptyset$ and $X \neq \{1\}$. Then, there is $a \in X$, $a \neq 1$. Applying Proposition 1.6, there is a prime filter P of A such that $a \notin P$. Thus, $X \nsubseteq P$, so $P \notin V(X)$. That is, $V(X) \neq Spec(A)$.
(v) "$\subseteq$" We have that $X_i \subseteq \bigcup_{i\in I} X_i$ for all $i \in I$. Applying (i), it follows that $V(\bigcup_{i\in I} X_i) \subseteq V(X_i)$ for all $i \in I$, hence $V(\bigcup_{i\in I} X_i) \subseteq \bigcap_{i\in I} V(X_i)$.
"$\supseteq$" If $P \in \bigcap_{i\in I} V(X_i)$, then $X_i \subseteq P$ for all $i \in I$. We get that $\bigcup_{i\in I} X_i \subseteq P$, that is $P \in V(\bigcup_{i\in I} X_i)$.
(vi) "$\supseteq$" Since $X \subseteq< X >$, from (i) we get that $V(< X >) \subseteq V(X)$.
"$\subseteq$" Let $P \in V(X)$, so $X \subseteq P$. It follows that $< X >\subseteq P$, i.e. $P \in V(< X >)$.
(vii) "$\subseteq$" Apply (i).
"$\supseteq$" Let $P \in V(< X > \cap < Y >)$ and suppose that $P \notin V(X) \cup V(Y)$. Hence, $P \notin V(X) =$

$V(< X >)$ and $P \notin V(Y) = V(< Y >)$, i.e. $< X > \not\subseteq P$ and $< Y > \not\subseteq P$. Thus, there are $x \in < X >$, $y \in < Y >$ such that $x,\, y \notin P$. Since $x, y \leq x \vee y$ and $< X >, < Y >$ are filters of A, we get that $x \vee y \in < X > \cap < Y > \subseteq P$. Hence, we have obtained $x,\, y \in A$ such that $x \vee y \in P$ and $x, y \notin P$. This contradicts the fact that P is prime.
(viii) "$\Rightarrow$" Applying (vi), we get that $V(X) = V(< X >) = V(< Y >) = V(Y)$.
"$\Leftarrow$" If $< X >= A$, then $V(X) = \emptyset$, by (iii). Thus, $V(Y) = \emptyset$, so, applying again (iii), we get that $< Y >= A$. Hence, $< X >=< Y >= A$. Suppose now that $< X >, < Y >$ are proper filters of A. Applying twice Proposition 1.9 and (vi), it follows that

$$\begin{aligned} < X > &= \bigcap\{P \in Spec(A) \mid P \in V(< X >)\} \\ &= \bigcap\{P \in Spec(A) \mid P \in V(X)\} \\ &= \bigcap\{P \in Spec(A) \mid P \in V(Y)\} \\ &= \bigcap\{P \in Spec(A) \mid P \in V(< Y >)\} \\ &= < Y > . \end{aligned}$$

(ix) Apply (viii) and the fact that, since F, G are filters of A, we have $< F >= F$ and $< G >= G$. □

By Proposition 2.1(ii), (v) and (vii), it follows that the family $\{V(X)\}_{X\subseteq A}$ of subsets of $Spec(A)$ satisfies the axioms for closed sets in a topological space. The resulting topology is called the Zariski topology and the topological space $Spec(A)$ is called the prime spectrum of A.

For any $X \subseteq A$, let us denote the complement of $V(X)$ by $D(X)$. Hence,

$$D(X) = \{P \in Spec(A) \mid X \not\subseteq P\}.$$

It follows that the family $\{D(X)\}_{X\subseteq A}$ is the family of open sets of the Zariski topology. By duality, from Proposition 2.1 we get the following.

