Boolean products of BL-algebras

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Abstract

The representation of algebras by Boolean products is a very general problem in universal algebra. In this paper we shall characterize the Boolean products of BL-chains, the weak Boolean products of local BL-algebras and the weak Boolean products of perfect BL-algebras.

Introduction

BL-algebras constitute the algebraic structures for Hájek's Basic Logic [17]. MV-algebras, Gödel algebras and product algebras are particular cases of BL-algebras. Apart from their logical interest, BL-algebras have important algebraic properties [20, 21, 22, 12].

This paper is a contribution to the representation theory of BL-algebras by Boolean products. In universal algebra there exist some very general representation theorems of algebras by (weak) Boolean products [4]. If $\mathcal{K}$ is a class of universal algebras and $\mathcal{H} \subseteq \mathcal{K}$, then a standard problem is to represent the algebras of $\mathcal{K}$ as (weak) Boolean products of members of $\mathcal{H}$.

In this paper we shall characterize the Boolean products of BL-chains, the weak Boolean products of local BL-algebras and the weak Boolean products of perfect BL-algebras. Our results extend some theorems related to the representation of MV-algebras by (weak) Boolean products [13, 8, 18, 19].

1 Definitions and first properties

A *BL-algebra* [17] is an algebra $(A, \wedge, \vee, \odot, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, \odot, \rightarrow$ and two constants $0, 1$ such that:
(i) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
(ii) $(A, \odot, 1)$ is a commutative monoid;
(iii) $\odot$ and $\rightarrow$ form an adjoint pair, i.e.
$c \leq a \rightarrow b$ iff $a \odot c \leq b$ for all $a, b, c \in A$;
(iv) $a \wedge b = a \odot (a \rightarrow b)$;
(v) $(a \rightarrow b) \vee (b \rightarrow a) = 1$.
A BL-algebra A is nontrivial iff $0 \neq 1$.
For any BL-algebra A, the reduct $L(A) = (A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice.
A *BL-chain* is a linear BL-algebra, i.e. a BL-algebra such that its lattice order is total.
For any $a \in A$, we define $a^- = a \rightarrow 0$. We shall denote $(a^-)^-$ by $a^=$.
We denote the set of natural numbers by ω. We define $a^0 = 1$ and $a^n = a^{n-1} \odot a$ for $n \in \omega - \{0\}$. The *order* of $a \in A$, in symbols $ord(a)$, is the smallest $n \in \omega$ such that $a^n = 0$. If no such n exists, then $ord(a) = \infty$.
The following properties hold in any BL-algebra A and will be used in the sequel:
(1.1) $a \odot b \leq a$
(1.2) $a \leq b$ implies $a \odot c \leq b \odot c$
(1.3) $a \odot b \leq a \wedge b$
(1.4) $a \odot b = 0$ iff $a \leq b^-$
(1.5) $a \vee b = 1$ implies $a \odot b = a \wedge b$
(1.6) $a \odot a^- = 0$
(1.7) $a \rightarrow (b \rightarrow c) = (a \odot b) \rightarrow c$
(1.8) $a \odot (b \vee c) = (a \odot b) \vee (a \odot c)$
(1.9) $a \odot (b \wedge c) = (a \odot b) \wedge (a \odot c)$
(1.10) $(a \wedge b)^- = a^- \vee b^-$ and $(a \vee b)^- = a^- \wedge b^-$
(1.11) $a \leq a^=$
(1.12) $a \vee b = ((a \rightarrow b) \rightarrow b) \wedge ((b \rightarrow a) \rightarrow a)$
(1.13) $a \vee b = 1$ implies $a^n \vee b^n = 1$ for any $n \in \omega$.

We remind that a MV-algebra is an algebra $(A, \oplus, -, 0)$ with one binary operation $\oplus$, one unary operation - and one constant 0 such that:
(i) $(A, \oplus, 0)$ is a commutative monoid;
(ii) $a^= = a$;
(iii) $a \oplus 0^- = 0^-$;
(iv) $(a^- \oplus b)^- \oplus b = (b^- \oplus a)^- \oplus a$.
If A is a MV-algebra, then the binary operations $\odot$, $\wedge$, $\vee$ and the constant 1 are defined by the following relations:
$a \odot b = (a^- \oplus b^-)^-$, $a \wedge b = (a \oplus b^-) \odot b$, $a \vee b = (a \odot b^-) \oplus b$ and $1 = 0^-$.
It is well-known that $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. We also define $0a = 0$ and $na = (n-1)a \oplus a$ for $n \in \omega - \{0\}$. For a detailed exposition

of MV-algebras see [7].

MV-algebras and BL-algebras are closely related. Indeed, a BL-algebra A is a MV-algebra iff $a^{=} = a$ for all $a \in A$. If A is a BL-algebra consider, following [22], the subset

$MV(A) = \{a \in A \mid a^{=} = a\} = \{a^{-} \mid a \in A\}$.

If one defines $a^{-} \oplus b^{-} = (a \odot b)^{-}$, then $(MV(A), \oplus, -, 0)$ becomes a MV-algebra (see [22]). The next lemma is obvious.

Lemma 1.1 ([22])
Let A be a BL-algebra. Consider the function $\varphi : A \to MV(A)$, defined by $\varphi(a) = a^{-}$ for any $a \in A$. Then, for any $a, b \in A$ and $n \in \omega$, the following are true:
(i) $\varphi(a \odot b) = \varphi(a) \oplus \varphi(b)$;
(ii) $\varphi(a \wedge b) = \varphi(a) \vee \varphi(b)$;
(iii) $\varphi(a \vee b) = \varphi(a) \wedge \varphi(b)$;
(iv) $\varphi(a^n) = n\varphi(a)$;
(v) $\varphi(0) = 1$ and $\varphi(1) = 0$.

Let A be a BL-algebra. A *filter* of A is a nonempty set $F \subseteq A$ such that for all $a, b \in A$,
(i) $a, b \in F$ implies $a \odot b \in F$;
(ii) $a \in F$ and $a \leq b$ implies $b \in F$.
A filter F of A is *proper* iff $F \neq A$.
By (1.3) it is obvious that any filter of A is also a filter of the lattice $L(A)$. A proper filter P of A is called *prime* provided that it is prime as a filter of $L(A)$: $a \vee b \in P$ implies $a \in P$ or $b \in P$.
A proper filter U of A is called *ultrafilter* (or *maximal filter*) if it is not contained in any other proper filter.
We remind some properties of filters that will be used in the sequel.

Proposition 1.2 ([20], Proposition 8)
If A is a nontrivial BL-algebra, then any proper filter of A can be extended to an ultrafilter.

Proposition 1.3 ([17], Lemma 2.3.15)
Let $a \in A, a \neq 1$. Then there is a prime filter P of A such that $a \notin P$.

Proposition 1.4 If A is a nontrivial BL-algebra, then any proper filter F of A is the intersection of all prime filters containing F.

Proposition 1.5 ([20], Proposition 6)
Let P be a prime filter of a nontrivial BL-algebra A. Then the set

$\mathcal{F} = \{F \mid P \subseteq F \text{ and } F \text{ is a proper filter of } A\}$

is linearly ordered with respect to set-theoretical inclusion.

Proposition 1.6 If A is a nontrivial BL-algebra, then any prime filter of A is contained in a unique ultrafilter.

Proof: Apply Propositons 1.2 and 1.5. □
With any filter F of A we can associate a congruence relation $\sim_F$ on A by defining

$a \sim_F b$ iff $a \to b \in F$ and $b \to a \in F$ iff $(a \to b) \odot (b \to a) \in F$.

For any $a \in A$, let a/F be the equivalence class $a/\sim_F$. If we denote by A/F the quotient set $A/_{\sim_F}$, then A/F becomes a BL-algebra with the natural operations induced from those of A. If $a, b \in A$, then $a/F \le b/F$ iff $a \to b \in F$. Hájek proved [17] that A/F is a BL-chain iff F is a prime filter of A.

If $h : A \to B$ be a homomorphism of BL-algebras, then the *kernel* of h is the set $Ker(h) = \{a \in A \mid h(a) = 1\}$. It is easy to see that

Lemma 1.7 Let $h : A \to B$ a homomorphism of BL-agebras.
(i) if G is a (prime) filter of B, then $h^{-1}(G)$ is a (prime) filter of A. Thus, in particular, $Ker(h)$ is a filter of A;
(ii) if h is onto then, for any (prime) filter F of A, $h(F)$ is a (prime) filter of B.

Let $X \subseteq A$. The filter of A generated by X will be denoted by $< X >$. We have that $< \emptyset >= \{1\}$ and $< X >= \{a \in A \mid x_1 \odot \cdots \odot x_n \le a$ for some $n \in \omega - \{0\}$ and some $x_1, \cdots, x_n \in X\}$ if $\emptyset \neq X \subseteq A$. For any $a \in A$, $< a >$ denotes the principal filter of A generated by $\{a\}$. Then, $< a >= \{b \in A \mid a^n \le b$ for some $n \in \omega - \{0\}\}$.
We shall also denote by $[X)$ the filter of the lattice $L(A)$ generated by X and by $[a)$ the principal filter of the lattice $L(A)$ generated by $\{a\}$. The following lemma is obvious:

Lemma 1.8 Let $\emptyset \neq X \subseteq A$ and $a \in A$. Then
(i) $[X) \subseteq < X >$;
(ii) $[a) =< a >$ iff $a \odot a = a$;
(iii) If X is $\odot$-closed, then $[X) =< X >= \{a \in A \mid$ there exists $x \in X$ such that $x \le a\}$;
(iv) If $x \odot x = x$ for any $x \in X$ and X is $\wedge$-closed, then $[X) =< X >= \{a \in A \mid$ there exists $x \in X$ such that $x \le a\}$.

