

SOME CLASSES OF PSEUDO-BL ALGEBRAS

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Abstract

Pseudo-BL algebras are noncommutative generalizations of BL-algebras and they include pseudo-MV algebras, a class of structures that are categorically equivalent to l -groups with strong unit. In this paper we characterize directly indecomposable pseudo-BL algebras and we define and study different classes of these structures: local, good, perfect, peculiar, and (strongly) bipartite pseudo-BL algebras.

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Introduction

BL-algebras are the algebraic structures for Hájek's Basic Logic [14]. The main example of a BL-algebra is the interval $[0, 1]$ endowed with the structure induced by a t -norm. MV-algebras, Gödel algebras and product algebras are the most known classes of BL-algebras. Recent investigations are concerned with noncommutative generalizations for these structures.

In [4, 13], pseudo-BL algebras were defined as noncommutative generalizations of BL-algebras. The main source of examples of pseudo-BL algebras is l -group theory. In [10], there was introduced a notion of pseudo- t -norm in order to recapture some of the properties of pseudo-BL algebras. For the interval $[0, 1]$, this notion induces more general algebras named weak pseudo-BL algebras.

Pseudo-MV algebras were introduced as a noncommutative generalization of MV-algebras (see [11, 12]). Dvurecenskij proved in [9] that the category of pseudo-MV algebras is equivalent to the category of l -groups with strong unit. This theorem extends the fundamental result established by Mundici for the commutative case [16].

In [2], Belluce, Di Nola and Lettieri studied local MV-algebras, structures having a unique maximal ideal. An important class of local MV-algebras are perfect MV-algebras, which are MV-algebras generated by their radical. The category of perfect MV-algebras is equivalent to the category of abelian l -groups [6]. All these results were extended in [15] to pseudo-MV algebras. Following [2], in [19] local BL-algebras were defined and classified.

Bipartite MV-algebras, defined in [7], are another important class of MV-algebras. Bipartite BL-algebras and strongly bipartite BL-algebras were defined in [18]. In [8] bipartite BL-algebras were classified and it was proved that the

variety generated by perfect BL-algebras is exactly the variety of strongly bipartite BL-algebras. All these results are parallel to the ones already existing for MV-algebras (see [1, 7]).

In this paper we shall extend some of these results to pseudo-BL algebras. By [5], the congruences of a pseudo-BL algebra are in a bijective correspondence with the normal filters. Then, there are two possibilities to define a concept of *local* pseudo-BL algebra. The first one is to define a local pseudo-BL algebra as being a pseudo-BL algebra with a unique ultrafilter. This paper deals with this approach. Another way is to consider structures having a unique maximal normal filter. For the second case we obtain the notion of *normal local* pseudo-BL algebra. The investigation of normal local pseudo-BL algebras seems to be a difficult and only problem, since we do not have a characterization of the normal filter generated by a set of elements.

The paper is divided into four sections. In the first section we recall some facts concerning pseudo-BL algebras and pseudo-MV algebras and we prove some properties used in the sequel. Following [3], we characterize directly indecomposable pseudo-BL algebras.

In Section 2 we define and study local pseudo-BL algebras. Many of the results from local MV-algebras [2] and local BL-algebras [19] are extended to local pseudo-BL algebras.

In the next section we study good pseudo-BL algebras, an important class of pseudo-BL algebras. We associate with any good pseudo-BL algebra a pseudo-MV algebra in a natural way.

In Section 4 we investigate some classes of local pseudo-BL algebras, namely perfect, locally finite and peculiar pseudo-BL algebras. We give a classification of local pseudo-BL algebras and we give a simpler proof of the fact that locally finite pseudo-BL algebras are exactly locally finite MV-algebras.

In the last section of the paper, following [18] we study (strongly) bipartite pseudo-BL algebras.

1 Definitions and first properties

A *pseudo-BL algebra* ([4, 13]) is an algebra $\mathbf{A} = (A, \wedge, \vee, \odot, \rightsquigarrow, \rightarrow, 0, 1)$ with five binary operations $\wedge, \vee, \odot, \rightsquigarrow, \rightarrow$ and two constants $0, 1$ such that:

- (A1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
- (A2) $(A, \odot, 1)$ is a monoid;
- (A3) $a \odot b \leq c$ if and only if $a \leq b \rightsquigarrow c$ if and only if $b \leq a \rightarrow c$;
- (A4) $a \wedge b = (a \rightsquigarrow b) \odot a = a \odot (a \rightarrow b)$;
- (A5) $(a \rightsquigarrow b) \vee (b \rightsquigarrow a) = (a \rightarrow b) \vee (b \rightarrow a) = 1$.

In the sequel, we shall agree that the operations \wedge, \vee, \odot have priority towards the operations $\rightsquigarrow, \rightarrow$. Sometimes, we shall put parenthesis even if this is not necessary.

It is proved in [4] that commutative pseudo-BL algebras are BL-algebras. For

details on BL-algebras see [14, 17].

A pseudo-BL algebra \mathbf{A} is nontrivial if and only if $0 \neq 1$. For any pseudo-BL algebra \mathbf{A} , the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. A *pseudo-BL chain* is a linear pseudo-BL algebra, that is a pseudo-BL algebra such that its lattice order is total.

For any $a \in A$, we define $a^\sim = a \rightsquigarrow 0$ and $a^- = a \rightarrow 0$. We shall write a^\sim instead of $(a^\sim)^\sim$ and a^- instead of $(a^-)^-$.

We denote the set of natural numbers by ω . We define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for $n \in \omega - \{0\}$. The *order* of $a \in A$, in symbols $\text{ord}(a)$, is the smallest $n \in \omega$ such that $a^n = 0$. If no such n exists, then $\text{ord}(a) = \infty$.

The following properties hold in any pseudo-BL algebra \mathbf{A} and will be used in the sequel. See [4] for details.

- (1) $(a \odot b) \rightsquigarrow c = a \rightsquigarrow (b \rightsquigarrow c)$;
- (2) $(b \odot a) \rightarrow c = a \rightarrow (b \rightarrow c)$;
- (3) $a \leq b$ if and only if $a \rightsquigarrow b = 1$ if and only if $a \rightarrow b = 1$;
- (4) $a \leq b$ implies $a \odot c \leq b \odot c$ and $c \odot a \leq c \odot b$;
- (5) $a \odot b \leq a, b$;
- (6) $a \odot b \leq a \wedge b$;
- (7) $a \odot b = 0$ if and only if $a \leq b^\sim$ if and only if $b \leq a^-$;
- (8) $a \odot 0 = 0 \odot a = 0$;
- (9) $a^\sim \odot a = a \odot a^- = 0$;
- (10) $1 \rightsquigarrow a = 1 \rightarrow a = a$;
- (11) $a^\sim = 1$ if and only if $a^- = 1$ if and only if $a = 0$;
- (12) $1^\sim = 1^- = 0$;
- (13) $a \leq b$ implies $b^\sim \leq a^\sim$ and $b^- \leq a^-$;
- (14) $a \leq a^{\sim\sim}$ and $a \leq a^{-\sim}$;
- (15) $a \rightsquigarrow b \leq b^\sim \rightarrow a^\sim$ and $a \rightarrow b \leq b^- \rightsquigarrow a^-$;
- (16) $a^{\sim\sim\sim} = a^\sim$ and $a^{-\sim\sim} = a^-$;
- (17) $(a \odot b)^\sim = a \rightsquigarrow b^\sim$ and $(a \odot b)^- = b \rightarrow a^-$;
- (18) $(a \vee b)^\sim = a^\sim \wedge b^\sim$ and $(a \vee b)^- = a^- \wedge b^-$;
- (19) $(a \wedge b)^\sim = a^\sim \vee b^\sim$ and $(a \wedge b)^- = a^- \vee b^-$;
- (20) $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$;
- (21) $(b \vee c) \odot a = (b \odot a) \vee (c \odot a)$;
- (22) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Let \mathbf{A} be a pseudo-BL algebra. According to [4], a *filter* of \mathbf{A} is a nonempty subset F of A such that for all $a, b \in A$,

- (i) if $a, b \in F$, then $a \odot b \in F$;
- (ii) if $a \in F$ and $a \leq b$, then $b \in F$.

By (6), it is obvious that any filter of \mathbf{A} is also a filter of the lattice $L(A)$.

A filter F of \mathbf{A} is *proper* if $F \neq A$. A proper filter P of \mathbf{A} is *prime* if for all $a, b \in A$, $a \vee b \in P$ implies $a \in P$ or $b \in P$. We shall denote by $\text{Spec}(A)$ the set of prime filters of the pseudo-BL algebra \mathbf{A} .

A proper filter U of \mathbf{A} is an *ultrafilter* (or a *maximal filter*) if it is not contained

in any other proper filter. We shall denote by $\mathcal{M}(A)$ the intersection of all ultrafilters of \mathbf{A} . Obviously, $\mathcal{M}(A)$ is a proper filter of \mathbf{A} .

We recall some properties of filters that will be used in the sequel.

Proposition 1.1 ([4, Theorem 3.25])

Let F be a filter of the pseudo-BL algebra \mathbf{A} and let S be a \vee -closed subset of A (that is if $a, b \in S$, then $a \vee b \in S$) such that $F \cap S = \emptyset$. Then there exists a prime filter P of \mathbf{A} such that $F \subseteq P$ and $P \cap S = \emptyset$.

Proposition 1.2 *Any proper filter of \mathbf{A} can be extended to a prime filter.*

Proof: Apply [4, Corollary 3.26]. □

Proposition 1.3 ([4, Corollary 3.32])

Any ultrafilter of \mathbf{A} is a prime filter of \mathbf{A} .

Proposition 1.4 ([4, Remark 3.33])

Any proper filter of \mathbf{A} can be extended to an ultrafilter.

Proposition 1.5 *Let \mathbf{A} be a pseudo-BL algebra. The following are equivalent:*

- (i) \mathbf{A} is a pseudo-BL chain;
- (ii) any proper filter of \mathbf{A} is prime.

Lemma 1.6 *If \mathbf{A} is a pseudo-BL algebra, then the sets $A_0^\sim = \{a \in A \mid a^\sim = 0\}$ and $A_0^- = \{a \in A \mid a^- = 0\}$ are proper filters of \mathbf{A} .*

Proof: Let us prove that A_0^\sim is a proper filter of \mathbf{A} . By (12), $1 \in A_0^\sim$. Let $a, b \in A_0^\sim$, that is, $a^\sim = b^\sim = 0$. By (17), we get that $(a \odot b)^\sim = a \rightsquigarrow b^\sim = a \rightsquigarrow 0 = a^\sim = 0$, hence $a \odot b \in A_0^\sim$. Let $a \in A_0^\sim$ and $b \in A$ such that $a \leq b$. Then $a^\sim = 0$ and, by (13), $b^\sim \leq a^\sim$, so $b^\sim = 0$, that is, $b \in A_0^\sim$. Thus, A_0^\sim is a filter of \mathbf{A} . Since, by (11), $0^\sim = 1$, it follows that $0 \notin A_0^\sim$, hence A_0^\sim is proper. We obtain similarly that A_0^- is a proper filter of \mathbf{A} . □

Let $X \subseteq A$. The filter of \mathbf{A} generated by X will be denoted by $\langle X \rangle$. We have that $\langle \emptyset \rangle = \{1\}$ and $\langle X \rangle = \{a \in A \mid x_1 \odot \dots \odot x_n \leq a \text{ for some } n \in \omega - \{0\} \text{ and some } x_1, \dots, x_n \in X\}$ if $\emptyset \neq X \subseteq A$. For any $a \in A$, $\langle a \rangle$ denotes the principal filter of \mathbf{A} generated by $\{a\}$. Then, $\langle a \rangle = \{b \in A \mid a^n \leq b \text{ for some } n \in \omega - \{0\}\}$.