Proposition 2.2. *Let A be a nontrivial BL-algebra. Then*
(i) $X \subseteq Y \subseteq A$ implies $D(X) \subseteq D(Y) \subseteq Spec(A)$;
(ii) $D(\{0\}) = Spec(A)$ and $D(\emptyset) = D(\{1\}) = \emptyset$;
(iii) $D(X) = Spec(A)$ iff $< X >= A$;
(iv) $D(X) = \emptyset$ iff $X = \emptyset$ or $X = \{1\}$;
(v) if $\{X_i\}_{i\in I}$ is any family of subsets of A, then $D(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} D(X_i)$;
(vi) $D(X) = D(< X >)$;
(vii) $D(X) \cup D(Y) = D(< X > \cup < Y >)$;
(viii) if $X,\, Y \subseteq A$, then $< X >=< Y >$ iff $D(X) = D(Y)$;
(ix) if $F,\, G$ are filters of A, then $F = G$ iff $D(F) = D(G)$.

For any $a \in A$, let us denote $V(\{a\})$ by $V(a)$ and $D(\{a\})$ by $D(a)$. Then,

$$V(a) = \{P \in Spec(A) \mid a \in P\} \text{ and } D(a) = \{P \in Spec(A) \mid a \notin P\}.$$

Proposition 2.3. *Let $a,\, b \in A$. Then*
(i) $D(a) = Spec(A)$ iff $< a >= A$;
(ii) $D(a) = \emptyset$ iff $a = 1$;
(iii) $D(a) = D(b)$ iff $< a >=< b >$;
(iv) $V(a) \subseteq D(a^-)$;
(v) if $a \leq b$, then $D(b) \subseteq D(a)$;
(vi) $D(a) \cap D(b) = D(a \vee b)$;
(vii) $D(a) \cup D(b) = D(a \wedge b) = D(a \odot b)$.

Proof. (i), (ii), (iii) Apply Proposition 2.2(iii), (iv) and (viii).
(iv) Let $P \in V(a)$, hence $a \in P$. If $a^- \in P$, then $0 = a \odot a^- \in P$, so P is not proper. Thus, we must have $a^- \notin P$, that is $P \in D(a^-)$.
(v) Let $P \in D(b)$, so $b \notin P$. If $P \notin D(a)$, then $a \in P$ and from $a \leq b$ we get that $b \in P$, that

is, a contradiction.
(vi) For any prime filter P of A, we have that $a \vee b \notin P$ iff $a \notin P$ and $b \notin P$. Hence, $P \in D(a \vee b)$ iff $a \vee b \notin P$ iff $a \notin P$ and $b \notin P$ iff $P \in D(a)$ and $P \in D(b)$ iff $P \in D(a) \cap D(b)$.
(vii) Applying Remark 1.3, we get that for any filter F of A, ($a \notin F$ or $b \notin F$) iff $a \odot b \notin F$ iff $a \wedge b \notin F$. It follows that for any prime filter P of A, $P \in D(a) \cup D(b)$ iff $P \in D(a \odot b)$ iff $P \in D(a \wedge b)$. □

Proposition 2.4. *Let A be a nontrivial BL-algebra. The family $\{D(a)\}_{a \in A}$ is a basis for the topology of $Spec(A)$.*

Proof. Let $X \subseteq A$ and $D(X)$ an open subset of $Spec(A)$. Then $D(X) = D(\bigcup_{a \in X} \{a\}) = \bigcup_{a \in X} D(a)$, by Proposition 2.2(v). Hence, any open subset of $Spec(A)$ is the union of subsets from the family $\{D(a)\}_{a \in A}$. □

The sets $D(a)$ will be called *basic open sets* of $Spec(A)$.

Proposition 2.5. *For any $a \in A$, $D(a)$ is compact in $Spec(A)$.*

Proof. It is enough to prove that any cover of $D(a)$ with basic open sets contains a finite cover of $D(a)$. Let $D(a) = \bigcup_{i \in I} D(a_i) = D(\bigcup_{i \in I} a_i)$. By Proposition 2.2(viii), we get that $< a >=< \bigcup_{i \in I} a_i >$, so $a \in < \bigcup_{i \in I} a_i >$. Hence, there are $n \geq 1$ and $i_1, \ldots, i_n \in I$ such that $a_{i_1} \odot \cdots \odot a_{i_n} \leq a$. We shall prove that $D(a) = D(a_{i_1}) \cup \cdots \cup D(a_{i_n})$. Applying Proposition 2.3(v) and (vi), we obtain that $D(a) \subseteq D(a_{i_1} \odot \cdots \odot a_{i_n}) = D(a_{i_1}) \cup \cdots \cup D(a_{i_n})$. The other inclusion is obvious, since $D(a_{i_1}) \cup \cdots \cup D(a_{i_n}) \subseteq \bigcup_{i \in I} D(a_i) = D(a)$. □