For any BL-algebra A, $B(A)$ denotes the Boolean algebra of all complemented elements in $L(A)$. Hence, $B(A) = B(L(A))$.

Proposition 1.9 Let $e \in A$. The following are equivalent:
(i) $e \in B(A)$;
(ii) $e \odot e = e$ and $e = e^{=}$;
(iv) $e \odot e = e$ and $e^- \to e = e$.
(iv) $e \vee e^- = 1$.

Proof: (i)$\Rightarrow$(ii) Suppose that $e \in B(A)$. Then $e \vee a = 1$ and $e \wedge a = 0$, for some $a \in A$. From (1.5) and (1.4) we obtain $a \leq e^-$. Moreover, $e^- = 1 \odot e^- = (e \vee a) \odot e^- \overset{(1.8)}{=} (e \odot e^-) \vee (a \odot e^-) \overset{(1.6)}{=} a \odot e^- \leq a$. Hence $e^- \leq a$. Thus, $a = e^-$ is the complement of e. It follows that $e^- \in B(A)$ and, similarly, $e^=$ is the complement of e^-. But the complement of e^- is also e. Since $L(A)$ is distributive, we get $e = e^=$.
(ii)$\Rightarrow$(iii) We have that $e \rightarrow e^- = e \rightarrow (e \rightarrow 0) \overset{(1.7)}{=} (e \odot e) \rightarrow 0 = e \rightarrow 0 = e^-$. Hence, $e \wedge e^- = e \odot (e \rightarrow e^-) = e \odot e^- \overset{(1.6)}{=} 0$.
Since $e \wedge e^- = e^- \wedge e = e^- \odot (e^- \rightarrow e) = 0$, by (1.4), we get $e^- \rightarrow e \leq e^= = e$. But $e \odot e^- \leq e$, so $e \leq e^- \rightarrow e$. We have got $e^- \rightarrow e = e$.
(iii)$\Rightarrow$(iv) Applying (1.12), $e \vee e^- = 1$ iff $(e \rightarrow e^-) \rightarrow e^- = 1$ and $(e^- \rightarrow e) \rightarrow e = 1$. By (iii), $e^- \rightarrow e = e$, hence $(e^- \rightarrow e) \rightarrow e = 1$. We also have that $e \rightarrow e^- = e \rightarrow (e \rightarrow 0) \overset{(1.7)}{=} (e \odot e) \rightarrow 0 = e \rightarrow 0 = e^-$. So, $(e \rightarrow e^-) \rightarrow e^- = 1$.
(iv)$\Rightarrow$(i) From $e \vee e^- = 1$ it follows by (1.5) and (1.6) that $e \wedge e^- = e \odot e^- = 0$. Hence, e^- is the complement of e. That is, $e \in B(A)$. $\Box$

Corollary 1.10 Let A be a BL-algebra. Then B(A)=B(MV(A)).

Proof: We have that $B(A) = MV(A) \cap \{a \in A \mid a \odot a = a\} = B(MV(A))$, following [7], Theorem 1.5.3. $\Box$

Lemma 1.11 Suppose that $a \in A$ and $e \in B(A)$. Then $e \odot a = e \wedge a$.

Proof: $e \wedge a = e \odot (e \rightarrow a) = e \odot e \odot (e \rightarrow a) = e \odot (e \wedge a) \overset{(1.9)}{=} (e \odot e) \wedge (e \odot a) = e \wedge (e \odot a) \overset{(1.1)}{=} e \odot a$. $\Box$

A BL-algebra A is called *directly indecomposable* iff A is nontrivial and whenever $A \cong A_1 \times A_2$ then either A_1 or A_2 is trivial. In a similar manner with [7], Chapter 6.4 we can prove that

Proposition 1.12 A BL-algebra A is directly indecomposable iff $B(A) = \{0, 1\}$.

It follows immediately that

Proposition 1.13 Any BL-chain is directly indecomposable.

If F is a filter of (a Boolean subalgebra of) $B(A)$ and $\sim_{<F>}$ is the congruence on A associated with the filter $< F >$ of A, then we denote by a_F the equivalence class of $a \in A$ and by A_F the quotient BL-algebra $A/ < F >$. It follows that for all $a, b \in A$, $a \sim_F b$ iff $(a \rightarrow b) \odot (b \rightarrow a) \in < F >$ iff there is $e \in F$ such that $e \leq (a \rightarrow b) \odot (b \rightarrow a)$ iff there is $e \in F$ such that $e \leq a \rightarrow b$ and $e \leq b \rightarrow a$, since $e \in B(A)$, so $e \odot e = e$.
The next lemma is a technical result needed for proving an important proposition.

Lemma 1.14 Let $a \in A$ and $e \in B(A)$. If $e \leq a \vee a^-$, then $e \odot a \in B(A)$.

Proof: Since $e \in B(A)$, by Proposition 1.9 we have that $e \vee e^- = 1$. Applying Lemma 1.11 and (1.10) we get $(e \odot a) \vee (e \odot a)^- = (e \wedge a) \vee (e \wedge a)^- = (e \wedge a) \vee (e^- \vee a^-) = (e \wedge a) \vee ((e^- \vee a^-) \wedge 1) = (e \wedge a) \vee ((e^- \vee a^-) \wedge (e^- \vee e)) = (e \wedge a) \vee (e \wedge a^-) \vee e^- = (e \wedge (a \vee a^-)) \vee e^- = e \vee e^- = 1$. We apply again Proposition 1.9 to obtain that $e \odot a \in B(A)$. □

Proposition 1.15 If P is an ultrafilter of $B(A)$, then A_P is directly indecomposable.

Proof: Let P be an ultrafilter of $B(A)$. By Proposition 1.12, we have to show that $B(A_P) = \{0_P, 1_P\}$. Let $a_P \in B(A_P)$. Applying Proposition 1.9, we have that $(a \vee a^-)_P = 1_P$. Hence, $a \vee a^- \in P$. It follows that there is $e_1 \in P$ such that $e_1 \leq a \vee a^-$. The complement of a_P is $a_P^- \in B(A_P)$. We obtain similarly that there is $e_2 \in P$ such that $e_2 \leq a^- \vee a^=$. Let $e = e_1 \wedge e_2$. Then $e \in P \subseteq B(A)$. Applying Lemma 1.14, from $e \leq a \vee a^-$, we get $e \odot a \in B(A)$ and, also, from $e \leq a^- \vee a^=$ we get $e \odot a^- \in B(A)$. Now, $e \leq a \vee a^-$ and Lemma 1.11 give us $(e \odot a) \vee (e \odot a^-) \overset{(1.8)}{=} e \odot (a \vee a^-) = e \wedge (a \vee a^-) = e \in P$. Since P is a prime filter of $B(A)$ and $e \odot a, e \odot a^- \in B(A)$ it follows that $e \odot a \in P$ or $e \odot a^- \in P$. If $e \odot a \in P$, then $(e \odot a)_P = 1_P$. We obtain $e_P \odot a_P = 1_P$, hence $a_P = 1_P$, since $e \in P$. Similarly, from $e \odot a^- \in P$ we get $a_P^- = 1_P$, so $a_P = 0_P$. Hence, we have obtained that $a_P \in B(A_P)$ implies $a_P \in \{0_P, 1_P\}$, that is $B(A_P) = \{0_P, 1_P\}$. □

Let us now recall some facts from lattice theory. Let $(L, \vee, \wedge, 0, 1)$ be a bounded distributive lattice. With any filter F of L we can associate a congruence $\sim_F$ on L defined by:

$a \sim_F b$ iff there is $t \in F$ such that $a \wedge t = b \wedge t$.

For any $a \in L$, we denote by a/F the equivalence class of a. We denote by L/F the quotient lattice $L/_{\sim_F}$. The Boolean algebra of all complemented elements in L will be denoted by $B(L)$. The prime spectrum of L is the set $Spec(L)$ of prime filters of L and the maximal spectrum of L is the set $Max(L)$ of ultrafilters of L. We endow these two sets with the Stone topology.

A *Stone filter* of L is a filter of L generated by a filter of the Boolean algebra $B(L)$. In other words, a filter F of L is a Stone filter iff for any $a \in L$ there is $e \in F \cap B(L)$ such that $e \leq a$, that is $F = [F \cap B(L))$. A *Stone ultrafilter* of L is a Stone filter generated by an ultrafilter of $B(L)$, i.e. a Stone filter F such that $F \cap B(L)$ is an ultrafilter of $B(L)$. For a more detailed analysis of these notions see [5].

If F is a filter of $B(L)$, we denote by a_F the equivalence class $a/[F)$ and by L_F the quotient lattice $L/[F)$.

A bounded distributive lattice L is called *normal* ([24], [9]) if for all $a, b \in L$, $a \wedge b = 0$ implies there exist $u, v \in L$ such that $u \vee v = 1$ and $a \wedge u = b \wedge v = 0$.

Proposition 1.16 Let L be a bounded distributive lattice. The following are equivalent:

(i) L is normal;
(ii) any prime filter of L is contained in a unique ultrafilter.