Lemma 1.7 *Let $a, b \in A$. Then*

- (i) $\langle a \rangle$ is proper if and only if $\text{ord}(a) = \infty$;
- (ii) if $a \leq b$ and $\text{ord}(b) < \infty$, then $\text{ord}(a) < \infty$;
- (iii) if $a \leq b$ and $\text{ord}(a) = \infty$, then $\text{ord}(b) = \infty$.

Proof: (i) $\langle a \rangle$ is proper if and only if $0 \notin \langle a \rangle$ if and only if $a^n \neq 0$ for all $n \in \omega - \{0\}$ if and only if $\text{ord}(a) = \infty$.

(ii), (iii) Applying (4), $a \leq b$ implies $a^n \leq b^n$ for all $n \in \omega$. \square

A filter H of \mathbf{A} is called *normal* ([5]) if for every $a, b \in A$ we have the equivalence:

$$(N) \quad a \rightsquigarrow b \in H \quad \text{if and only if} \quad a \rightarrow b \in H.$$

It is easy to see that $\{1\}$ and A are normal filters of the pseudo-BL algebra \mathbf{A} . We remark that if \mathbf{A} is a BL-algebra, then $\rightsquigarrow = \rightarrow$, so the notions of filter and normal filter coincide.

For a filter F of \mathbf{A} and $a \in A$, let us denote

$$a \odot F = \{a \odot x \mid x \in F\} \quad \text{and} \quad F \odot a = \{x \odot a \mid x \in F\}.$$

Proposition 1.8 ([5]) *Let H be a filter of \mathbf{A} . The following are equivalent:*

- (i) H is a normal filter;
- (ii) $a \odot H = H \odot a$ for any $a \in A$.

With any normal filter H of \mathbf{A} we can associate a congruence relation \equiv_H on \mathbf{A} by defining

$$a \equiv_H b \quad \text{if and only if} \quad (a \rightsquigarrow b) \odot (b \rightsquigarrow a) \in H \quad \text{if and only if} \quad (a \rightarrow b) \odot (b \rightarrow a) \in H.$$

In [5] it is proved that the map $H \mapsto \equiv_H$ is an isomorphism between the lattice of normal filters of \mathbf{A} and the lattice of congruences of \mathbf{A} .

If we denote by A/H the quotient set A/\equiv_H , then A/H becomes a pseudo-BL algebra \mathbf{A}/H with the natural operations induced from those of \mathbf{A} .

Proposition 1.9 ([5]) *Let H be a normal filter of \mathbf{A} . Then*

A/H is a pseudo-BL chain if and only if H is a prime filter of \mathbf{A} .

The following lemma is implicitly contained in [5].

Lemma 1.10 *Let H be a normal filter of \mathbf{A} and $a, b \in A$. Then*

- (i) $a/H = 1/H$ if and only if $a \in H$;
- (ii) $a/H = 0/H$ if and only if $a^\sim \in H$ if and only if $a^- \in H$;
- (iii) $a/H \leq b/H$ if and only if $a \rightsquigarrow b \in H$ if and only if $a \rightarrow b \in H$.

Proof: (i) $a/H = 1/H$ if and only if $(a \rightsquigarrow 1) \odot (1 \rightsquigarrow a) \in H$ if and only if $1 \odot (1 \rightsquigarrow a) \in H$ if and only if $a \in H$, since $a \rightsquigarrow 1 = 1$ and $1 \rightsquigarrow a = a$, by (3) and (10).

(ii) $a/H = 0/H$ if and only if $(a \rightsquigarrow 0) \odot (0 \rightsquigarrow a) \in H$ if and only if $a^\sim \odot 1 \in H$ if and only if $a^\sim \in H$. Applying (N), $a^\sim \in H$ if and only if $a \rightsquigarrow 0 \in H$ if and only if $a \rightarrow 0 \in H$ if and only if $a^- \in H$.

(iii) By (3) and (i), $a/H \leq b/H$ if and only if $a/H \rightsquigarrow b/H = 1/H$ if and only if $(a \rightsquigarrow b)/H = 1/H$ if and only if $a \rightsquigarrow b \in H$. By (N), we have that $a \rightsquigarrow b \in H$ if and only if $a \rightarrow b \in H$. \square

If $h : \mathbf{A} \rightarrow \mathbf{B}$ is a homomorphism of pseudo-BL algebras, then the *kernel* of h is the set $\text{Ker}(h) = \{a \in A \mid h(a) = 1\}$.

For any normal filter H of \mathbf{A} , let us denote by \square_H the natural homomorphism

from \mathbf{A} onto \mathbf{A}/H , defined by $[\]_H(a) = a/H$ for any $a \in A$. Then $H = \text{Ker}([\]_H)$. The following propositions are easily obtained.

Proposition 1.11 *Let $h : \mathbf{A} \rightarrow \mathbf{B}$ be a homomorphism of pseudo-BL algebras. Then the following properties hold:*

- (i) *for any (normal) filter G of \mathbf{B} , the set $h^{-1}(G) =_{def} \{a \in A \mid h(a) \in G\}$ is a (normal) filter of \mathbf{A} . Thus, in particular $\text{Ker}(h)$ is a normal filter of \mathbf{A} .*
- (ii) *h is injective if and only if $\text{Ker}(h) = \{1\}$.*

Proposition 1.12 *Let \mathbf{A} be a pseudo-BL algebra and H be a normal filter of \mathbf{A} .*

- (i) *The map $F \mapsto [\]_H(F)$ is an inclusion-preserving bijective correspondence between the filters of \mathbf{A} containing H and the filters of \mathbf{A}/H . The inverse map is also inclusion-preserving.*
- (ii) *F is a proper filter of \mathbf{A} containing H if and only if $[\]_H(F)$ is a proper filter of \mathbf{A}/H . Hence, there is a bijection between the proper filters of \mathbf{A} containing H and the proper filters of \mathbf{A}/H .*
- (iii) *There is a bijection between the ultrafilters of \mathbf{A} containing H and the ultrafilters of \mathbf{A}/H .*

Let \mathbf{A} be a pseudo-BL algebra and F be a filter of \mathbf{A} . We shall use the following notations:

$$F_{\sim}^* = \{a \in A \mid a \leq x^{\sim} \text{ for some } x \in F\}, \text{ and}$$

$$F_{-}^* = \{a \in A \mid a \leq x^{-} \text{ for some } x \in F\}.$$

Remark 1.13 *Let \mathbf{A} be a pseudo-BL algebra. Then*

- (i) $F_{\sim}^* = \{a \in A \mid a \odot x = 0 \text{ for some } x \in F\}$;
- (i') $F_{-}^* = \{a \in A \mid x \odot a = 0 \text{ for some } x \in F\}$;
- (ii) $F_{\sim}^* = \{a \in A \mid a^{-} \in F\}$;
- (ii') $F_{-}^* = \{a \in A \mid a^{\sim} \in F\}$.

Proof: (i), (i') Apply (7).

(ii) Let $a \in A$. If $a \leq x^{\sim}$ for some $x \in F$ then, by (13) and (14), we get that $x \leq x^{\sim-} \leq a^{-}$. Since F is a filter, it follows that $a^{-} \in F$. Conversely, suppose that $a^{-} \in F$. Then, $a \leq (a^{-})^{\sim}$, hence $a \in F_{\sim}^*$.

(ii') Similarly to (ii). □

For any pseudo-BL algebra \mathbf{A} , $B(A)$ denotes the Boolean algebra of all complemented elements in $L(A)$. Hence, $B(A) = B(L(A))$.

Proposition 1.14 ([5]) *Let \mathbf{A} be a pseudo-BL algebra and $e \in A$. The following are equivalent:*

- (i) $e \in B(A)$;
- (ii) $e \odot e = e$ and $e = e^{\sim-} = e^{-\sim}$;
- (iii) $e \odot e = e$ and $e^{\sim} \rightsquigarrow e = e$;

- (iii') $e \odot e = e$ and $e^- \rightarrow e = e$;
- (iv) $e \vee e^\sim = 1$;
- (iv') $e \vee e^- = 1$.

Lemma 1.15 ([5]) *Let \mathbf{A} be a pseudo-BL algebra and $e \in B(A)$. Then*

- (i) $\langle e \rangle = \{a \in A \mid e \leq a\}$;
- (ii) $e \odot a = e \wedge a$ for any $a \in A$;
- (iii) $e \vee (a \odot b) = (e \vee a) \odot (e \vee b)$ for any $a, b \in A$;
- (iv) $e^\sim = e^-$ is the complement of e .

A pseudo-BL algebra \mathbf{A} is called *directly indecomposable* if and only if \mathbf{A} is nontrivial and whenever $\mathbf{A} \cong \mathbf{A}_1 \times \mathbf{A}_2$ then either \mathbf{A}_1 or \mathbf{A}_2 is trivial.

In the sequel, in a similar manner with [3, Chapter 6.4], we shall give a characterization of directly indecomposable pseudo-BL algebras.

Let \mathbf{A} be a pseudo-BL algebra. For each $x \in A$, let the functions $\rightsquigarrow_x: A \times A \rightarrow A$, $\rightarrow_x: A \times A \rightarrow A$ and $h_x: A \rightarrow A$ defined by

$$a \rightsquigarrow_x b = x \vee (a \rightsquigarrow b), a \rightarrow_x b = x \vee (a \rightarrow b), \text{ and } h_x(a) = x \vee a.$$

Proposition 1.16 *Let \mathbf{A} be a pseudo-BL algebra and $e \in B(A)$. Then*

- (i) $\langle \mathbf{e} \rangle = (\langle e \rangle, \wedge, \vee, \odot, \rightsquigarrow_e, \rightarrow_e, e, 1)$ is a pseudo-BL algebra;
- (ii) $h_e(A) = \langle e \rangle$;
- (iii) h_e is a homomorphism of pseudo-BL algebras from \mathbf{A} onto $\langle \mathbf{e} \rangle$;
- (iv) $\text{Ker}(h_e) = \langle e^- \rangle$;
- (v) $\langle \mathbf{e} \rangle$ is nontrivial if and only if $e \neq 1$;
- (vi) $\langle e \rangle$ is a subalgebra of \mathbf{A} if and only if $e=0$ if and only if $\langle e \rangle = A$;
- (vii) $B(\langle e \rangle) = \langle e \rangle \cap B(A)$.

Proof: (i) By Lemma 1.15(i), we have that $\langle e \rangle = \{a \in A \mid e \leq a\}$. Let us verify the axioms from the definition of a pseudo-BL algebra.

(A1) It follows immediately that $(\langle e \rangle, \wedge, \vee, e, 1)$ is a bounded lattice.

(A2) Since $\langle e \rangle$ is a filter of \mathbf{A} , $\langle e \rangle$ is \odot -closed and, obviously, $(\langle e \rangle, \odot, 1)$ is a monoid.

(A3) Let $a, b, c \geq e$. If $a \odot b \leq c$, then $a \leq b \rightsquigarrow c \leq e \vee (b \rightsquigarrow c) = b \rightsquigarrow_e c$ and $b \leq a \rightarrow c \leq e \vee (a \rightarrow c) = a \rightarrow_e c$.

Conversely, let us suppose that $a \leq b \rightsquigarrow_e c$, that is $a \leq e \vee (b \rightsquigarrow c)$. Applying (4), (21), Lemma 1.15(ii) and (A4), we get that $a \odot b \leq [e \vee (b \rightsquigarrow c)] \odot b = (e \odot b) \vee [(b \rightsquigarrow c) \odot b] = (e \wedge b) \vee (b \wedge c) = e \vee (b \wedge c) = b \wedge c \leq c$.

Now, let us suppose that $b \leq a \rightarrow_e c$, so $b \leq e \vee (a \rightarrow c)$. Then, by (4), (20), Lemma 1.15(ii) and (A4), $a \odot b \leq a \odot [e \vee (a \rightarrow c)] = (a \odot e) \vee [a \odot (a \rightarrow c)] = (a \wedge e) \vee (a \wedge c) = e \vee (a \wedge c) = a \wedge c \leq c$.