Proposition 2.6. *The compact open subsets of $Spec(A)$ are exactly the finite unions of basic open sets.*

Proof. Since any basic open set is compact open, then a finite union of basic open sets is also compact open. Let now $D(X)$, with $X \subseteq A$, be a compact open subset of $Spec(A)$. Since $D(X)$ is open, we get that $D(X)$ is a union of basic open sets. Since $D(X)$ is compact, it follows that $D(X)$ is a finite union of basic open sets. □

Theorem 2.7. *$Spec(A)$ is a compact T_0 topological space.*

Proof. Applying Proposition 2.2(ii), we have that $Spec(A) = D(0)$. Apply now Proposition 2.5 to get that $Spec(A)$ is compact. It remains to prove that $Spec(A)$ is a T_0 space, which means that for any two distinct prime filters $P \neq Q \in Spec(A)$ there is an open set U of $Spec(A)$ such that $P \in U, Q \notin U$ or $Q \in U, P \notin U$. Since $P \neq Q$, we have that $P \not\subseteq Q$ or $Q \not\subseteq P$. Assume that $P \not\subseteq Q$, so there is $a \in P$ such that $a \notin Q$. Take $U = D(a)$. Then $Q \in U$ and $P \notin U$. Similarly if $Q \not\subseteq P$. □

In the sequel, let $Max(A)$ be the set of maximal filters of A. Since, by Proposition 1.8, $Max(A) \subseteq Spec(A)$, we consider on $Max(A)$ the topology induced by the Zariski topology. Thus, we obtain a topological space called the *maximal spectrum* of A.

For any $X \subseteq A$ and $a \in A$ let us define

$$
\begin{aligned}
V_{Max}(X) &= V(X) \cap Max(A) &= \{M \in Max(A) \mid X \subseteq M\} \\
D_{Max}(X) &= D(X) \cap Max(A) &= \{M \in Max(A) \mid X \not\subseteq M\},
\end{aligned}
$$

and

$$
\begin{aligned}
V_{Max}(a) &= V(a) \cap Max(A) &= \{M \in Max(A) \mid a \in M\}, \\
D_{Max}(a) &= D(a) \cap Max(A) &= \{M \in Max(A) \mid a \notin M\}.
\end{aligned}
$$

It follows that the family $\{V_{Max}(X)\}_{X \subseteq A}$ is the family of closed sets of the maximal spectrum, the family $\{D_{Max}(X)\}_{X \subseteq A}$ is the family of open sets of the maximal spectrum and the family $\{D_{Max}(a)\}_{a \in A}$ is a basis for the topology of $Max(A)$.

Proposition 2.8. *Let A be a nontrivial BL-algebra, $X, Y \subseteq A$, $\{X_i\}_{i\in I}$ a family of subsets of A, and $a, b \in A$. Then*
(i) $X \subseteq Y \subseteq A$ implies $D_{Max}(X) \subseteq D_{Max}(Y) \subseteq Max(A)$;
(ii) $D_{Max}(0) = Max(A)$ and $D_{Max}(\emptyset) = D_{Max}(1) = \emptyset$;
(iii) $D_{Max}(X) = Max(A)$ iff $< X >= A$;
(iv) $D_{Max}(\bigcup_{i\in I} X_i) = \bigcup_{i\in I} D_{Max}(X_i)$;
(v) $D_{Max}(X) = D_{Max}(< X >)$;
(vi) $D_{Max}(X) \cap D_{Max}(Y) = D_{Max}(< X > \cap < Y >)$;
(vii) $D_{Max}(a) = Max(A)$ iff $< a >= A$;
(viii) if $a \leq b$, then $D_{Max}(b) \subseteq D_{Max}(a)$;
(ix) $V_{Max}(a) \subseteq D_{Max}(a^-)$;
(x) $D_{Max}(a) \cap D_{Max}(b) = D_{Max}(a \vee b)$;
(xi) $D_{Max}(a) \cup D_{Max}(b) = D_{Max}(a \wedge b) = D_{Max}(a \odot b)$.