Proof: By [9], Theorem 2.4, L is normal iff any prime ideal of L contains a unique minimal prime ideal of L. But (see [23], p.82) the condition
(1) any prime ideal of L contains a unique minimal prime ideal
is equivalent in a bounded distributive lattice to the condition
(2) any prime filter of L is contained in a unique ultrafilter.
□
A bounded distributive lattice L is called *B-normal* ([6]) if for all $a, b \in L$, $a \wedge b = 0$ implies there exist $u, v \in B(L)$ such that $u \vee v = 1$ and $a \wedge u = b \wedge v = 0$. Dually, we obtain the notions of *co-normal* and *B-co-normal* lattice.

Remark 1.17 We remark that normal (B-normal) lattices are called co-normal (B-co-normal) in [14]. In this paper we adopt a terminology on the line of [9], [6].

By dualizing Proposition 2.6 from [14] we get

Proposition 1.18 Let L be a bounded distributive lattice. The following are equivalent:
(i) L is B-normal;
(ii) L is normal and $Max(L)$ is zero-dimensional (i.e. it has a basis of clopen subsets).

A bounded distributive lattice L is called *dense* iff for all $a, b \in L$, $a \wedge b = 0$ implies $a = 0$ or $b = 0$. Then, L is dense iff it has a unique ultrafilter.

It is easy to check, using Lemma 1.8, that if F is a Stone filter of $L(A)$, then $F = [F \cap B(A)) =< F \cap B(A) >$. It follows that Stone filters of $L(A)$ are filters of A.

We remind that if B is a Boolean algebra, then $Spec(B)$ is a Boolean space and the clopen sets of the basis are all the sets of the form $N_a = \{P \in Spec(B) \mid a \in P\}$, for $a \in B$.

Proposition 1.19 Let C be a subalgebra of the Boolean algebra $B(A)$. Then $\bigcap_{P \in Spec(C)} < P >= \{1\}$.

Proof: Obviously, $1 \in< P >$ for any $P \in Spec(C)$. Let $a \neq 1 \in A$. Then, by Proposition 1.3, there is a prime filter Q of A such that $a \notin Q$. It is easy to show that $Q \cap C$ is a prime filter of C, i.e. $Q \cap C \in Spec(C)$. We have that $< Q \cap C > \subseteq Q$, so, $a \notin < Q \cap C >$. □
From this proposition it follows that the intersection of all Stone ultrafilters of $L(A)$ is the filter $\{1\}$. Using the above proposition and a general result of universal algebra (see, e.g., [3], Lemma II.8.2, p. 56) we get

Proposition 1.20 Let C be a subalgebra of the Boolean algebra $B(A)$. Then A is isomorphic to a subdirect product of the family $\{A_P\}_{P \in Spec(C)}$.

Let A be a BL-algebra. We shall denote by $Spec(A)$ the set of prime filters of A and by $Max(A)$ the set of ultrafilters of A. For any $a \in A$, we define

$d(a) = \{P \in Spec(A) \mid a \in P\}$ and $D(a) = \{U \in Max(A) \mid a \in U\}$.

It is easy to see that

$d(a) \cap d(b) = d(a \wedge b) = d(a \odot b), d(a \vee b) = d(a) \cup d(b), d(0) = \emptyset, d(1) = Spec(A)$, and

$D(a) \cap D(b) = D(a \wedge b) = D(a \odot b), D(a \vee b) = D(a) \cup D(b), D(0) = \emptyset, D(1) = Max(A)$.

It follows that the family $\{d(a) \mid a \in A\}$ is a basis for a topology on $Spec(A)$ and $\{D(a) \mid a \in A\}$ is a basis for a topology on $Max(A)$.

In [15] there was defined the reticulation of a quantale as a generalization of the reticulation of a MV-algebra [1]. In the sequel, we shall define the reticulation of a BL-algebra A and we shall present some results. For details see [15]. Let us define a binary relation $\equiv$ on A by: $a \equiv b$ iff $d(a) = d(b)$. Then, $\equiv$ is an equivalence relation on A compatible with the operations $\odot, \wedge$ and $\vee$. For $a \in A$ let us denote by $[a]$ the class of $a \in A$ with respect to $\equiv$. The lattice $\beta(A) = (A/\equiv, \vee, \wedge, [0], [1])$ is called the *reticulation* of the BL-algebra A. We shall denote by $\beta : A \to \beta(A)$ the natural homomorphism, defined by $\beta(a) = [a]$. If $h : A \to B$ is a homomorphism of BL-algebras, then $\beta(h) : \beta(A) \to \beta(B)$, defined by $\beta(h)(a) = [h(a)]$, is a homomorphism of bounded lattices. It follows that we can define a functor β from the category of BL-algebras to the category of bounded distributive lattices. The functor β is called the *reticulation functor*. If F is a filter of A, then $\beta(F) = \{[a] \mid a \in A\}$ is a filter of the lattice $\beta(A)$ and the mapping $F \mapsto \beta(F)$ is an isomorphism between the lattice $\mathcal{F}(A)$ of filters of A and the lattice $\mathcal{F}(\beta(A))$ of filters of the reticulation of A. If $P \in Spec(A)$, then $\beta(P)$ is a prime filter of $\beta(A)$ and the mapping $P \mapsto \beta(P)$ is a homeomorphism between $Spec(A)$ and $Spec(\beta(A))$. Similarly, $Max(A)$ is homeomorphic to $Max(\beta(A))$. Applying Propositions 1.6 and 1.16 we obtain

Proposition 1.21 $\beta(A)$ is a normal lattice.

Proposition 1.22 Let $a, b \in A$. Then
(i) $[a] \leq [b]$ iff $d(a) \subseteq d(b)$;
(ii) $[a] = [b]$ iff $< a >=< b >$;
(iii) $[a] = [1]$ iff $a = 1$;
(iv) $[a] = [0]$ iff $a^n = 0$ for some $n \in \omega - \{0\}$;
(v) $[a^n] = [a]$ for any $n \in \omega - \{0\}$.

Proof: (i) Obviously.
(ii) If $< a >=< b >$, then for any prime filter P of A, $a \in P$ iff $< a >\subseteq P$ iff $< b >\subseteq P$ iff $b \in P$. Hence, $d(a) = d(b)$, i.e. $[a] = [b]$. If $[a] = [b]$, then, by Proposition 1.4, $< a >= \cap\{P \in Spec(A) \mid a \in P\} = \cap\{P \mid P \in d(a)\} = \cap\{P \mid P \in d(b)\} =< b >$.
(iii), (iv) and (v) follow from (ii). □

Proposition 1.23 $\beta|_{B(A)} : B(A) \to B(\beta(A))$ is an isomorphism of Boolean algebras.

Proof: If $e \in B(A)$, then $e \vee e^- = 1$ and $e \wedge e^- = 0$. It follows that $[e] \vee [e^-] = [1]$ and $[e] \wedge [e^-] = 0$. Hence, $\beta(e) = [e] \in B(\beta(A))$ and $[e]^- = [e^-]$. That is, $\beta|_{B(A)} : B(A) \to B(\beta(A))$ is well-defined and $(\beta(e))^- = \beta(e^-)$. It is easy to see that $\beta_{B(A)}$ is a homomorphism of Boolean algebras. Let $e_1, e_2 \in B(A)$. If $[e_1] = [e_2]$, then $< e_1 >=< e_2 >$, by Proposition 1.22(ii). Appying Proposition 1.8(ii), we get that $[e_1) = [e_2)$, that is $e_1 = e_2$. Hence, $\beta|_{B(A)}$ is one-to-one. Let us now prove that $\beta|_{B(A)}$ is onto. Let $[a] \in B(\beta(A))$. We get $b \in A$ such that $[a] \vee [b] = [1]$ and $[a] \wedge [b] = [0]$. By Proposition 1.22(ii),(iii), it follows that $a \vee b = 1$ and $(a \wedge b)^n = 0$ for some $n \in \omega - \{0\}$. From $a \vee b = 1$ and (1.13) we get that $a^n \vee b^n = 1$. Using twice (1.5) we have that $a^n \wedge b^n = a^n \odot b^n = (a \odot b)^n = (a \wedge b)^n = 0$. Hence, $a^n \in B(A)$ and, by Proposition 1.22(v), $\beta(a^n) = [a^n] = [a]$. $\Box$

Proposition 1.24 If F is a filter of A, then the lattices $\beta(A/F)$ and $\beta(A)/\beta(F)$ are isomorphic.

Proof: We shall prove that for every $a, b \in A$, the following equivalence holds:

(*) $[a/F] = [b/F]$ iff $[a]/\beta(F) = [b]/\beta(F)$.