(A4) Let $a, b \geq e$. We have that $(a \rightsquigarrow_e b) \odot a = [e \vee (a \rightsquigarrow b)] \odot a = (e \odot a) \vee [(a \rightsquigarrow b) \odot a] = (e \wedge a) \vee (a \wedge b) = e \vee (a \wedge b) = a \wedge b$ and, similarly, $a \odot (a \rightarrow_e b) = a \odot [e \vee (a \rightarrow b)] = (a \odot e) \vee [a \odot (a \rightarrow b)] = (a \wedge e) \vee (a \wedge b) = a \wedge b$.

(A5) Let $a, b \in A$. By (A5), we get that $(a \rightsquigarrow_e b) \vee (b \rightsquigarrow_e a) = e \vee (a \rightsquigarrow b) \vee e \vee (b \rightsquigarrow a) = e \vee 1 = 1$ and, similarly, $(a \rightarrow_e b) \vee (b \rightarrow_e a) = e \vee (a \rightarrow b) \vee e \vee (b \rightarrow a) = e \vee 1 = 1$.

$$b) \vee e \vee (b \rightarrow a) = e \vee 1 = 1.$$

Hence, $\langle e \rangle, \wedge, \vee, \odot, \rightsquigarrow_e, \rightarrow_e, e, 1$ is a pseudo-BL algebra.

(ii) For any $a \in \langle e \rangle$, we have that $h_e(a) = e \vee a = a$. Hence, $\langle e \rangle \subseteq h_e(A)$. The other inclusion is obvious.

(iii) Let $a, b \in A$. It follows immediately that $h_e(a \rightsquigarrow b) = e \vee (a \rightsquigarrow b) = a \rightsquigarrow_e b$, $h_e(a \rightarrow b) = e \vee (a \rightarrow b) = a \rightarrow_e b$, $h_e(0) = 0 \vee e = e$, $h_e(1) = e \vee 1 = 1$, $h_e(a \vee b) = e \vee (a \vee b) = h_e(a) \vee h_e(b)$. By (22), $h_e(a \wedge b) = e \vee (a \wedge b) = (e \vee a) \wedge (e \vee b) = h_e(a) \wedge h_e(b)$. Applying Lemma 1.15(iii), we also get that $h_e(a \odot b) = e \vee (a \odot b) = (e \vee a) \odot (e \vee b) = h_e(a) \odot h_e(b)$.

(iv) If $a \in \text{Ker}(h_e)$, then $h_e(a) = a \vee e = 1$, so $e^- = e^- \wedge (a \vee e) = (e^- \wedge a) \vee 0 = e^- \wedge a$. It follows that $a \geq e^-$, hence $a \in \langle e^- \rangle$. Conversely, if $a \geq e^-$, we get that $h_e(a) = e \vee a \geq e \vee e^- = 1$, hence $h_e(a) = 1$, that is $a \in \text{Ker}(h_e)$.

(v),(vi) They are obvious.

(vii) Let $a \in \langle e \rangle$, that is $e \leq a$. If $a \in B(\langle e \rangle)$, then there is $b \geq e$ such that $a \wedge b = e$ and $a \vee b = 1$. Taking $c = b \wedge e^\sim$, we get that $a \wedge c = 0$ and $a \vee c = a \vee (b \wedge e^\sim) = (a \vee b) \wedge (a \vee e^\sim) = 1 \wedge (a \vee e^\sim) = a \vee e^\sim \geq e \vee e^\sim = 1$, by (22) and Lemma 1.15(iv). Conversely, suppose that $a \in B(A)$, hence there is $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$. Let $c = e \vee b$. Then $c \geq e$ and $a \vee c = 1$, $a \wedge c = a \wedge (e \vee b) = (a \wedge e) \vee (a \wedge b) = e \vee 0 = e$. \square

Proposition 1.17 *Let $\{\mathbf{A}_i\}_{i \in I}$ be a nonempty family of pseudo-BL algebras and let $\mathbf{P} = \prod_{i \in I} \mathbf{A}_i$. Then there is a set $\{\delta_i \mid i \in I\} \subseteq B(\mathbf{P})$ satisfying the following conditions:*

- (i) $\bigwedge_{i \in I} \delta_i = 0$;
- (ii) $\delta_i \vee \delta_j = 1$, whenever $i \neq j$;
- (iii) each \mathbf{A}_i is isomorphic to $\langle \delta_i \rangle$.

Proof: Similar to the proof of [3, Lemma 6.4.4]. \square

Proposition 1.18 *Let \mathbf{A} be a pseudo-BL algebra and $e_1, \dots, e_n \in B(A)$, $n \geq 2$ such that*

- (i) $e_1 \wedge \dots \wedge e_n = 0$, and
 - (ii) $e_i \vee e_j = 1$ for $i \neq j$, $i, j = 1, \dots, n$.
- Then $\mathbf{A} \cong \langle \mathbf{e}_1 \rangle \times \dots \times \langle \mathbf{e}_n \rangle$.*

Proof: Similar to the proof of [3, Lemma 6.4.5]. \square .

Proposition 1.19 *A pseudo-BL algebra \mathbf{A} is directly indecomposable if and only if $B(A) = \{0, 1\}$.*

Proof: Similar to the proof of [3, Theorem 6.4.7]. \square

It follows immediately that

Proposition 1.20 *Any pseudo-BL chain is directly indecomposable.*

Proof: Let \mathbf{A} be a pseudo-BL chain and $e \in B(A)$. By Proposition 1.14, we get that $e \vee e^\sim = 1$. But $e \leq e^\sim$ or $e^\sim \leq e$, hence $e = 1$ or $e^\sim = 1$. By (11), it follows that $e \in \{0, 1\}$. \square

In the sequel we shall recall some facts about pseudo-MV algebras, which are noncommutative generalizations of MV-algebras (see [11, 12]).

A *pseudo-MV algebra* is an algebra $(A, \oplus, ^-, \sim, 0, 1)$ with one binary operation \oplus , two unary operations $^-, \sim$ and two constants $0, 1$ such that:

- (i) $(A, \oplus, 0)$ is a monoid;
- (ii) $a \oplus 1 = 1 \oplus a = a$;
- (iii) $1^\sim = 1^- = 0$;
- (iv) $(a^- \oplus b^-)^\sim = (a^\sim \oplus b^\sim)^-$;
- (v) $a \oplus (a^\sim \odot b) = b \oplus (b^\sim \odot a) = (a \odot b^-) \oplus b = (b \odot a^-) \oplus a$;
- (vi) $a \odot (a^- \oplus b) = (a \oplus b^\sim) \odot b$;
- (vii) $a^{-\sim} = a$,

where $a \odot b \stackrel{def}{=} (b^- \oplus a^-)^\sim$.

Let \mathbf{A} be a pseudo-MV algebra. On A one can define an order relation " \leq " by $a \leq b$ if and only if $a^- \oplus b = 1$ if and only if $b \oplus a^\sim = 1$.

Proposition 1.21 ([11, Proposition 1.13])

Let \mathbf{A} be a pseudo-MV algebra. Then (A, \leq) is a lattice in which for all $a, b \in A$, $a \vee b = a \oplus (a^\sim \odot b) = b \oplus (b^\sim \odot a) = (a \odot b^-) \oplus b = (b \odot a^-) \oplus a$, and $a \wedge b = (a \oplus b^\sim) \odot b = (b \oplus a^\sim) \odot a = a \odot (a^- \oplus b) = b \odot (b^- \oplus a)$.

For any $a \in A$, we define $0a = 0$ and $na = (n-1)a \oplus a$ for $n \in \omega - \{0\}$. The *MV-order* of $a \in A$, in symbols $\text{MV-ord}(a)$, is the smallest $n \in \omega$ such that $na = 1$. If no such n exists, then $\text{MV-ord}(a) = \infty$.

Lemma 1.22 ([15, Lemma 14])

Let \mathbf{A} be a pseudo-MV algebra. For any $a \in A$, $\text{MV-ord}(a^-) = \text{MV-ord}(a^\sim)$.

We shall denote by $D(A)$ the set $\{a \in A \mid \text{MV-ord}(a) = \infty\}$.

A pseudo-MV algebra \mathbf{A} is *locally finite* if for all $a \in A$,

$$a \neq 0 \text{ implies } \text{MV-ord}(a) < \infty.$$

According to [15], a pseudo-MV algebra \mathbf{A} is *strong* if for all $a \in A$,

$$a^- = a^\sim.$$

According to [11], an *ideal* of \mathbf{A} is a nonempty subset I of A such that for all $a, b \in A$,

- (i) if $a, b \in I$, then $a \oplus b \in I$;
- (ii) if $b \in I$ and $a \leq b$, then $a \in I$.

An ideal I is *proper* if $I \neq A$. A proper ideal of \mathbf{A} is called a *maximal ideal* if it is not contained in any other proper ideal.

An ideal H of a pseudo-MV algebra \mathbf{A} is called *normal* (see [12]) if for all $a, b \in A$,

$$a^\sim \odot b \in H \text{ if and only if } b \odot a^- \in H.$$

Lemma 1.23 ([12, Lemma 3.2])

Let H be a normal ideal of \mathbf{A} and $a \in A$. Then

$a \in H$ if and only if $a^- \in H$ if and only if $a^\sim \in H$.

Proposition 1.24 ([4, Corollary 2.34])

A pseudo-BL algebra \mathbf{A} is a pseudo-MV algebra if and only if

$a^{\sim-} = a^{-\sim} = a$ for all $a \in A$.

Following [2], in [15] local pseudo-MV algebras were defined and some classes of local pseudo-MV algebras were studied. Thus, a pseudo-MV algebra is *local* if and only if it has a unique maximal ideal and a local pseudo-MV algebra is:

-*perfect* if for any $a \in A$, $\text{MV-ord}(a) < \infty$ if and only if $\text{MV-ord}(a^-) = \infty$;

-*singular* if there are $a, b \in A$ such that $\text{MV-ord}(a) < \infty$, $\text{MV-ord}(b) < \infty$ and $\text{MV-ord}(a \odot b) = \infty$.

By Lemma 1.22, it follows that a local pseudo-MV algebra \mathbf{A} is perfect if and only if for any $a \in A$, $\text{MV-ord}(a) < \infty$ if and only if $\text{MV-ord}(a^\sim) = \infty$

Proposition 1.25 ([15]) *Every local pseudo-MV algebra is either perfect or singular. There is no local pseudo-MV algebra which is both perfect and singular.*

Proposition 1.26 ([15]) *Every locally finite pseudo-MV algebra different from $\{0, 1\}$ is singular.*

2 Local pseudo-BL algebras

Local rings play an important role in ring theory. On the other hand, the study of local objects became a standard problem for other classes of structures (MV-algebras [2], BL-algebras [19], pseudo-MV algebras [15]). In this section we shall study local pseudo-BL algebras.

A pseudo-BL algebra is called *local* if and only if it has a unique ultrafilter.

Lemma 2.1 *Let \mathbf{A} be a local pseudo-BL algebra. Then*

(i) *any proper filter of \mathbf{A} is included in the unique ultrafilter of \mathbf{A} ;*

(ii) *A_0^\sim, A_0^- are included in the unique ultrafilter of \mathbf{A} .*

Proof: (i) Apply Proposition 1.4 and the fact that \mathbf{A} has a unique ultrafilter.

(ii) Apply Lemma 1.6 and (i). \square

In the sequel, we shall use the following notations:

$D(A) = \{a \in A \mid \text{ord}(a) = \infty\}$, and

$D(A)^* = \{a \in A \mid \text{ord}(a) < \infty\}$.

Obviously, $D(A) \cap D(A)^* = \emptyset$ and $D(A) \cup D(A)^* = A$.