Proof. We have only to prove (iii), the other ones being immediate consequences of the corresponding properties for $Spec(A)$.
(iii) "$\Rightarrow$" If $< X >\neq A$, then $< X >$ is a proper filter of A, hence, applying Proposition 1.7, there is a maximal filter M of A such that $< X >\subseteq M$. It follows that $X \subseteq M$, that is, $M \notin D_{Max}(X)$. This contradicts the fact that $D_{Max}(X) = Max(A)$.
"$\Leftarrow$" If $< X >= A$, then $D(X) = Spec(A)$, by Proposition 2.2(iii), so $D_{Max}(X) = Max(A)$. □

Theorem 2.9. *$Max(A)$ is a compact Hausdorff topological space.*

Proof. Let us prove first that $Max(A)$ is compact. Let $Max(A) = \bigcup_{i\in I} D_{Max}(a_i) = D_{Max}(\bigcup_{i\in I} a_i)$, by Proposition 2.8(iv). Applying now Proposition 2.8(iii), we get that $A =< \bigcup_{i\in I} a_i >$, hence $0 \in< \bigcup_{i\in I} a_i >$. It follows that there are $n \geq 1$ and $i_1, \ldots, i_n \in I$ such that $a_{i_1} \odot \ldots \odot a_{i_n} = 0$. By Proposition 2.8(ii) and (ix), we get that $Max(A) = D_{Max}(0) = D_{Max}(a_{i_1} \odot \ldots \odot a_{i_n}) = D_{Max}(a_{i_1}) \cup \ldots \cup D_{Max}(a_{i_n})$. Hence, $Max(A)$ is compact.

Let M and N be two distinct maximal filters of A. Since $M \neq N$, there are $x \in M \setminus N$ and $y \in N \setminus M$. Let $a = x \to y$ and $b = y \to x$. Then, using Proposition 1.2(ii), we infer immediately that $a \notin M$ and $b \notin N$. Hence, $M \in D_{Max}(a)$ and $N \in D_{Max}(b)$. Moreover, by Proposition 2.8(x), (ii), and (1.3), $D_{Max}(a) \cap D_{Max}(b) = D_{Max}(a \vee b) = D_{Max}(1) = \emptyset$. Hence, $Max(A)$ is Hausdorff. □

3 The reticulation of a BL-algebra

In this section we shall use lattice-theoretical concepts without defining them. For a detailed analysis of these notions see, for example, [9].

Let A be a nontrivial BL-algebra. For any $a, b \in A$ define

$$a \equiv b \text{ iff } D(a) = D(b).$$

Hence, $a \equiv b$ iff for any $P \in Spec(A)$, ($a \notin P$ iff $b \notin P$) iff for any $P \in Spec(A)$, ($a \in P$ iff $b \in P$).

Proposition 3.1. *The relation $\equiv$ is a congruence relation on A with respect to $\odot, \wedge$, and $\vee$.*

Proof. It is obvious that $\equiv$ is an equivalence relation on A. Let a, b, c, $d \in A$ such that $a \equiv b$ and $c \equiv d$. We shall prove that $a \odot c \equiv b \odot d$, $a \wedge c \equiv b \wedge d$ and $a \vee c \equiv b \vee d$. Let $P \in Spec(A)$. Then $a \odot c \in P$ iff $a \in P$ and $c \in P$ iff $b \in P$ and $d \in P$ iff $b \odot d \in P$. That is, $a \odot c \equiv b \odot d$. We obtain similarly that $a \wedge c \equiv b \wedge d$. Since P is a prime filter, we get that $a \vee c \in P$ iff $a \in P$ or $c \in P$ iff $b \in P$ or $d \in P$ iff $b \vee d \in P$. Hence, $a \vee c \equiv b \vee d$. □

Let us denote by $[a]$ the equivalence class of $a \in A$ and let $A/_{\equiv}$ be the quotient set. We also denote by $\beta : A \to A/_{\equiv}$ the canonical surjection defined by $\beta(a) = [a]$.