Assume $[a/F] = [b/F]$, so $< a/F >=< b/F >$. Then, there is a $n \in \omega - \{0\}$ such that $a^n/F \leq b/F$ and $b^n/F \leq a/F$, hence $a^n \to b \in F$ and $b^n \to a \in F$. If we take $t = (a^n \to b) \wedge (b^n \to a) \in F$, then $t \leq a^n \to b$ and $t \leq b^n \to a$, so $a^n \odot t \leq b$ and $b^n \odot t \leq a$. If $a \wedge t \in F$, then $a, t \in F$, hence $a \odot t \in F$. We get that $a^n \odot t \in F$, since $(a \odot t)^n \in F$ and $(a \odot t)^n \leq a^n \odot t$. It follows that $b \in F$, hence $b \odot t \in F$, so $b \wedge t \in F$. Similarly, we prove that $b \wedge t \in F$ implies $a \wedge t \in F$. Thus, we have obtained that $a \wedge t \in F$ iff $b \wedge t \in F$ Hence, there is $[t] \in \beta(F)$ such that $[a] \wedge [t] \in \beta(F)$ iff $[b] \wedge [t] \in \beta(F)$. That is, $[a]/\beta(F) = [b]/\beta(F)$.
Conversely, suppose that $[a]/\beta(F) = [b]/\beta(F)$, i.e. $[a] \wedge [t] = [b] \wedge [t]$ for some $t \in F$. It follows that for any $P \in Spec(A)$ such that $F \subseteq P$, we have that $a \in P$ iff $a \wedge t \in P$ iff $b \wedge t \in P$ iff $b \in P$. Let $Q \in Spec(A/F)$ and let $h : A \to A/F$ be the natural homomorphism. Since h is onto, by Lemma 1.7(ii), we get that $h^{-1}(Q) \in Spec(A)$. We also have that $F \subseteq h^{-1}(Q)$. It folows that $a/F \in Q$ iff $a \in h^{-1}(Q)$ iff $b \in h^{-1}(Q)$ iff $b/F \in Q$. Hence, $[a/F] = [b/F]$. Thus, (*) has been proved.
It follows that $[a/F] \mapsto [a]/\beta(F)$ is the desired isomorphism. $\Box$

Proposition 1.25 Let P be a Stone ultrafilter of A. Then $\beta(A_P)$ and $\beta(A)_{\beta(P)}$ are isomorphic.

Proof: We have that $\beta(A)_{\beta(P)} = \beta(A)/ < \beta(P) >$ and, by Proposition 1.24, we get that $\beta(A_P) = \beta(A/ < P >) \cong \beta(A)/\beta(< P >)$. Hence, it suffices to show that $\beta(< P >) =< \beta(P) >$. Let $a \in< P >$. Then, there is $e \in P$ such that

$e \leq a$. It follows that $[e] \in \beta(P)$ and $[e] \leq [a]$. Hence, $\beta(a) = [a] \in < \beta(P) >$. Conversely, suppose that $[a] \in < \beta(P) >$, that is there exist $e \in P$ such that $[e] \leq [a]$. Since $[e] \vee [e^-] = [1]$, we get that $[e^- \vee a] = [1]$, so $e^- \vee a = 1$. Hence, $e \wedge a = 0 \vee (e \wedge a) = (e \wedge e^-) \vee (e \wedge a) = e \wedge (e^- \vee a) = e \wedge 1 = e$. We have got that $e \leq a$ and $e \in P$, so $a \in < P >$. Thus, $[a] \in \beta(< P >)$. $\square$

2 Boolean products of BL-algebras

A *weak Boolean product* of a nonempty family $\{A_x\}_{x \in X}$ of BL-algebras is a subdirect product A of the given family, in such a way that X can be endowed with a Boolean space topology having the following two properties:
(i) if $a, b \in A$, then the set $\|a = b\| = \{x \in X \mid a(x) = b(x)\}$ is open in X;
(ii) if $a, b \in A$ and Z is a clopen subset of X, then $a|_Z \cup b|_{X-Z} \in A$.
By requiring in condition (i) that $\|a = b\|$ be clopen we obtain the notion of *Boolean product.*
It is easy to see that for all $a, b \in A$ $\|a = b\| = \|(a \to b) \odot (b \to a) = 1\|$. Then, condition (i) above can be replaced by
$(i^{'})$ if $a \in A$, then the set $\|a = 1\| = \{x \in X \mid a(x) = 1_x\}$ is open (respectively, clopen) in X.
From the fact that BL-algebras form a variety it results that if A is a (weak) Boolean product of a nonempty family $\{A_x\}_{x \in X}$ of BL-algebras, then A is also a BL-algebra.
A *(weak) Boolean representation* of a BL-algebra A is an isomorphism from A onto a (weak) Boolean product of BL-algebras.
The following theorem is the analogue of the Theorem 6.5.2 for MV-algebras from [7].

Theorem 2.1 Let A be a weak Boolean product of a nonempty family $\{A_x\}_{x \in X}$ of nontrivial BL-algebras. Let C be defined by
$C = \{a \in A \mid a(x) \in \{0_x, 1_x\}$ for all $x \in X\}$.
Then
(i) C is a subalgebra of the Boolean algebra $B(A)$;
(ii) the correspondence $x \overset{\sigma_C}{\mapsto} P_x = \{a \in C \mid a(x) = 1_x\}$ is a homeomorphism from X onto $Spec(C)$;
(iii) for all $x \in X$, A_x is isomorphic to A_{P_x};
(iv) C coincides with $B(A)$ iff all algebras A_x are directly indecomposable.
Conversely, suppose that A is a nontrivial BL-algebra and C is a subalgebra of $B(A)$. Then A is representable as the weak Boolean product of the family $\{A_P\}_{P \in Spec(C)}$.

Proof: (i) By the fact that A is a subalgebra of $\prod_{x \in X} A_x$ it follows that $0, 1 \in A$ and that $a, b \in A$ implies $a \odot b, a \to b, a \wedge b, a \vee b \in A$. Then, it is easy to see that C is a subalgebra of $B(A)$. We remark that for proving (i) it is sufficiently to have only that A is a subalgebra of $\prod_{x \in X} A_x$.

(ii) Let $x \in X$. Let us first prove that $P_x \in Spec(C)$. It is obvious that P_x is a filter of C. Because $0 \in C - P_x$, we obtain that P_x is a proper filter of C. Let $a, b \in C$ such that $a \vee b \in P_x$. Then $a(x) \vee b(x) = 1_x$ and, since $a(x), b(x) \in \{0_x, 1_x\}$, we must have $a(x) = 1_x$ or $b(x) = 1_x$. Hence, $a \in P_x$ or $b \in P_x$. It follows that $\sigma_C : X \to Spec(C)$ is well-defined.
Let $x \neq y \in X$. Since X is Hausdorff and has a basis of clopen sets, there exists a clopen set N such that $x \in N$ and $y \notin N$. Let $a = 1|_N \cup 0|_{X-N}$. Then $a \in A$, because $0, 1 \in A$ and N is clopen. Moreover, since $a(z) \in \{0_z, 1_z\}$ for any $z \in X$, we have that $a \in C$. From $x \in N$ it follows that $a(x) = 1_x$, hence $a \in P_x$. But $y \notin N$ implies $a(y) = 0_y$, that is $a \notin P_y$. We get $P_x \neq P_y$. Hence, σ_C is one-to-one.
Suppose that σ_C is not onto. Then there is $P \in Spec(C)$ such that $P \neq P_x$ for any $x \in X$. Because P, P_x are ultrafilters of C, $P_x \not\subseteq P$. That is, for any $x \in X$, there exists $a_x \in C$ such that $a_x \in P_x$ and $a_x \notin P$. Then, $a_x(x) = 1_x$, i.e. $x \in \|a_x = 1\|$. We have obtained that $X = \bigcup_{x \in X} \|a_x = 1\|$. By the fact that $\|a_x = 1\|$ is open in X for any $x \in X$ and by the compactness of X we get that there are $x_1, \cdots, x_n \in X$ such that $X = \|a_{x_1} = 1\| \cup \cdots \cup \|a_{x_n} = 1\| \subseteq \|a_{x_1} \vee \cdots \vee a_{x_n} = 1\|$. Then $X = \|a_{x_1} \vee \cdots \vee a_{x_n} = 1\|$, hence $a_{x_1} \vee \cdots \vee a_{x_n} = 1$. Since P is a prime filter of C and $1 \in P$ it follows that $a_{x_i} \in P$ for some $i \in \{1, \cdots, n\}$. This is a contradiction with the fact that $a_x \notin P$ for any $x \in X$. Hence, σ_C is onto.
Let $c \in C$. Then, for any $x \in X$, $P_x \in N_c$ iff $c \in P_x$ iff $c(x) = 1_x$ iff $x \in \|c = 1\|$. It results that $\sigma_C^{-1}(N_c) = \|c = 1\|$, hence $\sigma_C^{-1}(N_c)$ is open in X for any $c \in C$. Since $\{N_c \mid c \in C\}$ form a basis for the topology of $Spec(C)$, we get that σ_C is continuous.
Now (ii) follows from the well-known fact that continuous bijections between compact Hausdorff spaces are homeomorphisms.
(iii) For any $x \in X$, let $h_x : A \to A_x$ be the natural homomorphism. Since h_x is onto, we have that $A/Ker(h_x) \cong A_x$. It remains to prove that $< P_x >= Ker(h_x)$. For any $a \in A$, $a \in Ker(h_x)$ iff $a(x) = 1_x$, hence $P_x = Ker(h_x) \cap C$. It follows that $< P_x > \subseteq Ker(h_x)$.
Let now $a \in Ker(h_x)$. It follows that $x \in \|a = 1\|$, so $P_x = \sigma_C(x) \in \sigma_C(\|a = 1\|)$. Since $\|a = 1\|$ is open in X and σ_C is a homeomorphism, we get that $\sigma_C(\|a = 1\|)$ is open in $Spec(C)$. But $Spec(C)$ is a Boolean space, so there is $c \in C$ such that $P_x \in N_c \subseteq \sigma_C(\|a = 1\|)$. Let us prove that $c \leq a$. Since $c \in C$, we have that $c(z) \in \{0_z, 1_z\}$ for any $z \in X$. Let $z \in X$ such that $c(z) = 1_z$. Then $\sigma_C(z) = P_z \in N_c \subseteq \sigma_C(\|a = 1\|)$ and from the fact that σ_C is one-to-one we get $z \in \|a = 1\|$, i.e. $a(z) = 1_z$. We have obtained $c \leq a$ and, since $c \in P_x$, it follows that $a \in < P_x >$.
(iv) If $C = B(A)$, then P_x is an ultrafilter of $B(A)$ for any $x \in X$. Applying Proposition 1.15 we get that A_{P_x} is directly indecomposable for any $x \in X$. From (iii) it follows that A_x is directly indecomposable for any $x \in X$.
Conversely, suppose that A_x is directly indecomposable for any $x \in X$ and $C \neq B(A)$, hence there is $a \in B(A) - C$. It follows that there is $x \in X$