Proposition 2.2 *Let \mathbf{A} be a pseudo-BL algebra. The following are equivalent:*

(i) *$D(A)$ is a filter of \mathbf{A} ;*

- (ii) $D(A)$ is a proper filter of \mathbf{A} ;
- (iii) \mathbf{A} is local;
- (iv) $D(A)$ is the unique ultrafilter of \mathbf{A} ;
- (v) for all $a, b \in A$,
 $\text{ord}(a \odot b) < \infty$ implies $\text{ord}(a) < \infty$ or $\text{ord}(b) < \infty$.

Proof: (i) if and only if (ii) We have that $\text{ord}(0) = 1$, hence $0 \notin D(A)$.
(i) implies (v) Let $a, b \in A$ such that $\text{ord}(a \odot b) < \infty$, so $a \odot b \notin D(A)$. Since $D(A)$ is a filter of \mathbf{A} , we get that $a \notin D(A)$ or $b \notin D(A)$. Hence, $\text{ord}(a) < \infty$ or $\text{ord}(b) < \infty$.
(v) implies (i) Since $1 \in D(A)$, we have that $D(A)$ is nonempty. Let $a, b \in D(A)$, that is $\text{ord}(a) = \text{ord}(b) = \infty$. It follows that $\text{ord}(a \odot b) = \infty$, that is $a \odot b \in D(A)$. If $a \leq b$ and $a \in D(A)$, then $a^n > 0$ for all $n \in \omega$. Since $a^n \leq b^n$, we have that $b^n > 0$ for all $n \in \omega$. That is, $\text{ord}(b) = \infty$, hence $b \in D(A)$. Thus, we have proved that $D(A)$ is a filter of \mathbf{A} .
(iv) implies (iii) It is immediate.
(iii) implies (iv) Let U be the unique ultrafilter of \mathbf{A} . Applying Lemma 2.1(i) and Lemma 1.7(i), we get that $a \in U$ if and only if $\langle a \rangle \subseteq U$ if and only if $\langle a \rangle$ is proper if and only if $\text{ord}(a) = \infty$ if and only if $a \in D(A)$. Hence, $U = D(A)$.
(iv) implies (i) It is obvious.
(i) implies (iv) Since $0 \notin D(A)$, we have that $D(A)$ is proper. Let F be a proper filter of \mathbf{A} . If $a \in F$, then $\langle a \rangle \subseteq F$, so $\langle a \rangle$ is a proper filter of \mathbf{A} . Applying Lemma 1.7(i), it follows that $\text{ord}(a) = \infty$, hence $a \in D(A)$. Thus, we have got that any proper filter F of \mathbf{A} is included in $D(A)$. From this fact it follows that $D(A)$ is the unique ultrafilter of \mathbf{A} . \square

Corollary 2.3 *Let \mathbf{A} be a local pseudo-BL algebra. Then*

- (i) for any $a \in A$, $\text{ord}(a) < \infty$ or $(\text{ord}(a^\sim) < \infty$ and $\text{ord}(a^-) < \infty)$;
- (ii) $D(A)^\sim_* \subseteq D(A)^*$ and $D(A)^*_+ \subseteq D(A)^*$;
- (iii) $D(A) \cap D(A)^\sim_* = D(A) \cap D(A)^*_+ = \emptyset$.

Proof: (i) Let $a \in A$. By (9), we have that $a^\sim \odot a = a \odot a^- = 0$, so $\text{ord}(a^\sim \odot a) = \text{ord}(a \odot a^-) = \text{ord}(0) = 1 < \infty$. Apply now Proposition 2.2(v) to get (i).
(ii) Let $a \in D(A)^\sim_*$, so there is $x \in D(A)$ such that $a \leq x^\sim$. Since $\text{ord}(x) = \infty$, applying (i), we get that $\text{ord}(x^\sim) < \infty$. Applying now Lemma 1.7(ii), we get that $\text{ord}(a) < \infty$. Hence, $a \in D(A)^*$. We obtain similarly that $D(A)^*_+ \subseteq D(A)^*$.
(iii) Apply (ii) and the fact that $D(A) \cap D(A)^* = \emptyset$. \square

Proposition 2.4 *Any pseudo-BL chain is a local pseudo-BL algebra.*

Proof: Let \mathbf{A} be a pseudo-BL chain. We apply Proposition 2.2(v) to obtain that \mathbf{A} is local. Let $a, b \in A$ such that $\text{ord}(a \odot b) < \infty$. Since \mathbf{A} is a chain, we have that $a \leq b$ or $b \leq a$. Suppose that $a \leq b$. Then $a \odot a \leq a \odot b$, so, by

Lemma 1.7(ii), we get that $\text{ord}(a \odot a) < \infty$, hence $\text{ord}(a) < \infty$. Similarly, from $b \leq a$ it follows that $\text{ord}(b) < \infty$. \square

A proper normal filter P of a pseudo-BL algebra \mathbf{A} is called *primary* if for all $a, b \in A$,

$((a \odot b)^n)^\sim \in P$ for some $n \in \omega$ implies $(a^m)^\sim \in P$ or $(b^m)^\sim \in P$ for some $m \in \omega$.

Applying the definition of a normal filter, we get that a proper normal filter P of \mathbf{A} is primary if and only if for all $a, b \in A$,

$((a \odot b)^n)^- \in P$ for some $n \in \omega$ implies $(a^m)^- \in P$ or $(b^m)^- \in P$ for some $m \in \omega$.

Remark 2.5 *Suppose that \mathbf{A} is a BL-algebra and let P be a proper filter of \mathbf{A} . The following are equivalent:*

(i) P is primary;

(ii) for all $a, b \in A$,

$(a \odot b)^- \in P$ implies $(a^m)^- \in P$ or $(b^m)^- \in P$ for some $m \in \omega$.

Proof: (i) implies (ii) It follows immediately from the definition of a primary filter.

(ii) implies (i) Let $a, b \in A$ such that $((a \odot b)^n)^- \in P$ for some $n \in \omega$. Since \odot is commutative, we get that $((a \odot b)^n)^- = (a^n \odot b^n)^- \in P$. Applying now (ii), it follows that there is $p \in \omega$ such that $(a^{np})^- \in P$ or $(b^{np})^- \in P$. Hence, letting $m = np$, we have that $(a^m)^- \in P$ or $(b^m)^- \in P$. \square

Hence, in the case that \mathbf{A} is a BL-algebra, the notion of primary filter defined here coincides with the notion of primary filter defined in [19].

Proposition 2.6 *Let \mathbf{A} be a pseudo-BL algebra and P be a proper normal filter of \mathbf{A} . The following are equivalent:*

(i) \mathbf{A}/P is a local pseudo-BL algebra;

(ii) P is a primary filter of \mathbf{A} .

Proof: Applying Proposition 2.2(v) and Lemma 1.10(ii), we have that \mathbf{A}/P is local if and only if

for all $a, b \in A$, $\text{ord}(a/P \odot b/P) < \infty$ implies $\text{ord}(a/P) < \infty$ or $\text{ord}(b/P) < \infty$ if and only if for all $a, b \in A$, $(a/P \odot b/P)^n = 0/P$ for some $n \in \omega$ implies $(a/P)^m = 0/P$ or $(b/P)^m = 0/P$ for some $m \in \omega$ if and only if

for all $a, b \in A$, $((a \odot b)^n)/P = 0/P$ for some $n \in \omega$ implies $a^m/P = 0/P$ or $b^m/P = 0/P$ for some $m \in \omega$ if and only if

for all $a, b \in A$, $((a \odot b)^n)^\sim \in P$ for some $n \in \omega$ implies $(a^m)^\sim \in P$ or $(b^m)^\sim \in P$ for some $m \in \omega$ if and only if P is primary. \square

Proposition 2.7 *Any prime normal filter of a pseudo-BL algebra \mathbf{A} is primary.*

Proof: Let P be a prime normal filter of \mathbf{A} . Applying Proposition 1.9, we get that \mathbf{A}/P is a pseudo-BL chain, hence \mathbf{A}/P is local, by Proposition 2.4. Apply now Proposition 2.6 to get that P is primary. \square

Proposition 2.8 *Let \mathbf{A} be a pseudo-BL algebra. A proper normal filter of \mathbf{A} is primary if and only if it is contained in a unique ultrafilter of \mathbf{A} .*

Proof: Let H be a proper normal filter of \mathbf{A} . By Proposition 2.6, H is primary if and only if \mathbf{A}/H is a local algebra if and only if \mathbf{A}/H has a unique ultrafilter. Applying Proposition 1.12(iii), there is a bijection between the set of ultrafilters of \mathbf{A}/H and the set of ultrafilters of \mathbf{A} that contain H . Hence, H is primary if and only if there is a unique ultrafilter of \mathbf{A} that contains H . \square

Proposition 2.9 *Let \mathbf{A} be a pseudo-BL algebra. The following are equivalent:*

- (i) \mathbf{A} is local;
- (ii) any proper normal filter of \mathbf{A} is primary;
- (iii) $\{1\}$ is a primary filter of \mathbf{A} .

Proof: (i) implies (ii) Let H be a proper normal filter of \mathbf{A} . Since \mathbf{A} is local, by Lemma 2.1(i) and Proposition 2.2(iv) it follows that $D(A)$ is the unique ultrafilter of \mathbf{A} containing H . Applying Proposition 2.8, we get that H is primary.

(ii) implies (iii) Apply the fact that $\{1\}$ is a proper normal filter of \mathbf{A} .

(iii) implies (i) Since $\{1\}$ is a primary filter of \mathbf{A} , by Proposition 2.6, we get that $\mathbf{A}/\{1\}$ is local. But $\mathbf{A} \cong \mathbf{A}/\{1\}$, hence \mathbf{A} is local. \square

Proposition 2.10 *Any local pseudo-BL algebra is directly indecomposable.*

Proof: Let \mathbf{A} be a local pseudo-BL algebra. We shall prove that $B(A) = \{0, 1\}$ and then apply Proposition 1.19. Let $e \in B(A)$. Applying Corollary 2.3(i), we get that $\text{ord}(e) < \infty$ or $\text{ord}(e^\sim) < \infty$, that is there is $n \in \omega - \{0\}$ such that $e^n = 0$ or $(e^\sim)^n = 0$. But $e^n = e$ and $(e^\sim)^n = e^\sim$, by Proposition 1.14(ii) and the fact that e^\sim is the complement of e , so $e, e^\sim \in B(A)$. It follows that $e = 0$ or $e^\sim = 0$. By Proposition 1.14(ii) and (11), from $e^\sim = 0$ we get that $e = (e^\sim)^- = 0^- = 1$. That is, $e \in \{0, 1\}$. Hence, $B(A) = \{0, 1\}$. \square

3 Good pseudo-BL algebras

A *good* pseudo-BL algebra is a pseudo-BL algebra \mathbf{A} satisfying the following identity

$$(*) \quad a^{\sim-} = a^{-\sim}.$$

Pseudo-MV algebras are particular cases of good pseudo-BL algebras. In [5] it is proved that any pseudo-product algebra is also a good pseudo-BL algebra.

A *strong* pseudo-BL algebra is a pseudo-BL algebra \mathbf{A} such that

$$a^\sim = a^- \text{ for all } a \in A.$$

Obviously, every strong pseudo-BL algebra is a good pseudo-BL algebra.

In the sequel, if not otherwise specified, \mathbf{A} is a good pseudo-BL algebra. Let us consider the subset

$$M(A) = \{a \in A \mid a^{\sim-} = a^{-\sim} = a\}.$$

Lemma 3.1 *Let \mathbf{A} be a good pseudo-BL algebra. Then*

- (i) $0, 1 \in M(A)$;
- (ii) $a^\sim, a^- \in M(A)$ for all $a \in A$;
- (iii) if $a, b \in M(A)$, then $a \rightsquigarrow b = b^\sim \rightarrow a^\sim$ and $a \rightarrow b = b^- \rightsquigarrow a^-$;
- (iv) if $a, b \in M(A)$, then $(a^\sim \odot b^\sim)^- = (a^- \odot b^-)^\sim = a^- \rightsquigarrow b = b^\sim \rightarrow a$.

Proof: (i) Apply (11) and (12).