Proposition 3.2. *The algebra* $(A/_{\equiv}, \wedge, \vee, [0], [1])$ *is a bounded distributive lattice, where*

$$[a] \vee [b] = [a \vee b],$$
$$[a] \wedge [b] = [a \wedge b].$$

Proof. By Proposition 3.1, the operations $\vee, \wedge$ on $A/_{\equiv}$ are well-defined. The rest of the proof is routine. We shall prove, for example, that $A/_{\equiv}$ is distributive. If $a, b, c \in A$, then

$$[a] \wedge ([b] \vee [c]) = [a \wedge (b \vee c)] = [(a \wedge b) \vee (a \wedge c)] = [a \wedge b] \vee [a \wedge c] = ([a] \wedge [b]) \vee ([a] \wedge [c]).$$

□

Proposition 3.3. *Let* $a, b \in A$.
(i) $[a] \leq [b]$ *iff* $D(b) \subseteq D(a)$
(ii) if $a \leq b$*, then* $[a] \leq [b]$*;*
(iii) $[a] = [b]$ *iff* $< a >=< b >$*;*
(iv) $[a] = [1]$ *iff* $a = 1$*;*
(v) $[a] = [0]$ *iff* $a^n = 0$ *for some* $n \in \omega - \{0\}$*;*
(vi) $[a^n] = [a]$ *for any* $n \in \omega - \{0\}$*;*
(vii) $[a \wedge b] = [a \odot b]$*;*
(viii) if $e \in B(A)$*, then* $[e] \leq [a]$ *iff* $e \leq a$.

Proof. (i) Applying Proposition 2.3(vii), $[a] \leq [b]$ iff $[a] = [a] \wedge [b]$ iff $[a] = [a \wedge b]$ iff $D(a) = D(a \wedge b) = D(a) \cup D(b)$ iff $D(b) \subseteq D(a)$.
(ii) By Proposition 2.3(v), $a \leq b$ implies $D(b) \subseteq D(a)$. Apply now (i).
(iii) We have that $[a] = [b]$ iff $D(a) = D(b)$ iff $< a >=< b >$, by Proposition 2.3(iii).
(iv) By (ii) and Lemma 1.11, we get that $[a] = [1]$ iff $< a >=< 1 >$ iff $< a >= \{1\}$ iff $a = 1$.
(v), Again, by (ii). we get that $[a] = [0]$ iff $< a >=< 0 >= A$ iff $0 \in < a >$ iff $a^n = 0$ for some $n \in \omega - \{0\}$.
(vi), (vii) Apply Proposition 2.3(vii).
(viii) "$\Leftarrow$" By (ii).
"$\Rightarrow$" From $[e] \leq [a]$, we get that $[e \wedge a] = [e]$, so $< e \wedge a >=< e >$, by (iii). Hence, $e \wedge a \geq e$. Since, obviously, $e \wedge a \leq e$, we get that $e \wedge a = e$, that is, $e \leq a$. □

The lattice $\beta(A) = A/_{\equiv}$ will be called the *reticulation* of A.

Lemma 3.4. *Let* $h : A \to B$ *be a BL-morphism. For any* $a, b \in A$,

$$D(a) = D(b) \textit{ implies } D(h(a)) = D(h(b)).$$

Proof. Let $Q \in Spec(B)$. Applying Proposition 1.13(ii), we have that $h^{-1}(Q) \in Spec(A)$. It follows that $Q \in D(h(a))$ iff $h(a) \notin Q$ iff $a \notin h^{-1}(Q)$ iff $h^{-1}(Q) \in D(a)$ iff $h^{-1}(Q) \in D(b)$ iff $b \notin h^{-1}(Q)$ iff $h(b) \notin Q$ iff $Q \in D(h(b))$. Hence, $D(h(a)) = D(h(b))$. □