such that $a(x) \not\in \{0_x, 1_x\}$. Since $a \in B(A)$, $a(x) \in B(A_x)$. It follows that $B(A_x) \neq \{0_x, 1_x\}$. This contradicts the fact that A_x is directly indecomposable.
Conversely, let A be a nontrivial BL-algebra and C be a subalgebra of $B(A)$. By Proposition 1.20, we get that A is isomorphic to a subdirect product of the family $\{A_P\}_{P \in Spec(C)}$. To simplify the notation, we can safely identify A with its corresponding subalgebra of $\prod_{P \in Spec(C)} A_P$. Let us verify (i) and (ii) from the definition of the weak Boolean product.
Let $a, b \in A$ and $P \in Spec(C)$ such that $P \in \|a = b\|$. Then $a_P = b_P$, i.e. there is $c \in P$ such that $c \leq a \to b$ and $c \leq b \to a$. It follows that $P \in N_c \subseteq \|a = b\|$ and N_c is clopen in $Spec(C)$. Hence, $\|a = b\|$ is open in $Spec(C)$.
Let $a, b \in A$ and $Z = N_c$ a clopen set in $Spec(C)$, where $c \in C$. We shall prove that $a|_Z \cup b|_{Spec(C)-Z} = (a \wedge c) \vee (b \wedge c^-) \in A$. Let $P \in Z$. Then $c \in P$, hence $c_P = 1_P$ and $c_P^- = 0_P$. We obtain $((a \wedge c) \vee (b \wedge c^-))_P = (a_P \wedge c_P) \vee (b_P \wedge c_P^-) = a_P \vee 0_P = a_P$. We prove similarly that for $P \in Spec(C) - Z$, $((a \wedge c) \vee (b \wedge c^-))_P = b_P$. $\Box$
Applying the above theorem and Proposition 1.15 we get the next corrolary.

Corollary 2.2 Any nontrivial BL-algebra A is representable as a weak Boolean product of directly indecomposable BL-algebras.

3 Boolean products of BL-chains

From Theorem 2.1 we obtain the next characterization of weak Boolean products of BL-chains.

Proposition 3.1 A nontrivial BL-algebra A is a weak Boolean product of BL-chains iff the Stone ultrafilters of $L(A)$ are prime filters of A.

Proof: "$\Rightarrow$" Suppose that A is a weak Boolean product of a nonempty family $\{A_x\}_{x \in X}$ of BL-chains. Because, by Proposition 1.13, any BL-chain is directly indecomposable, we obtain from Theorem 2.1.(iv) that $C = B(A)$. Hence, by Theorem 2.1.(ii),(iii), we get that $A_P = A/ < P >$ is a BL-chain for every ultrafilter P of $B(A)$. It follows that $< P >$ is a prime filter of A for any ultrafilter P of $B(A)$. Hence, any Stone ultrafilter of $L(A)$ is a prime filter of A.
"$\Leftarrow$" Suppose that any Stone ultrafilter of $L(A)$ is a prime filter of A. We know from Theorem 2.1 that if A is a nontrivial BL-algebra and C is a subalgebra of $B(A)$, then A is representable as the weak Boolean product of the family $\{A_P\}_{P \in Spec(C)}$. Let $C = B(A)$ and $P \in Spec(B(A))$. It follows that $< P >$ is a Stone ultrafilter of $L(A)$ and, by hypothesis, it is a prime filter of A. Hence, $A_P = A/ < P >$ is a BL-chain. $\Box$

In the sequel we shall give a characterization of BL-algebras representable by Boolean products of BL-chains. But, first, let us recall some facts from lattice theory (see [16]).

Let $(L, \vee, \wedge, 0)$ a lattice with 0. An element $a^* \in L$ is a *pseudocomplement* of $a \in L$ iff $a \wedge a^* = 0$ and $a \wedge x = 0$ implies that $x \leq a^*$. A bounded lattice L is called *pseudocomplemented* iff every element has a pseudocomplement.
Let $(L, \vee, \wedge, 0, 1)$ be a distributive pseudocomplemented lattice. L is called a *Stone lattice* iff it satisfies the *Stone identity*:
$a^* \vee a^{**} = 1$.

Proposition 3.2 ([16], Theorem 3.14.3, p.161)
Let L be a distributive pseudocomplemented lattice. Then L is a Stone lattice iff $(a \wedge b)^* = a^* \vee b^*$ for all $a, b \in L$.

By dualizing, we get the concepts of *dual pseudocomplement*, *dual pseudocomplemented lattice* and *dual Stone lattice*. The following result is a consequence of [5], Theorems 3.3 and 3.4:

Proposition 3.3 If L is a dual Stone lattice, then any Stone ultrafilter of L is a prime filter of L.

Now we are ready to prove the most important result of this chapter.

Theorem 3.4 A nontrivial BL-algebra A is a Boolean product of BL-chains iff $L(A)$ is a dual Stone lattice.

Proof: "$\Rightarrow$" Suppose that A is a Boolean product of BL-chains. In particular, A is a weak Boolean product of BL-chains, so we can apply Proposition 3.1 to obtain that any Stone ultrafilter of $L(A)$ is a prime filter of A. By Theorem 2.1, we can suppose that A is a Boolean product of the family $\{A_P\}_{P \in Spec(B(A))}$.
Let $a \in A$. Because A is a Boolean product of the family $\{A_P\}_{P \in Spec(B(A))}$, we obtain that $Z = \|a = 1\|$ is clopen in $Spec(B(A))$. Since $0, 1 \in A$ and Z is clopen, applying (ii) from the definition of the weak Boolean product we get that $a^\circ = 0|_Z \cup 1|_{Spec(B(A))-Z} \in A$. We shall prove that a° is the dual pseudocomplement of a. It is clear that $a \vee a^\circ = 1$. Now, let $c \in A$ such that $a \vee c = 1$. Then $(a \vee c)_P = 1_P$ for any $P \in Spec(B(A))$, i.e. $a \vee c \in < P >$ for any $P \in Spec(B(A))$. But $< P >$ is a Stone ultrafilter of $L(A)$, hence it is a prime filter of A. It follows that for any $P \in Spec(B(A))$, $a \in < P >$ or $c \in < P >$. Let us now prove that $a^\circ \leq c$, i.e. $a^\circ_P \leq c_P$ for any $P \in Spec(B(A))$. If $P \in Z$, then $a^\circ_P = 0_P \leq c_P$. If $P \in Spec(B(A)) - Z$, then $a^\circ_P = 1_P$. Since $P \notin Z$, we have that $a_P \neq 1_P$, i.e. $a \notin < P >$. It follows that $c \in < P >$, hence $c_P = 1_P$.
Hence, $L(A)$ is a distributive dual pseudocomplemented lattice and for any $a \in A$, the dual pseudocomplement of a is
$a^\circ = 0|_Z \cup 1|_{Spec(B(A))-Z}$, where $Z = \|a = 1\|$.
By the dual of Proposition 3.2, $L(A)$ is a dual Stone lattice iff $(a \vee b)^\circ = a^\circ \wedge b^\circ$ for all $a, b \in L$. In the sequel we shall prove this identity. Let $P \in Spec(B(A))$. Then $(a \vee b)^\circ_P = 0_P$ iff $(a \vee b)_P = 1_P$ iff $a \vee b \in < P >$ iff $a \in < P >$ or $b \in < P >$ iff $a_P = 1_P$ or $b_P = 1_P$ iff $a^\circ_P = 0_P$ or $b^\circ_P = 0_P$. It is clear that if $a^\circ_P = 0_P$ or