(ii) Let $a \in A$. Applying (*) for a^\sim and a^- and (16), we have that $(a^\sim)^\sim^- = (a^\sim)^{-\sim} = a^\sim$ and $(a^-)^{-\sim} = (a^-)^\sim^- = a^-$. It follows that $a^\sim, a^- \in M(A)$.

(iii), (iv) See [4, Lemma 2.31]. \square

For any $a, b \in A$, let us define

$$a \oplus b \stackrel{def}{=} (b^\sim \odot a^\sim)^-.$$

Lemma 3.2 *Let \mathbf{A} be a good pseudo-BL algebra. Then*

- (i) $a \oplus b \in M(A)$ for any $a, b \in A$;
- (ii) if $a, b \in M(A)$, then $a \oplus b = (b^\sim \odot a^\sim)^- = (b^- \odot a^-)^\sim = b^- \rightsquigarrow a = a^\sim \rightarrow b$;
- (iii) if $a, b \in M(A)$, then $a \oplus b^- = a^\sim \rightarrow b^-$, $a \oplus b^\sim = b \rightsquigarrow a$, $a^- \oplus b = a \rightarrow b$ and $a^\sim \oplus b = b^- \rightsquigarrow a^\sim$;
- (iv) if $a, b \in M(A)$, then $a^\sim \oplus b^\sim = (b \odot a)^\sim$ and $a^- \oplus b^- = (b \odot a)^-$.

Proof: (i) Apply Lemma 3.1(ii).

(ii) Apply Lemma 3.1(iv).

(iii) Apply (ii).

(iv) By (iii), (1) and (2), we have that $a^\sim \oplus b^\sim = b \rightsquigarrow a^\sim = b \rightsquigarrow (a \rightsquigarrow 0) = (b \odot a) \rightsquigarrow 0 = (b \odot a)^\sim$ and $a^- \oplus b^- = a \rightarrow b^- = a \rightarrow (b \rightarrow 0) = (b \odot a) \rightarrow 0 = (b \odot a)^-$. \square

The following proposition extends a result from [19].

Proposition 3.3 *Let \mathbf{A} be a good pseudo-BL algebra. The structure*

$$\mathbf{M}(\mathbf{A}) = (M(A), \oplus, ^-, ^\sim, 0, 1)$$

is a pseudo-MV algebra. The order on \mathbf{A} agrees with the one of $\mathbf{M}(\mathbf{A})$, defined by $a \leq_{M(A)} b$ if and only if $a^- \oplus b = 1$.

Proof: By Lemma 3.1 and Lemma 3.2, it follows that the operations $\oplus, ^-, ^\sim$ are well defined on $M(A)$ and that $0, 1 \in M(A)$. Let us denote by $\odot_{M(A)}$ the product on $M(A)$. Hence, for all $a, b \in M(A)$ we have that $a \odot_{M(A)} b = (b^- \oplus a^-)^\sim = (b^\sim \oplus a^\sim)^- \in M(A)$. We shall verify the axioms from the definition of a pseudo-MV algebra. In the proof we shall use without mention Lemmas 3.1 and 3.2. Let $a, b, c \in M(A)$.

(i) We have that $(a \oplus b) \oplus c = (b^\sim \odot a^\sim)^- \oplus (c^\sim)^- = (c^\sim \odot (b^\sim \odot a^\sim))^- = ((c^\sim \odot b^\sim) \odot a^\sim)^- = (a^\sim)^- \oplus (c^\sim \odot b^\sim)^- = a \oplus (b \oplus c)$. We also get that $a \oplus 0 = (0^\sim \odot a^\sim)^- = (1 \odot a^\sim)^- = a^\sim^- = a$. Similarly, $0 \oplus a = a$. Hence $(M(A), \oplus, 0)$ is a monoid.

(ii) By (8), (11) and (12), $a \oplus 1 = (1^\sim \odot a^\sim)^- = (0 \odot a^\sim)^- = 0^- = 1$. We obtain

similarly that $1 \oplus a = 1$

(iii) Apply (12).

(iv) By (*), $(a^- \oplus b^-)^\sim = (b \odot a)^{-\sim} = (b \odot a)^{\sim-} = (a^\sim \oplus b^\sim)^-$.

(v) We have to prove that $a \oplus (a^\sim \odot_{M(A)} b) = b \oplus (b^\sim \odot_{M(A)} a) = (a \odot_{M(A)} b^-) \oplus b = (b \odot_{M(A)} a^-) \oplus a$. Applying (18) and (A4), we get that

$$\begin{aligned} a \oplus (a^\sim \odot_{M(A)} b) &= a \oplus (b^- \oplus a)^\sim = (a^-)^\sim \oplus (b^- \oplus a)^\sim = ((b^- \oplus a) \odot a^-)^\sim = \\ &= ((b \rightarrow a) \odot a^-)^\sim = ((a^- \rightsquigarrow b^-) \odot a^-)^\sim = (a^- \wedge b^-)^\sim = a^{-\sim} \vee b^{-\sim} = a \vee b \text{ and} \\ (b \odot_{M(A)} a^-) \oplus a &= (a \oplus b^\sim)^- \oplus a = (a \oplus b^\sim)^- \oplus (a^\sim)^- = (a^\sim \odot (a \oplus b^\sim))^- = \\ &= (a^\sim \odot (b \rightsquigarrow a))^- = (a^\sim \odot (a^\sim \rightarrow b^\sim))^- = (a^\sim \wedge b^\sim)^- = a^{\sim-} \vee b^{\sim-} = a \vee b. \end{aligned}$$

We get similarly that $b \oplus (b^\sim \odot_{M(A)} a) = (a \odot_{M(A)} b^-) \oplus b = b \vee a = a \vee b$.

(vi) $a \odot_{M(A)} (a^- \oplus b) = a \odot_{M(A)} (a \rightarrow b) = ((a \rightarrow b)^- \oplus a^-)^\sim = (a \odot (a \rightarrow b))^{-\sim} = (a \wedge b)^{-\sim} = (b \wedge a)^{-\sim} = ((b \rightsquigarrow a) \odot b)^{-\sim} = (b^- \oplus (b \rightsquigarrow a))^\sim = (b \rightsquigarrow a) \odot_{M(A)} b = (a \oplus b^\sim) \odot_{M(A)} b$.

(vii) It follows from the definition of $M(A)$.

Hence, $\mathbf{M}(\mathbf{A})$ is a pseudo-MV algebra. By Lemma 3.2(iii), we have that for all $a, b \in M(A)$, $a \leq_{M(A)} b$ if and only if $b \oplus a^\sim = 1$ if and only if $a \rightsquigarrow b = 1$ if and only if $a \leq b$. \square

As a consequence of this proposition we obtain Corollary 2.34 from [4]:

Corollary 3.4 *A pseudo-BL algebra \mathbf{A} is a pseudo-MV algebra if and only if $a^{\sim-} = a^{-\sim} = a$ for all $a \in A$.*

Remark 3.5 *Let \mathbf{A} be a good pseudo-BL algebra. For any $a, b \in M(A)$,*

$$a \odot_{M(A)} b = (b^\sim \oplus a^\sim)^- = (b^- \oplus a^-)^\sim = (a \odot b)^{\sim-} = (a \odot b)^{-\sim}.$$

Proof: Apply the definitions of \oplus and $\odot_{M(A)}$. \square

Since, $a, b \in M(A)$ does not imply $a \odot b \in M(A)$, it follows that, generally, $(a \odot b)^{-\sim} \neq a \odot b$. Hence, the product on the pseudo-MV algebra $\mathbf{M}(\mathbf{A})$ does not coincide with the product on the pseudo-BL algebra \mathbf{A} . In the case of BL-algebras, the product is the same (see [19]).

Proposition 3.6 *Let \mathbf{A} be a good pseudo-BL algebra. Then*

\mathbf{A} is a strong pseudo-BL algebra if and only if $\mathbf{M}(\mathbf{A})$ is a strong pseudo-MV algebra.

Proof: If $a^\sim = a^-$ for all $a \in A$, then $a^\sim = a^-$ for all $a \in M(A)$. Conversely, suppose that $a^\sim = a^-$ for all $a \in M(A)$. Let $a \in A$. By (16) and (*), $a^\sim = a^{\sim-\sim} = (a^{\sim-})^\sim = (a^{-\sim})^\sim$. But, by Lemma 3.1(ii), we have that $a^{-\sim} \in M(A)$, so $(a^{-\sim})^\sim = (a^{-\sim})^- = a^-$, by (16). Thus, for all $a \in A$, we have that $a^\sim = a^-$. \square

Let \mathbf{A} be a good pseudo-BL algebra. Since, by Lemma 3.1(ii), $a^-, a^\sim \in M(A)$ for any $a \in A$, we can define the maps

$$\begin{aligned} \varphi_1 : A &\rightarrow M(A) \text{ by } \varphi_1(a) = a^\sim \text{ for any } a \in A, \text{ and} \\ \varphi_2 : A &\rightarrow M(A) \text{ by } \varphi_2(a) = a^- \text{ for any } a \in A. \end{aligned}$$

Lemma 3.7 *Let \mathbf{A} be a good pseudo-BL algebra. The following properties hold for all $a, b \in A$:*

- (i) φ_1, φ_2 are onto;
- (ii) $\varphi_1(a \vee b) = \varphi_1(a) \wedge \varphi_1(b)$ and $\varphi_2(a \vee b) = \varphi_2(a) \wedge \varphi_2(b)$;
- (iii) $\varphi_1(a \wedge b) = \varphi_1(a) \vee \varphi_1(b)$ and $\varphi_2(a \wedge b) = \varphi_2(a) \vee \varphi_2(b)$;
- (iv) $\varphi_1(a) \leq \varphi_1(b)$ if and only if $\varphi_2(a) \leq \varphi_2(b)$;
- (v) $a \leq b$ implies $\varphi_1(a) \geq \varphi_1(b)$ and $\varphi_2(a) \geq \varphi_2(b)$;
- (vi) $\varphi_1(a) = 1$ if and only if $\varphi_2(a) = 1$ if and only if $a = 0$;
- (vii) $\varphi_1(1) = \varphi_2(1) = 0$;
- (viii) $\varphi_1(a) = 0$ if and only if $\varphi_2(a) = 0$;
- (ix) $\varphi_1(a \odot b) = \varphi_1(b) \oplus \varphi_1(a)$ and $\varphi_2(a \odot b) = \varphi_2(b) \oplus \varphi_2(a)$;
- (x) for any $n \in \omega$, $\varphi_1(a^n) = n\varphi_1(a)$ and $\varphi_2(a^n) = n\varphi_2(a)$.

Proof: (i) Let $a \in M(A)$. Then $a = a^{-\sim} = \varphi_1(a^-)$ and $a = a^{\sim-} = \varphi_2(a^{\sim})$.
(ii) Apply (18).
(iii) Apply (19).
(iv) Suppose that $\varphi_1(a) \leq \varphi_1(b)$, that is $a^{\sim} \leq b^{\sim}$. Applying (13) and (16), it follows that $b^{-\sim} = b^{\sim-} \leq a^{\sim-} = a^{-\sim}$, so $a^- = a^{-\sim-} \leq b^{-\sim-} = b^-$. Hence, $\varphi_2(a) \leq \varphi_2(b)$. We prove similarly that $\varphi_2(a) \leq \varphi_2(b)$ implies $\varphi_1(a) \leq \varphi_1(b)$.
(v) Apply (13).
(vi) Apply (11).
(vii) Apply (12).
(viii) Suppose that $a^{\sim} = 0$, so $a^{-\sim} = a^{\sim-} = 1$, hence, $a^- = a^{-\sim-} = 0$. We get similarly that $a^- = 0$ implies $a^{\sim} = 0$.
(ix) Apply Lemma 3.2(iv).
(x) By induction on n . For $n = 0$, we have that $a^0 = 1$, so $\varphi_1(1) = 0$ and $0\varphi_1(a) = 0$. Suppose that $\varphi_1(a^n) = n\varphi_1(a)$. By (ix), it follows that $\varphi_1(a^{n+1}) = \varphi_1(a^n \odot a) = \varphi_1(a) \oplus \varphi_1(a^n) = \varphi_1(a) \oplus n\varphi_1(a) = (n+1)\varphi_1(a)$. Similarly for φ_2 . \square

Lemma 3.8 *Let \mathbf{A} be a nontrivial good pseudo-BL algebra. Then*

- (i) $A_0^- = A_0^{\sim} \stackrel{\text{not}}{=} A_0$;
- (ii) if $a \in A_0$, then $\text{ord}(a) = \infty$;
- (iii) for any $a \in A$, $\text{ord}(a) = \text{MV-ord}(\varphi_1(a)) = \text{MV-ord}(\varphi_2(a))$;
- (iv) for any $a \in A$, $\text{ord}(a^{\sim}) = \text{ord}(a^-)$;
- (v) $\varphi_1(D(A)) = \varphi_2(D(A)) = D(M(A))$ and $\varphi_1^{-1}(D(M(A))) = \varphi_2^{-1}(D(M(A))) = D(A)$.