Lemma 3.5. *Let* A *be a BL-algebra,* F *a filter of* A *and* $a, b \in A$ *such that* $[a] = [b]$*. Then*

$$a \in F \textit{ iff } b \in F.$$

Proof. If $F = A$, it is obvious. Let us assume that F is a proper filter of A. Suppose that $a \in F$ and $b \notin F$. Applying Prime Filter Theorem with F and $S = \{b\}$, we get a prime filter P such that $F \subseteq P$ and $b \notin P$. Hence, $P \in D(b)$, but $P \notin D(a)$, since $a \in F \subseteq P$. We have got that $D(a) \neq D(b)$, i.e. $[a] \neq [b]$. This is a contradiction with the hypothesis. □

Let $h : A \to B$ be a BL-morphism. of BL-algebras and let us define

$$\beta(h) : \beta(A) \to \beta(B) \text{ by } \beta(h)[a] = [h(a)]$$

Proposition 3.6. $\beta(h)$ *is a bounded lattice morphism.*

Proof. If a, $b \in A$ then

$$\begin{aligned}\beta(h)([a]\wedge[b]) &= \beta(h)([a\wedge b]) = [h(a\wedge b)] = [h(a)\wedge h(b)]\\ &= [h(a)]\wedge[h(b)] = \beta(h)([a])\wedge\beta(h)([b]).\end{aligned}$$

We get similarly that $\beta(h)([a]\vee[b]) = \beta(h)([a])\vee\beta(h)([b])$. Finally, $\beta(h)([0]) = [h(0)] = [0]$ and $\beta(h)([1]) = [h(1)] = [1]$. □

Hence, we have defined a functor

$$\beta : \mathcal{BL} \to \mathcal{LD}01,$$

called the *reticulation functor*.

For any $F \in \mathcal{F}(A)$, let

$$\beta(F) = \{[a] \mid a \in F\}.$$

For any $H \in \mathcal{F}(\beta(A))$, let

$$H_* = \beta^{-1}(H).$$

Lemma 3.7. *Let A be a nontrivial BL-algebra. Then*
(i) if $F \in \mathcal{F}(A)$, then for any $a \in A$,
$[a] \in \beta(F)$ iff $a \in F$;
(ii) If $F \in \mathcal{F}(A)$, then $\beta(F) \in \mathcal{F}(\beta(A))$;
(iii) If $H \in \mathcal{F}(\beta(A))$, then $H_* \in \mathcal{F}(A)$;
(iv) If $F \in \mathcal{F}(A)$, then $(\beta(F))_* = F$;
(v) If $H \in \mathcal{F}(\beta(A))$, then $\beta(H_*) = H$.
(vi) If $F, G \in \mathcal{F}(A)$, then
$F \subseteq G$ iff $\beta(F) \subseteq \beta(G)$.

Proof. (i) Suppose that $[a] \in \beta(F)$. Then, there is $b \in F$ such that $[a] = [b]$. Applying now Lemma 3.5, it follows that $a \in F$, too.
(ii) We have that $1 \in F$, so $[1] \in \beta(F)$, hence, $\beta(F)$ is nonempty. Let $a, b \in A$ such that $[a], [b] \in \beta(F)$. Then, by (i), $a, b \in F$, so $a \wedge b \in F$, since F is a filter of A. It follows that $[a] \wedge [b] = [a \wedge b] \in \beta(F)$. Let $a, b \in A$ such that $[a] \leq [b]$ and $[a] \in \beta(F)$. It follows that $[a \vee b] = [a] \vee [b] = [b]$ and $a \vee b \in F$, since $a \leq a \vee b$ and $a \in F$. We have got that $[b] \in \beta(F)$. Hence, $\beta(F)$ is a filter of the lattice $\beta(A)$.
(iii) Since $[1] \in H$, it follows that $1 \in H_*$. Let $a, b \in H_*$, so $[a], [b] \in H$. By Proposition 3.3 (vii), we get $[a \odot b] = [a \wedge b] = [a] \wedge [b] \in H$, that is $a \odot b \in H_*$. Let $a \in H_*$ and $b \in A$ such that $a \leq b$. Applying Proposition 3.3 (ii), it follows that $[a] \leq [b]$ and, since $[a] \in H$, we get that $[b] \in H$, that is $b \in H_*$.
(iv) Let $a \in A$. By (i), we get that $a \in (\beta(F))_*$ iff $[a] \in \beta(F)$ iff $a \in F$. Hence, $(\beta(F))_* = F$.
(v) Let $a \in A$. By (i), it follows that $[a] \in \beta(H_*)$ iff $a \in H_*$ iff $[a] \in H$.
(vi) Applying (i), we get $F \subseteq G$ iff for any $a \in A$, $a \in F$ implies $a \in G$ iff for any $a \in A$, $[a] \in \beta(F)$ implies $[a] \in \beta(G)$ iff $\beta(F) \subseteq \beta(G)$. □