$b_P^\circ = 0_P$, then $(a^\circ \wedge b^\circ)_P = a_P^\circ \wedge b_P^\circ = 0_P$. Suppose now that $(a^\circ \wedge b^\circ)_P = 0_P$ and $a_P^\circ \neq 0_P$ and $b_P^\circ \neq 0_P$. Then, $a_P^\circ = 1_P$ and $b_P^\circ = 1_P$. Hence, $(a^\circ \wedge b^\circ)_P = 1_P$, a contradiction. We have thus proved that $(a \vee b)_P^\circ = 0_P$ iff $(a^\circ \wedge b^\circ)_P = 0_P$. Since $(a \vee b)_P^\circ, (a^\circ \wedge b^\circ)_P \in \{0_P, 1_P\}$, it follows that $(a \vee b)_P^\circ = (a^\circ \wedge b^\circ)_P$ for any $P \in Spec(B(A))$.
"$\Leftarrow$" Conversely, suppose that $L(A)$ is a dual Stone lattice. By Theorem 2.1, A is a weak Boolean product of the family $\{A_P\}_{P \in Spec(B(A))}$. For any $a \in A$ let a° be the dual pseudocomplement of a. Then, $a^\circ \vee a^{\circ\circ} = 1$ by the definition of the dual pseudocomplement and $a^\circ \wedge a^{\circ\circ} = 0$ by the fact that $L(A)$ is a dual Stone lattice. It follows that $a^\circ \in B(A)$ and its complement is $a^{\circ\circ} \in B(A)$. In the sequel we shall prove that $\|a = 1\| = N_{a^{\circ\circ}}$ for any $a \in A$. Let $P \in Spec(B(A))$. If $P \in \|a = 1\|$, then $a \in < P >$, i.e. there is $e \in P$ such that $e \leq a$. Since $L(A)$ is a Stone lattice we can apply the dual of Proposition 3.2 to obtain $a^\circ = (e \vee a)^\circ = e^\circ \wedge a^\circ$, hence $a^\circ \leq e^\circ$. Applying this again we get $e^{\circ\circ} \leq a^{\circ\circ}$. Since $e \in B(A)$ we have that the dual pseudocomplement of e is just the complement of e. Hence, $e^\circ = e^-$, so $e^{\circ\circ} = e^{=} = e$. Thus, we get $e \leq a^{\circ\circ}$, so $a^{\circ\circ} \in < P >$. Since $a^{\circ\circ} \in B(A)$ also, it follows that $a^{\circ\circ} \in < P > \cap B(A) = P$. Hence, $P \in N_{a^{\circ\circ}}$.
Conversely, let $P \in N_{a^{\circ\circ}}$, i.e. $a^{\circ\circ} \in P$. Since $a^\circ \vee a^{\circ\circ} = a^\circ \vee a = 1$, we have that $a^{\circ\circ} \leq a$. Hence $a \in < P >$, i.e. $P \in \|a = 1\|$.
We have got that $\|a = 1\|$ is a clopen set of $Spec(B(A))$ for any $a \in A$. It follows that A is a Boolean product of the family $\{A_P\}_{P \in Spec(B(A))}$. By Proposition 3.3, any Stone ultrafilter of $L(A)$ is a prime filter of $L(A)$. But, any Stone filter of $L(A)$ is also a filter of A. It follows then that any Stone ultrafilter of $L(A)$ is a prime filter of A. Hence, A_P is a BL-chain for any $P \in Spec(B(A))$. □

4 Weak Boolean products of local BL-algebras

Local BL-algebras are studied in [21, 22] in the same way as local MV-algebras are analyzed in [2]. Thus, a BL-algebra is called *local* iff it has a unique ultrafilter. In the sequel, we remind some definitions and properties.

Proposition 4.1 ([22], Proposition 1)
A nontrivial BL-algebra A is local iff for all $a \in A$,
$ord(a) < \infty$ or $ord(a^-) < \infty$

Lemma 4.2 Let A be a nontrivial BL-algebra A. Then A is local iff $\beta(A)$ is a dense lattice.

Proof: It follows from the fact that $Max(A)$ and $Max(\beta(A))$ are homeomorphic. □
A proper filter P of A is called *primary* if, for all $a, b \in A$,
$(a \odot b)^- \in P$ implies $(a^n)^- \in P$ or $(b^n)^- \in P$ for some $n \in \omega$.

Proposition 4.3 ([22], Proposition 2)
Let P be a filter of A. The following are equivalent:
(i) A/P is a local BL-algebra;
(ii) P is a primary filter of A.

Proposition 4.4 Any local BL-algebra is directly indecomposable.

Proof: Let A be a local BL-algebra. We shall prove that $B(A) = \{0, 1\}$ and then apply Proposition 1.12. Let $e \in B(A)$. Then $e \odot e = e$ and, since the complement of e is $e^- \in B(A)$ we have also $e^- \odot e^- = e^-$. By Proposition 4.1 we get that $ord(e) < \infty$ or $ord(e^-) < \infty$, i.e. there is $n \in \omega - \{0\}$ such that $e^n = 0$ or $(e^-)^n = 0$. But $e^n = e$ and $(e^-)^n = e^-$. It follows that $e = 0$ or $e^- = 0$, i.e. $e \in \{0, 1\}$. Hence, $B(A) = \{0, 1\}$. □

Proposition 4.5 A filter of A is primary iff it is contained in a unique ultrafilter of A.

Proof: Let F be a filter of A. Applying Proposition 4.3, F is primary iff A/F is a local algebra iff A/F has a unique ultrafilter. But, from a general result of universal algebra we have that there is a bijection between the set of filters of A/F and the set of filters of A that contain F and this bijection preserves the ultrafilters. Hence, A/F has a unique ultrafilter iff there is a unique ultrafilter of A that contains F. □

Using Proposition 4.3, we obtain from Theorem 2.1 in a similar manner with Proposition 3.1:

Proposition 4.6 A nontrivial BL-algebra A is a weak Boolean product of local BL-algebras iff the Stone ultrafilters of $L(A)$ are primary filters of A.

Let us recall that a *quasi-local* MV-algebra [13] is a nontrivial MV-algebra A such that for any $a \in A$ there are $e \in B(A)$ and $n \in \omega - \{0\}$ such that $na \oplus e = 1$ and $na^- \oplus e^- = 1$. In [13] it is shown that quasi-local MV-algebras are exactly the weak Boolean products of local algebras. In order to extend this result to BL-algebras we shall define the appropriate concept of quasi-local BL-algebra. A nontrivial BL-algebra A is called *quasi-local* if, for any $a \in A$, there are $e \in B(A)$ and $n \in \omega - \{0\}$ such that $a^n \odot e = 0$ and $(a^-)^n \odot e^- = 0$.

Remark 4.7 Using the reticulation of a BL-algebra we obtain that

A is quasi-local iff for any $a \in A$, there is $e \in B(A)$ such that $[a] \wedge [e] = [0]$ and $[a^-] \wedge [e^-] = [0]$.

Proposition 4.8 For a BL-algebra A the following are equivalent:
(i) A is a quasi-local BL-algebra;
(ii) $MV(A)$ is a quasi-local MV-algebra.

Proof: (i)⇒(ii) Suppose that A is a quasi-local BL-algebra and let $a \in A$. Then, there are $e \in B(A)$ and $n \in \omega - \{0\}$ such that $a^n \odot e = 0$ and $(a^-)^n \odot e^- = 0$. Since $e \in B(A)$, we have also $e^- \in B(A)$ and, by Corrolary 1.10, $e^- \in B(MV(A))$. Applying Lemma 1.1, we get $na^- \oplus e^- = n\varphi(a) \oplus \varphi(e) = \varphi(a^n \odot e) = \varphi(0) = 1$ and, similarly, $na^= \oplus e^= = 1$. Hence, for any $a^- \in MV(A)$ there are $e^- \in B(MV(A))$ and $n \in \omega - \{0\}$ such that $na^- \oplus e^- = 1$ and $na^= \oplus e^= = 1$. That is, $MV(A)$ is a quasi-local MV-algebra.
(ii)⇒(i) Suppose that $MV(A)$ is a quasi-local MV-algebra and let $a \in A$. Then $a^- \in MV(A)$, so there are $e \in B(MV(A))$ and $n \in \omega - \{0\}$ such that $na^- \oplus e = 1$ and $na^= \oplus e^- = 1$. Applying again Lemma 1.1, we obtain $(a^n \odot e^-)^- = \varphi(a^n \odot e^-) = n\varphi(a) \oplus e^= = na^- \oplus e = 1$. Hence, $(a^n \odot e^-)^= = 0$ and, from (1.11), we get $a^n \odot e^- = 0$. We prove similarly that $(a^-)^n \odot e^= = 0$. Hence, for any $a \in A$ there are $e^- \in B(A)$ and $n \in \omega - \{0\}$ such that $a^n \odot e^- = 0$ and $(a^-)^n \odot e^= = 0$. So, A is a quasi-local BL-algebra. □
The next proposition establishes the relationship between local and quasi-local BL-algebras.

Proposition 4.9 For a BL-algebra A the following are equivalent:
(i) A is local;
(ii) A is quasi-local and directly indecomposable.

Proof: (i)⇒(ii) Suppose that A is local. By Proposition 4.4, A is directly indecomposable. Let $a \in A$. From Proposition 4.1 it results that there is $n \in \omega - \{0\}$ such that $a^n = 0$ or $(a^-)^n = 0$. If $a^n = 0$, then letting $e = 1$ we obtain $a^n \odot e = 0$ and $(a^-)^n \odot e^- = (a^-)^n \odot 0 = 0$. If $(a^-)^n = 0$, then with $e = 0$ we get $a^n \odot e = 0$ and $(a^-)^n \odot e^- = 0$. That is, there are $e \in B(A)$ and $n \in \omega - \{0\}$ such that $a^n \odot e = 0$ and $(a^-)^n \odot e^- = 0$. Hence, A is quasi-local.
(ii)⇒(i) Suppose that A is quasi-local and directly indecomposable. Let $a \in A$. Then there are $e \in B(A)$ and $n \in \omega - \{0\}$ such that $a^n \odot e = 0$ and $(a^-)^n \odot e^- = 0$. But, by the fact that A is directly indecomposable, we have $B(A) = \{0, 1\}$. We get $a^n = 0$ or $(a^-)^n = 0$. Applying again Proposition 4.1 it follows that A is local. □

Following [13], we define the quasi-primary filters of a BL-algebra. Thus, a proper filter F of a BL-algebra A is called *quasi-primary* if, for all $a, b \in A$,

$(a \odot b)^- \in F$ implies that there are $n \in \omega - \{0\}$ and $u \in A$ such that $u \vee u^- \in B(A)$, $(a^n \odot u)^- \in F$ and $(b^n \odot u^-)^- \in F$.

Proposition 4.10 Any primary filter of a BL-algebra A is a quasi-primary filter of A.