Proof: (i) Apply Lemma 3.7(viii).
(ii) Suppose that there is $n \in \omega$ such that $a^n = 0$. Then, applying Lemma 3.7 (vi) and (x), we get that $\varphi_1(a^n) = \varphi_1(0) = 1$ and $\varphi_1(a^n) = n\varphi_1(a) = na^{\sim} = n0 = 0$. We have got that $0 = 1$, that is a contradiction. Hence, $a^n \neq 0$ for all $n \in \omega$, so $\text{ord}(a) = \infty$.
(iii) Let $a \in A$ and $n \in \omega$. By Lemma 3.7(vi), we have that $a^n = 0$ if and

only if $\varphi_1(a^n) = 1$ if and only if $n\varphi_1(a) = 1$. Hence, $\text{ord}(a) = \text{MV-ord}(\varphi_1(a))$. Similarly for φ_2 .

(iv) Let $a \in A$. Applying (i) and (*), we get that $\text{ord}(a^\sim) = \text{MV-ord}(\varphi_2(a^\sim)) = \text{MV-ord}(a^{\sim\sim}) = \text{MV-ord}(a^{-\sim}) = \text{MV-ord}(\varphi_1(a^-)) = \text{ord}(a^-)$.

(v) Apply (iii) and the fact that φ_1, φ_2 are onto. \square

Lemma 3.9 *Let \mathbf{A} be a good pseudo-BL algebra. Suppose that I is an ideal of $\mathbf{M}(\mathbf{A})$ and F is a filter of \mathbf{A} . Then*

- (i) $\varphi_1^{-1}(I), \varphi_2^{-1}(I)$ are filters of \mathbf{A} ;
- (ii) $\varphi_1(F), \varphi_2(F)$ are ideals of $\mathbf{M}(\mathbf{A})$;
- (iii) $F \subseteq \varphi_1^{-1}(\varphi_1(F))$ and $F \subseteq \varphi_2^{-1}(\varphi_2(F))$;
- (iv) $I = \varphi_1(\varphi_1^{-1}(I))$ and $I = \varphi_2(\varphi_2^{-1}(I))$;
- (v) I is proper if and only if $\varphi_1^{-1}(I)$ is proper if and only if $\varphi_2^{-1}(I)$ is proper;
- (vi) F is proper if and only if $\varphi_1(F)$ is proper if and only if $\varphi_2(F)$ is proper;
- (vii) if F is an ultrafilter of \mathbf{A} , then $F = \varphi_1^{-1}(\varphi_1(F))$ and $F = \varphi_2^{-1}(\varphi_2(F))$;
- (viii) if I is a maximal ideal of $\mathbf{M}(\mathbf{A})$, then $\varphi_1^{-1}(I), \varphi_2^{-1}(I)$ are ultrafilters of \mathbf{A} ;
- (ix) if F is an ultrafilter of \mathbf{A} , then $\varphi_1(F), \varphi_2(F)$ are maximal ideals of $\mathbf{M}(\mathbf{A})$.

Proof: (i) Let us prove that $\varphi_1^{-1}(I)$ is a filter of \mathbf{A} . Since $\varphi_1(1) = 0 \in I$, we have that $1 \in \varphi_1^{-1}(I)$. Let $a_1, a_2 \in \varphi_1^{-1}(I)$. It follows that $\varphi_1(a_1), \varphi_1(a_2) \in I$, so $\varphi_1(a_1 \odot a_2) = \varphi_1(a_2) \oplus \varphi_1(a_1) \in I$. Hence, $a_1 \odot a_2 \in \varphi_1^{-1}(I)$. Let $a_1 \in \varphi_1^{-1}(I), a_2 \in A$ such that $a_1 \leq a_2$. By Lemma 3.7(v), we get that $\varphi_1(a_2) \leq \varphi_1(a_1) \in I$, so $\varphi_1(a_2) \in I$, that is $a_2 \in \varphi_1^{-1}(I)$. Thus, we have proved that $\varphi_1^{-1}(I)$ is a filter of \mathbf{A} . We get similarly that $\varphi_2^{-1}(I)$ is a filter of \mathbf{A} .

(ii) Let us prove that $\varphi_1(F)$ is an ideal of $\mathbf{M}(\mathbf{A})$. We have that $0 = \varphi_1(1) \in \varphi_1(F)$. Let $b_1, b_2 \in \varphi_1(F)$. That is, there are $a_1, a_2 \in F$ such that $b_1 = \varphi_1(a_1)$ and $b_2 = \varphi_1(a_2)$. We have that $a_2 \odot a_1 \in F$ and $b_1 \oplus b_2 = \varphi_1(a_2 \odot a_1) \in \varphi_1(F)$. Let $b_1, b_2 \in M(\mathbf{A})$ such that $b_1 \leq b_2$ and $b_2 \in \varphi_1(F)$. It follows that $b_2 = \varphi_1(a_2)$ with $a_2 \in F$ and, since φ_1 is onto, there is $a \in A$ such that $\varphi_1(a) = b_1$. Let $a_1 = a \vee a_2$. Then $a_2 \leq a_1$ and $a_2 \in F$, so $a_1 \in F$ and $\varphi_1(a_1) = \varphi_1(a) \wedge \varphi_1(a_2) = b_1 \wedge b_2 = b_1$, by Lemma 3.7(iii). Hence, $b_1 \in \varphi_1(F)$. We obtain in the same manner that $\varphi_2(F)$ is an ideal of $\mathbf{M}(\mathbf{A})$.

(iii) It is obvious.

(iv) It follows from the fact that φ_1 and φ_2 are onto.

(v) I is not proper if and only if $1 \in I$ if and only if $\varphi_1(0) \in I$ if and only if $0 \in \varphi_1^{-1}(I)$ if and only if $\varphi_1^{-1}(I)$ is not proper.

(vi) If $0 \in F$, then $1 = \varphi_1(0) \in \varphi_1(F)$. Suppose that $1 \in \varphi_1(F)$. Then, there is $a \in F$ such that $\varphi_1(a) = 1$. Applying Lemma 3.7(vi), we get that $a = 0$, hence $0 \in F$.

(vii) Suppose that F is an ultrafilter of \mathbf{A} . Then, by (v) and (vi), $\varphi_1^{-1}(\varphi_1(F))$ is a proper filter of \mathbf{A} and, by (iii), $F \subseteq \varphi_1^{-1}(\varphi_1(F))$. Since F is ultrafilter, we get that $F = \varphi_1^{-1}(\varphi_1(F))$.

(viii) Suppose that $\varphi_1^{-1}(I) \subseteq F$, where F is a proper filter of \mathbf{A} . It follows that

$I = \varphi_1(\varphi_1^{-1}(I)) \subseteq \varphi_1(F)$. Since $\varphi_1(F)$ is proper, we get that $I = \varphi_1(F)$, so $\varphi_1^{-1}(I) = \varphi_1^{-1}(\varphi_1(F)) \supseteq F$. Hence, $\varphi_1^{-1}(I) = F$.

(ix) Suppose that $\varphi_1(F) \subseteq I$, where I is a proper ideal of $\mathbf{M}(\mathbf{A})$. It follows that $F = \varphi_1^{-1}(\varphi_1(F)) \subseteq \varphi_1^{-1}(I)$. Since $\varphi_1^{-1}(I)$ is proper, we get that $F = \varphi_1^{-1}(I)$, so $\varphi_1(F) = \varphi_1(\varphi_1^{-1}(I)) = I$. \square

The next result is a consequence of the above proposition.

Proposition 3.10 *The maps φ_1, φ_2 are bijections between the set of ultrafilters of \mathbf{A} and the set of maximal ideals of $\mathbf{M}(\mathbf{A})$.*

Corollary 3.11 *Let \mathbf{A} be a good pseudo-BL algebra. Then \mathbf{A} is local if and only if $\mathbf{M}(\mathbf{A})$ is local.*

We remark that if \mathbf{A} is a BL-algebra, then $\varphi_1 = \varphi_2$ and the results obtained above extend some results from [19, 8].

Proposition 3.12 *Let \mathbf{A} be a good pseudo-BL algebra. Suppose that I is an ideal of $\mathbf{M}(\mathbf{A})$ and F is a filter of \mathbf{A} . Then*

- (i) *if F is normal in \mathbf{A} , then $\varphi_1(F) = \varphi_2(F) \stackrel{\text{not}}{=} \varphi(F)$;*
- (ii) *if F is normal in \mathbf{A} , then $\varphi(F)$ is normal in $\mathbf{M}(\mathbf{A})$;*
- (iii) *if I is normal in $\mathbf{M}(\mathbf{A})$, then $\varphi_1^{-1}(I) = \varphi_2^{-1}(I)$.*

Proof: (i) Let $b \in \varphi_1(F)$, that is $b = \varphi_1(a)$ with $a \in F$. By (14), we have that $a \leq a^{\sim-}$, so $a^{\sim-} \in F$, hence $a^{\sim} \in F$, since F is a normal filter of \mathbf{A} . Since $b \in M(A)$, we also get that $b = b^{\sim-} = a^{\sim-} = \varphi_2(a^{\sim})$, hence $b \in \varphi_2(F)$. Thus, $\varphi_1(F) \subseteq \varphi_2(F)$. The other inclusion is proved similarly.

(ii) Let $b, c \in M(A)$. By Lemma 3.2(iii), we have that $b^{\sim} \odot_{M(A)} c = (c^- \oplus b)^{\sim} = (c \rightarrow b)^{\sim}$ and $c \odot_{M(A)} b^- = (b \oplus c^{\sim})^- = (c \rightsquigarrow b)^-$. Suppose that $b^{\sim} \odot_{M(A)} c \in \varphi(F)$, so there is $a \in F$ such that $(c \rightarrow b)^{\sim} = a^{\sim}$. But $c \rightarrow b = c^- \oplus b \in M(A)$, hence $c \rightarrow b = (c \rightarrow b)^{\sim-} = a^{\sim-} \geq a$, by (14). Since $a \in F$ and F is a filter, we get that $c \rightarrow b \in F$. But F is normal, hence $c \rightsquigarrow b \in F$. We obtain that $c \odot_{M(A)} b^- = (c \rightsquigarrow b)^- \in \varphi(F)$. We get similarly that $c \odot_{M(A)} b^- \in \varphi(F)$ implies $b^{\sim} \odot_{M(A)} c \in \varphi(F)$.