Proposition 3.8. *The mapping $F \mapsto \beta(F)$ is an isomorphism between the lattices $\mathcal{F}(A)$ and $\mathcal{F}(\beta(A))$.*

Proof. Let us define

$$u : \mathcal{F}(A) \to \mathcal{F}(\beta(A)), \quad u(F) = \beta(F)$$

for any filter F of A and

$$v : \mathcal{F}(\beta(A)) \to \mathcal{F}(A), \quad v(H) = H_*$$

for every filter H of $\beta(A)$. By Lemma 3.7 (ii) and (iii), it follows that u and v are well-defined. Applying Lemma 3.7 (iv) and (v), we get that u is a bijection and its inverse is v. Finally, from Lemma 3.7 (vi) we obtain that u is a lattice homomorphism. Hence, u is a bijective homomorphism of lattices, that is an isomorphism of lattices. □

Thus, the BL-algebra A and its associated lattice $\beta(A)$ have the same filter structure. This is not the case with the lattice $L(A) = (A, \wedge, \vee, 0, 1)$, whose filter structure is in general quite diferent from that of A. For example if A is the Łukasiewicz structure, then A has only two filters as a BL-algebra: $\{1\}$, and $[0, 1]$, while every interval $[x, 1]$, with $x \in [0, 1]$ is a filter of the lattice $L(A)$. Hence, the lattices $\beta(A)$ and $L(A)$ are in general different lattices.

Lemma 3.9. *Let A be a BL-algebra and $F \in \mathcal{F}(A)$. Then*
(i) F is a proper filter of A iff $\beta(F)$ is a proper filter of $\beta(A)$;
(ii) $F \in Spec(A)$ iff $\beta(P) \in Spec(\beta(A))$;
(ii) $F \in Max(A)$ iff $\beta(F) \in Max(\beta(A))$.

Proof. By Proposition 3.8, it follows that $F \in \mathcal{F}(A)$ iff $\beta(F) \in \mathcal{F}(\beta(A))$. In the proof, we shall apply more times Lemma 3.7 (i).
(i) F is a proper filter of A iff $0 \notin F$ iff $[0] \notin \beta(F)$ iff $\beta(F)$ is a proper filter of $\beta(A)$.
(ii) $F \in Spec(A)$ iff F is proper and for any $a, b \in A$, $a \vee b \in F$ implies $a \in F$ or $b \in F$ iff $\beta(F)$ is proper and for any $a, b \in A$, $[a \vee b] \in \beta(F)$ implies $[a] \in \beta(F)$ or $[b] \in \beta(F)$ iff $\beta(F)$ is proper and for any $a, b \in A$, $[a] \vee [b] \in \beta(F)$ implies $[a] \in \beta(F)$ or $[b] \in \beta(F)$ iff $\beta(F) \in Spec(\beta(A))$.
(iii) Applying (i) and Proposition 3.8, we get that $F \in Max(A)$ iff F is proper and for any proper filter G of A, $F \subseteq G$ implies $F = G$ iff $\beta(F)$ is proper and for any proper filter $\beta(G)$ of $\beta(A)$, $\beta(F) \subseteq \beta(G)$ implies $\beta(F) = \beta(G)$. Using now that any proper filter H of $\beta(A)$ is $\beta(G)$ for some proper filter G of A, we get that $F \in Max(A)$ iff $\beta(F)$ is proper and for any proper filter H of $\beta(A)$, $\beta(F) \subseteq H$ implies $\beta(F) = H$ iff $\beta(F) \in Max(\beta(A))$. □