Proof: Let P be a primary filter of A and $a, b \in A$ such that $(a \odot b)^- \in F$. Then $(a^n)^- \in P$ or $(b^n)^- \in P$ for some $n \in \omega - \{0\}$. If $(a^n)^- \in P$, then letting $u = 1$, we have that $u \vee u^- = 1$, $(a^n \odot u)^- = (a^n)^- \in P$ and $(b^n \odot u^-)^- = 0^- = 1 \in P$. If $(b^n)^- \in P$, then taking $u = 0$ we obtain that $u \vee u^- = 1$, $(a^n \odot u)^- = 0^- = 1 \in P$ and $(b^n \odot u^-)^- = (b^n)^- \in P$. Hence, P is quasi-primary. □

Proposition 4.11 Let F be a filter of A. The following are equivalent:
(i) A/F is a quasi-local BL-algebra;
(ii) F is a quasi-primary filter of A.

Proof: (i)$\Rightarrow$(ii) Let $a, b \in F$ such that $(a \odot b)^- \in F$. Since A/F is quasi-local, there are $u/F \in B(A/F)$ and $n \in \omega - \{0\}$ such that $(a^n \odot u)/F = 0/F$ and $((a^-)^n \odot u^-)/F = 0/F$. By Proposition 1.9, $u/F \in B(A/F)$ iff $(u \vee u^-)/F = 1/F$ iff $u \vee u^- \in F$. From $(a^n \odot u)/F = 0/F$ we get $(a^n \odot u) \to 0 \in F$, so $(a^n \odot u)^- \in F$. By (1.7), $(a \odot b)^- = b \to a^-$. Hence, $(a \odot b)^- \in F$ iff $b \to a^- \in F$ iff $b/F \to a^-/F = 1/F$ iff $b/F \leq a^-/F$. It follows that $(b^n \odot u^-)/F \leq ((a^-)^n \odot u^-)/F = 0/F$. Hence, $(b^n \odot u^-)/F = 0/F$, i.e. $(b^n \odot u^-)^- \in F$. Thus, we have got that F is quasi-primary.
(ii)$\Rightarrow$(i) Let $a \in A$. We have that $(a \odot a^-)^- = 0^- = 1 \in F$. Since F is quasi-primary, there are $n \in \omega - \{0\}$ and $u \in A$ such that $u \vee u^- \in B(A)$, $(a^n \odot u)^- \in F$ and $((a^-)^n \odot u^-)^- \in F$. From $u \vee u^- \in F$ we get that $u/F \in B(A/F)$. We have also that from $(a^n \odot u)^- \in F$ it folows that $(a^n \odot u) \to 0 \in F$, hence $(a^n \odot u)/F = 0/F$. Similarly, from $((a^-)^n \odot u^-)^- \in F$ we get $((a^-)^n \odot u^-)/F = 0/F$. Thus, we have got that A/F is quasi-local. □

Proposition 4.12 Let A be a BL-algebra. The following are equivalent:
(i) A is quasi-local;
(ii) any proper filter of A is quasi-primary.

Proof: (i)$\Rightarrow$(ii) Let F be a filter of A. We shall prove that A/F is quasi-local and then apply Proposition 4.11. Since A is quasi-local, there are $e \in B(A)$ and $n \in \omega - \{0\}$ such that $a^n \odot e = 0$ and $(a^-)^n \odot e^- = 0$. It follows that $(a^n \odot e)/F = 0/F$ and $(a^-)^n \odot e^-/F = 0/F$. Since $e \in B(A)$, we have that $e \vee e^- = 1 \in F$, hence $e/F \in B(A/F)$.
(i)$\Rightarrow$(ii) Since $\{1\}$ is a filter of A, it follows that $\{1\}$ is a quasi-primary filter of A. Applying Proposition 4.11, we get that $A/\{1\}$ is quasi-local. But $A \cong A/\{1\}$, hence A is quasi-local. □

Before proving the most important result of this chapter, we shall remind that any bounded distributive lattice $(L, \vee, \wedge, 0, 1)$ can be represented as the weak Boolean product of the family $\{L_P\}_{P \in Spec(B(L)}$ (see [6], [10] for details).

Proposition 4.13 ([14], Proposition 2.5)
Let L be a bounded distributive lattice. The following are equivalent:
(i) L is B-normal;
(ii) L_P is a dense lattice for any $P \in Spec(B(L)$.

Proposition 4.14 Let A be a nontrivial BL-algebra A. The following are equivalent:
(i) each Stone ultrafilter of $L(A)$ is a primary filter of A;
(ii) each Stone ultrafilter of $L(A)$ is contained in a unique ultrafilter of A;
(iii) each prime filter of $B(A)$ is contained in a unique ultrafilter of A;

(iv) A is quasi-local;
(v) $Max(A)$ is zero-dimensional.

Proof: (i)$\Leftrightarrow$(ii) See Proposition 4.5.
(ii)$\Rightarrow$(iii) Let P be a prime filter of $B(A)$. Then $< P >$ is a Stone ultrafilter of $L(A)$, hence it is contained in a unique ultrafilter U of A. But $P \subseteq < P >$, so P is contained in U. If there are two ultrafilters U_1 and U_2 of A that contain P, then they contain also $< P >$. We obtain that the Stone ultrafilter $< P >$ of $L(A)$ is contained in two ultrafilters of A.
(iii)$\Rightarrow$(ii) Let P be a Stone ultrafilter of $L(A)$. Then $P \cap B(A)$ is an ultrafilter, hence a prime filter of $B(A)$. By (iii), it is contained in a unique ultrafilter U of A. It follows that $P =< P \cap B(A) > \subseteq U$.If there are two ultrafilters U_1 and U_2 of A that contain P, then they contain also $P \cap B(A)$. We obtain that the prime filter $P \cap B(A)$ of $B(A)$ is contained in two ultrafilters of A.
(iii)$\Rightarrow$(iv) Let $a \in A$. We have to show that there are $e \in B(A)$ and $n \in \omega - \{0\}$ such that $a^n \odot e = 0$ and $(a^-)^n \odot e^- = 0$.
We define $F =< a^- > \cap B(A)$ and $I = \{e \in B(A) \mid a^n \odot e = 0$ for some $n \in \omega - \{0\}\}$. Then, it is clear that F is a filter of $B(A)$ and $e \in F$ iff $e \in B(A)$ and there is $n \in \omega - \{0\}$ such that $(a^-)^n \odot e^- = 0$. Let us prove that I is an ideal of $B(A)$. It is easy to see that $0 \in I$ and if $e \leq f$, $e \in B(A), f \in I$, then $e \in I$. Let $e, f \in I$. Then there are $n, m \in \omega - \{0\}$ such that $a^n \odot e = 0$ and $a^m \odot f = 0$. Letting $k = max\{n, m\}$, it follows that $a^k \odot e = a^k \odot f = 0$. We get $a^k \odot (e \vee f) = (a^k \odot e) \vee (a^k \odot f) = 0 \vee 0 = 0$. Hence, $e \vee f \in I$.
If $e \in I \cap F$, then $e \in B(A)$ and there are $m, n \in \omega - \{0\}$ such that $a^m \odot e = 0$ and $(a^-)^n \odot e^- = 0$. Taking $k = max\{n, m\}$ we have $a^k \odot e = 0$ and $(a^-)^k \odot e^- = 0$. Hence, A is quasi-local.
Therefore, to complete the proof we need to show that $I \cap F \neq \emptyset$. Suppose not, i.e. $I \cap F = \emptyset$. Then there is a prime filter P of $B(A)$ such that $F \subseteq P$ and $P \cap I = \emptyset$. Then, if $< a, P >$ is the filter of A generated by $\{a\} \cup P$, we have that $a^- \notin < a, P >$. Indeed, if $a^- \in < a, P >$, then there would be $n \in \omega$ and $p \in P$ such that $a^n \odot p \leq a^-$. If $n = 0$, then $p \leq a^-$, hence $a \odot p = 0$. It follows that $p \in I$, which is a contradiction, because $P \cap I = \emptyset$. Suppose that $n > 0$. Using that $a^n \odot p \leq a$, from $a^n \odot p \leq a^-$, we obtain $a^{2n} \odot p^n \leq a \odot a^- = 0$. But $p^n = p$, because $p \in B(A)$. It follows that $a^{2n} \odot p = 0$, hence $p \in I$. We have got again $p \in P \cap I$, that is a contradiction.
Let $< a^-, P >$ be the filter of A generated by $\{a^-\} \cup P$. If $a \in < a^-, P >$, then there are $n \in \omega$ and $p \in P$ such that $(a^-)^n \odot p \leq a$. If $n = 0$, then $p \leq a$, hence $a^- \odot p = 0$. Because $p \in B(A)$, we have that $p = p^{=}$. We obtain $a^- \odot p^{=} = 0$, hence $p^- \in F \subseteq P$. Since $p \in P$ also, it follows that $0 = p \odot p^- = p \wedge p^- \in P$. This is a contradiction with the fact that P is a prime filter, hence it must be a proper filter of $B(A)$. Suppose that $n > 0$. Then, using that $(a^-)^n \odot p \leq a^-$, we obtain $(a^-)^{2n} \odot p^n \leq a \odot a^- = 0$. Since $p^n = p$, we get $(a^-)^{2n} \odot p = 0$, hence $(a^-)^{2n} \odot p^{=} = 0$. We have obtained again that $p^- \in F$ and we get similarly a contradiction.