(iii) Let $a \in \varphi_1^{-1}(I)$, so $a^{\sim} \in I$. Since I is normal, from $a^{\sim} \in I$ and Lemma 1.23 we get that $a^{\sim=} \in I$. But, by (*) and (16), $a^{\sim=} = a^{-\sim-} = a^- = \varphi_2(a)$. We have got that $\varphi_2(a) \in I$, that is $a \in \varphi_2^{-1}(I)$. We prove similarly that $a \in \varphi_2^{-1}(I)$ implies $a \in \varphi_1^{-1}(I)$. \square

4 Some classes of local pseudo-BL algebras

Perfect pseudo-BL algebras

A pseudo-BL algebra \mathbf{A} is called *perfect* if

- (i) \mathbf{A} is a local good pseudo-BL algebra, and
- (ii) for any $a \in A$,

$\text{ord}(a) < \infty$ if and only if $\text{ord}(a^\sim) = \infty$.

Proposition 4.1 *Let \mathbf{A} be a good pseudo-BL algebra. Then \mathbf{A} is perfect if and only if $\mathbf{M}(\mathbf{A})$ is perfect.*

Proof: We have that \mathbf{A} is local if and only if $\mathbf{M}(\mathbf{A})$ is local, by Corollary 3.11. In the sequel, we shall apply repeatedly Proposition 3.8(iii). Suppose that \mathbf{A} is perfect and let $a \in M(A)$, so $a = a^{\sim-}$. We get that $\text{MV-ord}(a) < \infty$ if and only if $\text{MV-ord}(a^{\sim-}) < \infty$ if and only if $\text{ord}(a^\sim) < \infty$ if and only if $\text{ord}(a) = \infty$ if and only if $\text{MV-ord}(a^\sim) = \infty$. Hence, $\mathbf{M}(\mathbf{A})$ is perfect. Conversely, suppose that $\mathbf{M}(\mathbf{A})$ is perfect and let $a \in A$. Then, by Lemma 3.1(i), $a^\sim \in M(A)$. It follows that $\text{ord}(a) < \infty$ if and only if $\text{MV-ord}(a^\sim) < \infty$ if and only if $\text{MV-ord}(a^\sim) = \infty$ if and only if $\text{ord}(a^\sim) = \infty$. Hence, \mathbf{A} is perfect. \square

Proposition 4.2 *Let \mathbf{A} be a local good pseudo-BL algebra. The following are equivalent:*

- (i) \mathbf{A} is perfect;
- (ii) for any $a \in A$, $\text{ord}(a) < \infty$ implies $\text{ord}(a^\sim) = \infty$;
- (ii') for any $a \in A$, $\text{ord}(a) < \infty$ implies $\text{ord}(a^-) = \infty$;
- (iii) $D(A)_\sim^* = D(A)^*$;
- (iii') $D(A)_-^* = D(A)^*$.

Proof: (i) if and only if (ii) Let $a \in A$. Since \mathbf{A} is local, by Corollary 2.3, we have that $\text{ord}(a) = \infty$ implies $\text{ord}(a^\sim) < \infty$, hence $\text{ord}(a^\sim) = \infty$ implies $\text{ord}(a) < \infty$. It follows that \mathbf{A} is perfect if and only if $(\text{ord}(a) < \infty$ implies $\text{ord}(a^\sim) = \infty)$.

(ii) if and only if (ii') Apply Lemma 3.8(iv).

(ii') implies (iii) Since \mathbf{A} is local, $D(A)_\sim^* \subseteq D(A)^*$, by Corollary 2.3(ii). Let us prove the converse inclusion. Let $a \in A$ such that $\text{ord}(a) < \infty$. From (ii') we get that $\text{ord}(a^-) = \infty$, so $a^- \in D(A)$ and, by (14), $a \leq a^{\sim-}$. Hence, $a \in D(A)_\sim^*$.

(iii) implies (ii') Suppose that $D(A)_\sim^* = D(A)^*$ and let $a \in A$ with $\text{ord}(a) < \infty$, that is $a \in D(A)^*$. It follows that there is $x \in D(A)$ such that $a \leq x^\sim$, so $x^{\sim-} \leq a^-$, by (13). Since $x \leq x^{\sim-}$ and $\text{ord}(x) = \infty$, applying Lemma 1.7(iii), we get that $\text{ord}(x^{\sim-}) = \infty$. Applying again Lemma 1.7(iii), from $x^{\sim-} \leq a^-$ it follows that $\text{ord}(a^-) = \infty$.

(ii) if and only if (iii') It is similar to (ii') if and only if (iii). \square

A primary filter P of a pseudo-BL algebra \mathbf{A} is called *perfect* if for all $a \in A$, $(a^n)^\sim \in P$ for some $n \in \omega$ implies $((a^\sim)^m)^\sim \notin P$ for all $m \in \omega$.

Lemma 4.3 *Let \mathbf{A} be a pseudo-BL algebra and P be a perfect filter of \mathbf{A} . Then for all $a \in A$,*

$(a^n)^\sim \in P$ for some $n \in \omega$ if and only if $((a^\sim)^m)^\sim \notin P$ for all $m \in \omega$.

Proof: Let $a \in A$ such that $((a^\sim)^m)^\sim \notin P$ for all $m \in \omega$. We have to prove that $(a^n)^\sim \in P$ for some $n \in \omega$. By (9), $a^\sim \odot a = 0$, hence $((a^\sim \odot a)^n)^\sim = 0^\sim = 1 \in P$

for all $n \in \omega$. Apply now the fact that P is primary and the hypothesis to get that $(a^n)^\sim \in P$ for some $n \in \omega$. \square

Proposition 4.4 *Let \mathbf{A} be a good pseudo-BL algebra and P be a proper normal filter of \mathbf{A} . The following are equivalent:*

- (i) \mathbf{A}/P is a perfect pseudo-BL algebra;
- (ii) P is a perfect filter of \mathbf{A} ;
- (iii) P is primary and for all $a \in A$,
 $(a^n)^- \in P$ for some $n \in \omega$ implies $((a^-)^m)^- \notin P$ for all $m \in \omega$.

Proof: Since good pseudo-BL algebras form a variety, it follows that \mathbf{A}/P is a good pseudo-BL algebra. By Proposition 2.6, we have that \mathbf{A}/P is local if and only if P is primary. Let $a \in A$. Applying Lemma 1.10, we get that $\text{ord}(a/P) < \infty$ if and only if $(a/P)^n = 0/P$ for some $n \in \omega$ if and only if $(a^n)^\sim \in P$ for some $n \in \omega$ if and only if $(a^n)^- \in P$ for some $n \in \omega$, that $\text{ord}((a/P)^\sim) = \infty$ if and only if $((a/P)^\sim)^m \neq 0/P$ for all $m \in \omega$ if and only if $((a^\sim)^m)^\sim \notin P$ for all $m \in \omega$ and that $\text{ord}((a/P)^-) = \infty$ if and only if $((a/P)^-)^m \neq 0/P$ for all $m \in \omega$ if and only if $((a^-)^m)^- \notin P$ for all $m \in \omega$. Apply now Proposition 4.2(ii) and (iii) to get that (i) if and only if (ii) and (i) if and only if (iii). \square

Proposition 4.5 *Let \mathbf{A} be a BL-algebra and P be a proper filter of \mathbf{A} . The following are equivalent:*

- (i) P is a perfect filter of \mathbf{A} ;
- (ii) for all $a \in A$,
 $(a^n)^- \in P$ for some $n \in \omega$ if and only if $((a^-)^m)^- \notin P$ for all $m \in \omega$.

Proof: (i) implies (ii) Apply Lemma 4.3.

(ii) implies (i) We shall prove that \mathbf{A}/P is local and apply Proposition 2.6 to get that P is a primary filter. Let $a \in A$ and suppose that $\text{ord}(a^-/P) = \infty$. As in the proof of Proposition 4.4, we get that $((a^-)^m)^- \notin P$ for all $m \in \omega$. Applying (i), it follows that $(a^n)^- \in P$ for some $n \in \omega$, that is, $\text{ord}(a/P) < \infty$. Thus, we have proved that for all $a \in A$, $\text{ord}(a/P) < \infty$ or $\text{ord}(a^-/P) < \infty$. Apply now [19, Proposition 1] to obtain that \mathbf{A}/P is local. \square

Hence, in the case that \mathbf{A} is a BL-algebra, the notion of perfect filter defined above coincides with the notion of perfect filter defined in [19].

Proposition 4.6 *Let \mathbf{A} be a good pseudo-BL algebra. The following are equivalent:*

- (i) \mathbf{A} is perfect;
- (ii) any proper normal filter of \mathbf{A} is perfect;
- (iii) $\{1\}$ is a perfect filter of \mathbf{A} .

Proof: (i) implies (ii) Let F be a proper normal filter of \mathbf{A} . Since \mathbf{A} is local, by Proposition 2.9 it follows that F is primary. Let $a \in A$ such that $(a^n)^\sim \in F$

for some $n \in \omega$. Suppose that $((a^\sim)^k)^\sim \in F$ for some $k \in \omega$. We get that $\langle (a^n)^\sim \rangle, \langle ((a^\sim)^k)^\sim \rangle \subseteq F$ and, since F is proper, it follows that $\langle (a^n)^\sim \rangle$ and $\langle ((a^\sim)^k)^\sim \rangle$ are also proper filters of \mathbf{A} . Applying Lemma 1.7(i), we get that $\text{ord}((a^n)^\sim) = \text{ord}(((a^\sim)^k)^\sim) = \infty$. Since \mathbf{A} is perfect, we obtain that $\text{ord}(a^n) < \infty$ and $\text{ord}((a^\sim)^k) < \infty$, hence, $\text{ord}(a) < \infty$ and $\text{ord}(a^\sim) < \infty$, a contradiction with the fact that \mathbf{A} is perfect. Thus, $(a^n)^\sim \in F$ for some $n \in \omega$ implies $((a^\sim)^m)^\sim \notin F$ for all $m \in \omega$.

(ii) implies (iii) It is obvious, since $\{1\}$ is a proper normal filter of \mathbf{A} .

(iii) implies (i) Since $\{1\}$ is a perfect filter of \mathbf{A} , applying Proposition 4.4, we get that $\mathbf{A}/\{1\}$ is perfect. But $\mathbf{A} \cong \mathbf{A}/\{1\}$, hence \mathbf{A} is perfect. \square

Locally finite pseudo-BL algebras

According to [5], a pseudo-BL algebra \mathbf{A} is *locally finite* if for any $a \in A$, $a \neq 1$ implies $\text{ord}(a) < \infty$.

Proposition 4.7 *Let \mathbf{A} be a pseudo-BL algebra. The following are equivalent:*

- (i) \mathbf{A} is locally finite;
- (ii) $\{1\}$ is the unique proper filter of \mathbf{A} .

Proof: Applying Lemma 1.7(i), it follows that A is locally finite if and only if for every $a \in A$, if $a \neq 1$ then $\langle a \rangle = A$ if and only if $\{1\}$ is the unique proper filter of A . \square

Proposition 4.8 *Every locally finite pseudo-BL algebra \mathbf{A} is a local pseudo-BL algebra.*

Proof: We have that $D(A) = \{1\}$, hence $D(A)$ is a filter of \mathbf{A} . Apply Proposition 2.2 to get that \mathbf{A} is local. \square

In [5] it is proved that locally finite pseudo-BL algebras are locally finite MV-algebras. We shall give a simpler proof of this fact.

Proposition 4.9 *Let \mathbf{A} be a locally finite pseudo-BL algebra. Then for all $a \in A$,*

$$a^{\sim-} = a^{-\sim} = a.$$

Hence, $A = M(A)$.