Proposition 3.10. *The mapping $P \mapsto \beta(P)$ is a homeomorphism between the topological spaces $Spec(A)$ and $Spec(\beta(A))$.*

Proof. Let us consider the restriction of u to $Spec(A)$, denoted also by u. By Proposition 3.8 and Lemma 3.9(ii), we get that $u : Spec(A) \to Spec(\beta(A))$ is bijective. In order to obtain that u is a homeomorphism, we shall prove that u is continuous and open. Let $a \in A$. Then

$$\begin{aligned} u^{-1}(D([a])) &= \{P \in Spec(A) \mid u(P) \in D([a])\} \\ &= \{P \in Spec(A) \mid \beta(P) \in D([a])\} \\ &= \{P \in Spec(A) \mid [a] \notin \beta(P)\} \\ &= \{P \in Spec(A) \mid a \notin P\} \\ &= D(a). \end{aligned}$$

Hence, u is continuous.

$$\begin{aligned} u(D(a)) &= \{\beta(P) \mid P \in Spec(A),\ P \in D(a)\} \\ &= \{\beta(P) \mid P \in Spec(A),\ a \notin P\} \\ &= \{\beta(P) \mid P \in Spec(A),\ [a] \notin \beta(P)\} \\ &= \{T \in Spec(\beta(A)) \mid [a] \notin T\} \\ &= D([a]). \end{aligned}$$

We have got also that u is open. □

Proposition 3.11. *The mapping $M \mapsto \beta(M)$ is a homeomorphism between the topological spaces $Max(A)$ and $Max(\beta(A))$.*

Proof. We consider now the restriction of u to $Max(A)$, denoted also by u. By Proposition 3.8 and Lemma 3.9(iii), we get that $u : Max(A) \to Max(\beta(A))$ is bijective. By the proof of the above proposition, it follows that for any $a \in A$, $u^{-1}(D_{Max}([a])) = u^{-1}(D([a]) \cap Max(\beta(A))) = u^{-1}(D([a])) \cap Max(A) = D_{Max}(A)$, and $u(D_{Max}(a)) = u(D(a) \cap Max(A)) = u(D(A)) \cap Max(\beta(A)) = D([a]) \cap Max(\beta(A)) = D_{Max}([a])$. Hence, u is continuous and open. □

Let us remind that a bounded distributive lattice $(L, \wedge, \vee, 0, 1)$ is called *normal* [17, 4] if for all $a, b \in L$, $a \wedge b = 0$ implies there exist $u, v \in L$ such that $u \vee v = 1$ and $a \wedge u = b \wedge v = 0$. Normal lattices were introduced by Wallman[17] as an abstraction of the lattice of closed sets of a normal topological space.

The following proposition gives an equivalent characterization of normal lattices.

Proposition 3.12. [4]
Let L be a bounded distributive lattice. The following are equivalent:
(i) L is normal;
(ii) any prime filter of L is contained in a unique maximal filter of L.

Completely normal lattices were introduced as an abstraction of the lattice of closed sets of a completely normal topological space. Thus, a bounded distributive lattice L is called *completely normal* (or *relatively normal* in [4]) if each interval $[x, y]$ with $x < y$ is a normal lattice.

Proposition 3.13. [4, 11, 12]
Let L be a bounded distributive lattice.The following are equivalent:
(i) L is completely normal;
(ii) each proper filter of L which contains a prime filter is prime;
(iii) the set of filters of L including a given prime filter is linearly ordered by set-theoretical inclusion;
(iv) the set of prime filters of L including a given prime filter is linearly ordered by set-theoretical inclusion.

Proposition 3.14. *$\beta(A)$ is a normal and completely normal lattice.*

Proof. By Propositions 1.10, 3.8, and 3.12(ii), we get that $\beta(A)$ is a normal lattice. The fact that $\beta(A)$ is completely normal follows applying Propositions 3.8, 1.4, and 3.13(ii). □

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