Proof: If $a = 0$, then it follows immediately that $0^{\sim-} = 0^{-\sim} = 0$. Suppose that $a \neq 0$. Let us prove that $a^{-\sim} = a$. By (14), we have that $a < a^{-\sim}$. Suppose that $a^{-\sim} \not\leq a$, hence $a^{-\sim} \rightarrow a \neq 1$. Since \mathbf{A} is locally finite, it follows that $\text{ord}(a^{-\sim} \rightarrow a) < \infty$, hence $(a^{-\sim} \rightarrow a)^n = 0$ for some $n \in \omega - \{0\}$. By (16), (2), (A4) and (14), we get that $(a^{-\sim} \rightarrow a) \rightarrow a^- = (a^{-\sim} \rightarrow a) \rightarrow a^{\sim-} = (a^{-\sim} \rightarrow a) \rightarrow (a^{-\sim} \rightarrow 0) = a^{-\sim} \odot (a^{-\sim} \rightarrow a) \rightarrow 0 = (a \wedge a^{\sim-}) \rightarrow 0 = a \rightarrow 0 = a^-$. Applying repeatedly this procedure, it follows that $(a^{-\sim} \rightarrow a)^n \rightarrow a^- = a^-$, hence $a^- = 0 \rightarrow a^- = 1$, so, by (11), $a = 0$. We have got a contradiction, since $a \neq 0$. Hence, $a^{-\sim} = a$. We prove similarly that $a^{\sim-} = a$. \square

Corollary 4.10 ([5]) *Every locally finite pseudo-BL algebra \mathbf{A} is a locally finite MV-algebra.*

Proof: Applying Proposition 4.9 and Proposition 1.24, we get that \mathbf{A} is a pseudo-MV algebra. Let $a \in A$, $a \neq 0$, so $a^\sim \neq 1$, by (11). By Proposition 3.8(i), we obtain that $\text{MV-ord}(a) = \text{MV-ord}(a^{\sim-}) = \text{ord}(a^\sim) < \infty$. Thus, we have proved that \mathbf{A} is a locally finite pseudo-MV algebra. Apply now [15, Proposition 39] to get that \mathbf{A} is a locally-finite MV-algebra. \square

Peculiar pseudo-BL algebras

A pseudo-BL algebra \mathbf{A} is called *peculiar* if

- (i) \mathbf{A} is a local good pseudo-BL algebra;
- (ii) there is $a \in A - \{1\}$ such that $\text{ord}(a) = \infty$;
- (iii) there is $a \in A$ such that $\text{ord}(a) < \infty$ and $\text{ord}(a^\sim) < \infty$.

Let us denote by \mathcal{PF} the class of perfect pseudo-BL algebras, by \mathcal{LF} the class of locally finite pseudo-BL algebras and by \mathcal{PC} the class of peculiar pseudo-BL algebras. The following proposition is similar to [2, Theorem 5.1].

Proposition 4.11 *Let \mathbf{A} be a local good pseudo-BL algebra dif and only iferent from $\mathbf{L}_2 = \{0, 1\}$. Then exactly one of the following holds:*

- (i) $\mathbf{A} \in \mathcal{PF}$;
- (ii) $\mathbf{A} \in \mathcal{LF}$;
- (iii) $\mathbf{A} \in \mathcal{PC}$.

Proof: By the definitions, if $\mathbf{A} \notin \mathcal{PF} \cup \mathcal{LF}$, then $\mathbf{A} \in \mathcal{PC}$. Hence, one of (i), (ii) or (iii) holds. It is easy to see that $\mathcal{PC} \cap \mathcal{LF} = \mathcal{PC} \cap \mathcal{PF} = \emptyset$. Let us prove that $\mathcal{PF} \cap \mathcal{LF} = \{\mathbf{L}_2\}$. Obviously, \mathbf{L}_2 is perfect and locally finite. Now, let $\mathbf{A} \neq \mathbf{L}_2$ be a locally finite pseudo-BL algebra. Since $A \neq \{0, 1\}$, there is $a \in A$ such that $a \neq 0$ and $a \neq 1$. From $a \neq 0$ and (11) we get that $a^\sim \neq 1$. Applying now the fact that \mathbf{A} is locally finite, it follows that $\text{ord}(a) < \infty$ and $\text{ord}(a^\sim) < \infty$. Hence, \mathbf{A} is not perfect. That is, exactly one of (i), (ii), (iii) holds. \square

Proposition 4.12 *Let \mathbf{A} be a local good pseudo-BL algebra such that $A \neq M(A)$. Then*

\mathbf{A} is a peculiar pseudo-BL algebra if and only if $\mathbf{M}(\mathbf{A}) \neq \mathbf{L}_2$ is a singular pseudo-MV algebra.

Proof: Suppose that \mathbf{A} is peculiar. Then \mathbf{A} is not perfect, hence, by Proposition 4.1, $\mathbf{M}(\mathbf{A})$ is not a perfect pseudo-MV algebra. Since \mathbf{L}_2 is a perfect pseudo-MV algebra, it follows that $\mathbf{M}(\mathbf{A}) \neq \mathbf{L}_2$. Applying Proposition 1.25, we also get that $\mathbf{M}(\mathbf{A})$ is singular. Conversely, suppose that $\mathbf{M}(\mathbf{A}) \neq \mathbf{L}_2$ and that $\mathbf{M}(\mathbf{A})$ is a singular pseudo-MV algebra. Since $A \neq M(A)$, by Proposition 4.9 we get that \mathbf{A} is not locally finite. We also have that $\mathbf{M}(\mathbf{A})$ is not perfect, hence \mathbf{A} is not perfect. Applying Proposition 4.11, we get that \mathbf{A} is peculiar. \square

5 Bipartite pseudo-BL algebras

In this section, we shall define (strongly) bipartite pseudo-BL algebra and we shall prove some properties of them, following [18, 8].

A pseudo-BL algebra \mathbf{A} is called *bipartite* if $U \cup U_{\sim}^* = U \cup U_{-}^* = A$ for some ultrafilter U of \mathbf{A} . \mathbf{A} is called *strongly bipartite* if $U \cup U_{\sim}^* = U \cup U_{-}^* = A$ for any ultrafilter U of \mathbf{A} . Obviously, any strongly bipartite pseudo-BL algebra is bipartite.

A filter F of \mathbf{A} is called *Boolean* if for all $a \in A$,

$$a \vee a^{\sim} \in F \text{ and } a \vee a^{-} \in F.$$

It is obvious that if $F \subseteq G$ are two filters of \mathbf{A} and F is Boolean, then G is also Boolean.

Proposition 5.1 *Let \mathbf{A} be a pseudo-BL algebra and F be a filter of \mathbf{A} . The following are equivalent:*

- (i) F is a Boolean ultrafilter of \mathbf{A} ;
- (ii) F is a Boolean prime filter of \mathbf{A} ;
- (iii) F is proper and for all $a \in A$,
 $a \in F$ or $(a^{\sim} \in F \text{ and } a^{-} \in F)$.

Proof: (i) implies (ii) It is obvious, since, by Proposition 1.3, any ultrafilter of \mathbf{A} is a prime filter of \mathbf{A} .

(ii) implies (iii) Let $a \in A$. Since F is Boolean, we have that $a \vee a^{\sim} \in F$ and $a \vee a^{-} \in F$. Apply now the fact that F is prime to get (iii).

(iii) implies (ii) Let G be a proper filter of \mathbf{A} such that $F \subseteq G$ and suppose that $F \neq G$. Then there is $a \in G$ such that $a \notin F$. By (iii), it follows that $a^{\sim}, a^{-} \in F \subseteq G$, so by (8), $0 = a^{\sim} \odot a \in G$, hence G is not proper, that is a contradiction. Hence, $G = F$. Thus, F is an ultrafilter of \mathbf{A} . Let us prove now that F is Boolean. Let $a \in A$. If $a \in F$, since $a \leq a \vee a^{\sim}$ and $a \leq a \vee a^{-}$, we get that $a \vee a^{\sim}, a \vee a^{-} \in F$. If $a \notin F$, then $a^{\sim}, a^{-} \in F$ and from $a^{\sim} \leq a \vee a^{\sim}$, $a^{-} \leq a \vee a^{-}$ we also get that $a \vee a^{\sim}, a \vee a^{-} \in F$. \square

Lemma 5.2 *Let \mathbf{A} be a pseudo-BL algebra and U be an ultrafilter of \mathbf{A} . The following are equivalent:*

- (i) $U \cup U_{\sim}^* = U \cup U_{-}^* = A$;
- (ii) U is Boolean.

Proof: Applying Proposition 5.1(iii) and Remark 1.13(ii) and (ii'), we get that U is Boolean if and only if for all $a \in A$, $a \in U$ or $(a^{\sim} \in U \text{ and } a^{-} \in U)$ if and only if for all $a \in A$, $a \in U$ or $(a \in U_{\sim}^* \text{ and } a \in U_{-}^*)$ if and only if $U \cup U_{\sim}^* = U \cup U_{-}^* = A$. \square

Proposition 5.3 *Let \mathbf{A} be a pseudo-BL algebra \mathbf{A} . The following are equivalent:*

- (i) \mathbf{A} is bipartite;
- (ii) \mathbf{A} has a Boolean proper filter.

Proof: (i) implies (ii) Apply the above lemma.
(ii) implies (i) Suppose that \mathbf{A} has a Boolean proper filter F . By Proposition 1.4, we can extend F to an ultrafilter U and U is also Boolean. Applying again Lemma 5.2, we get that \mathbf{A} is bipartite. \square

Let \mathbf{A} be a pseudo-BL algebra. Following [18], we define

$$\mathcal{B}(A) = \cap \{F \mid F \text{ is a Boolean filter of } \mathbf{A}\}, \text{ and}$$

$$\text{sup}(A) = \{a \vee a^\sim \mid a \in A\} \cup \{a \vee a^- \mid a \in A\}.$$

The following remark is obvious.

Remark 5.4 *Let \mathbf{A} be a pseudo-BL algebra. Then*
(i) $\mathcal{B}(A)$ is the smallest Boolean filter of \mathbf{A} ;
(ii) if $\text{sup}(A)$ is a filter of \mathbf{A} , then it is a Boolean filter;
(iii) $\text{sup}(A) \subseteq \mathcal{B}(A)$.

Proposition 5.5 *Let \mathbf{A} be a pseudo-BL algebra. Then*

- (i) $\mathcal{B}(A) = \langle \text{sup}(A) \rangle$;
- (ii) $\text{sup}(A) = \{a \in A \mid a \geq a^\sim \text{ or } a \geq a^-\}$.

Proof: (i) By the above remark, we have that $\text{sup}(A) \subseteq \mathcal{B}(A)$ and $\mathcal{B}(A)$ is a filter of \mathbf{A} . Hence, $\langle \text{sup}(A) \rangle \subseteq \mathcal{B}(A)$. Obviously, $\langle \text{sup}(A) \rangle$ is a Boolean filter of \mathbf{A} , so $\mathcal{B}(A) \subseteq \langle \text{sup}(A) \rangle$.

(ii) Let $a \in \text{sup}(A)$. If $a = x \vee x^\sim$ for some $x \in A$ then, by (18), $a = x \vee x^\sim \geq x^\sim \geq x^\sim \wedge x^\approx = (x \vee x^\sim)^\sim = a^\sim$. We prove similarly that if $a = x \vee x^-$ for some $x \in A$, then $a \geq a^-$. Conversely, if $a \in A$ such that $a \geq a^\sim$, then $a = a \vee a^\sim$, hence $a \in \text{sup}(A)$. Similarly, if $a \geq a^-$, then $a = a \vee a^-$, that is $a \in \text{sup}(A)$. \square

Proposition 5.6 *Let \mathbf{A} be a pseudo-BL algebra \mathbf{A} . The following are equivalent:*

- (i) \mathbf{A} is strongly bipartite;
- (ii) any ultrafilter of \mathbf{A} is Boolean;
- (iii) $\mathcal{B}(A) \subseteq \mathcal{M}(A)$, where we remind that $\mathcal{M}(A)$ denotes the intersection of all ultrafilters of \mathbf{A} .

Proof: (i) if and only if (ii) Apply Lemma 5.2.
(ii) implies (iii) If U is an ultrafilter of \mathbf{A} then, by (ii), U is Boolean. Applying Remark 5.4(i), we get that $\mathcal{B}(A) \subseteq U$.
(iii) implies (ii) Let U be an ultrafilter of \mathbf{A} . Then $\mathcal{B}(A) \subseteq U$ and $\mathcal{B}(A)$ is a Boolean filter of \mathbf{A} . It follows that U is also Boolean. \square

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