Hence, $a \notin < a^-, P >$ and $a^- \notin < a, P >$. It follows that $< a, P >$ and $< a^-, P >$ are proper filters of nontrivial BL-algebra A. Applying Proposition 1.2, there are ultrafilters U_1 and U_2 of A such that $< a, P > \subseteq U_1$ and $< a^-, P > \subseteq U_2$. It is clear that $a \notin U_2$ and $a^- \notin U_1$. Then, $U_1 \neq U_2$. It follows that U_1 and U_2 are distinct ultrafilters of A that contain the prime filter P of $B(A)$. This contradicts (iii).
(iv)⇒(iii) Suppose that A is quasi-local and let P be a prime filter of $B(A)$. Then, $0 \notin P$, so by Proposition 1.2, it is contained in an ultrafilter of A. Suppose that there are two distinct ultrafilters U_1, U_2 of A such that $P \subseteq U_1$ and $P \subseteq U_2$. Since U_1, U_2 are ultrafilters, we have that $U_1 \not\subseteq U_2$ and $U_2 \not\subseteq U_1$. Hence, there is $a \in U_1$ such that $a \notin U_2$. It follows that $U_2 \subset < U_2 \cup \{a\} >$. Since U_2 is an ultrafilter of A, we must have $< U_2 \cup \{a\} > = A$, hence $0 \in < U_2 \cup \{a\} >$. Then, there are $p \in U_2$ and $n \in \omega$ such that $a^n \odot p = 0$. Since $p \in U_2$, we have that $p \neq 0$, so $n > 0$. From $a^n \odot p = 0$ we get $p \leq (a^n)^-$, hence $(a^n)^- \in U_2$. Let $b = a^n$. From the fact that A is quasi-local it follows that there are $m \in \omega - \{0\}$ and $e \in B(A)$ such that $b^m \odot e = 0$ and $(b^-)^m \odot e^- = 0$. Since $a \in U_1$, it follows that $b \in U_1$, so $b^m \in U_1$. From $b^m \odot e = 0$ iff $b^m \leq e^-$, we get $e^- \in U_1 \cap B(A)$. Similarly, from $b^- = (a^n)^- \in U_2$ and $(b^-)^m \odot e^- = 0$, we get $e^= = e \in U_2 \cap B(A)$. Hence, we have obtained $e \in B(A)$ such that $e^- \in U_1$ and $e \in U_2$. But $1 = e \vee e^- \in P$ and P is a prime filter of $B(A)$. It follows that $e \in P$ or $e^- \in P$. If $e \in P \subseteq U_1$, then $0 \in U_1$, since $e^- \in U_1$ too. Hence, U_1 is not a proper filter of A. Similarly, from $e^- \in P$ we obtain that U_2 is not a proper filter of A. Hence, by supposing that P is contained in two distinct ultrafilters of A we have got a contradiction. So, (iii) is proved.
(iv)⇒(v) Since for any $e \in B(A)$, we have that $D(e^-) = Max(A) - D(e)$, it follows that $D(e)$ is clopen for any $e \in B(A)$. We shall prove that the family $\{D(e) \mid e \in B(A)\}$ is a basis for $Max(A)$. Let $a \in A$. If $U \in D(a)$, then $a \in U$, hence $a^- \notin U$. We get that there are $u \in U$ and $n \in \omega - \{0\}$ such that $(a^-)^n \odot u = 0$. Since A is quasi-local, there are $e \in B(A)$ and $k \in \omega - \{0\}$ such that $u^k \odot e = 0$ and $(u^-)^k \odot e^- = 0$. Since $u^k \in U$ and $u^k \odot e = 0 \notin U$, we get that $e \notin U$, so $e^- \in U$, i.e. $U \in D(e^-)$. In the sequel, we prove that $D(e^-) \subseteq D(a)$. Let $M \in D(e^-)$, i.e. $e^- \in M$. From $e^- \leq ((u^-)^k)^-$, we obtain that $((u^-)^k)^- \in M$. Hence, $(u^-)^k \notin M$, so $u^- \notin M$. It follows that $u \in M$ and, since $(a^-)^n \odot u = 0 \notin M$, we get $(a^-)^n \notin M$. From this we obtain that $a^- \notin M$, hence $a \in M$. Thus, $M \in D(a)$. We have proved that for any $U \in D(a)$ there is $e_U \in B(A)$ such that $U \in D(e_U) \subseteq D(a)$. We get that $D(a) = \cup_{U \in D(a)} D(e_U)$. Hence, $\{D(e) \mid e \in B(A)\}$ is a basis of clopen subsets of $Max(A)$.
(v)⇒(i) Suppose that $Max(A)$ is zero-dimensional. Since $Max(A)$ is homeomorphic to $Max(\beta(A))$, it folows that $Max(\beta(A))$ is zero-dimensional. By Proposition 1.21, $\beta(A)$ is a normal lattice. Hence, by Proposition 1.18, $\beta(A)$ is a B-normal lattice. Applying Proposition 4.13 and Proposition 1.23 we get that $\beta(A)_{\beta(P)}$ is a dense lattice for any $P \in Spec(B(A))$. That is, by Proposition 1.25, $\beta(A_P)$ is dense for any $P \in Spec(B(A))$. We apply Proposition 4.2 to obtain that A_P is a local BL-algebra for any $P \in Spec(B(A)$. Hence, each Stone

ultrafilter of $L(A)$ is a primary filter of A. $\square$

From Propositions 4.6 and 4.14 we obtain

Theorem 4.15 A nontrivial BL-algebra A is a weak Boolean product of local BL-algebras iff A is quasi-local.

5 Weak Boolean products of perfect BL-algebras

As in the case of MV-algebras (see [2, 11]), a BL-algebra A is called *perfect* if it is local and for any $a \in A$,

$ord(a) < \infty$ implies $ord(a^-) = \infty$.

The filters corresponding to perfect BL-algebras are perfect filters. A proper filter P of A is *perfect* if, for all $a \in A$,

$(a^n)^- \in P$ for some $n \in \omega$ iff $((a^-)^m)^- \notin P$ for all $m \in \omega$.

Proposition 5.1 ([22], Proposition 14)
Let P be a filter of A. The following are equivalent:
(i) A/P is a perfect BL-algebra;
(ii) P is a perfect filter of A.

Proposition 5.2 ([22], Proposition 15)
Any perfect filter of A is a primary filter of A.

Using Proposition 5.1 and Theorem 2.1 we get the following result, similar to Proposition 4.6.

Proposition 5.3 A nontrivial BL-algebra A is a weak Boolean product of perfect BL-algebras iff the Stone ultrafilters of $L(A)$ are perfect filters of A.

In [13] there were also defined quasi-perfect MV-algebras and there was proved that quasi-perfect MV-algebras are exactly the weak Boolean products of perfect MV-algebras. A MV-algebra A is *quasi-perfect* if it is quasi-local and for any $a \in A, e \in B(A) - \{1\}$,

$na \oplus e = 1$ for some $n \in \omega$ implies $ma^- \oplus e \neq 1$ for all $m \in \omega$.

In the sequel, we shall extend all these to BL-algebras.
A BL-algebra A is *quasi-perfect* if it is quasi-local and satisfies
$(*)$ for any $a \in A, e \in B(A) - \{0\}$,

$a^n \odot e = 0$ for some $n \in \omega$ implies $(a^-)^m \odot e \neq 0$ for all $m \in \omega$.

Remark 5.4 Using the reticulation we get that

A is quasi-perfect iff for any $a \in A, e \in B(A) - \{0\}$, if $[a] \wedge [e] = [0]$ then $[a^-] \wedge [e] \neq [0]$.

In analogy with Propositions 4.8 and 4.9, we have that

Proposition 5.5 For a BL-algebra A the following are equivalent:
(i) A is a quasi-perfect BL-algebra;
(ii) $MV(A)$ is a quasi-perfect MV-algebra.

Proof: (i)$\Rightarrow$(ii) By Proposition 4.8, $MV(A)$ is a quasi-local MV-algebra. Let $a^- \in MV(A), e \in B(MV(A)) - \{1\}$ such that $na^- \oplus e = 1$ for some $n \in \omega$ and suppose that $ma^= \oplus e = 1$ for some $m \in \omega$. We get that $e^- \in B(A) - \{0\}$. Applying Lemma 1.1, it follows that $(a^n \odot e^-)^- = \varphi(a^n \odot e^-) = na^- \oplus e^= = na^- \oplus e = 1$, hence $(a^n \odot e^-)^= = 0$ and, finally, $a^n \odot e^- = 0$. Similarly, $((a^-)^m \odot e^-)^- = ma^= \oplus e = 1$, so $(a^-)^m \odot e^- = 0$. We have g'ot $e^- \in B(A) - \{0\}$ and $n, m \in \omega$ such that $a^n \odot e^- = 0$ and $(a^-)^m \odot e^- = 0$. This contradicts the fact that A is a quasi-perfect BL-algebra.
(ii)$\Rightarrow$(i) By Proposition 4.8, A is a quasi-local BL-algebra. Let $a \in A, e \in B(A) - \{0\}$ such that $a^n \odot e = 0$ for some $n \in \omega$ and suppose that $(a^-)^m \odot e = 0$ for some $m \in \omega$. We have $e^- \in B(MV(A)) - \{1\}$. Applying again Lemma 1.1, we get $na^- \oplus e^- = \varphi(a^n \odot e) = \varphi(0) = 1$ and, similarly, $ma^= \oplus e^- = \varphi((a^-)^m \odot e) = \varphi(0) = 1$. Hence there are $a^- \in MV(A)$, $e^- \in B(MV(A)) - \{1\}$ and $n, m \in \omega$ such that $na^- \oplus e^- = 1$ and $ma^= \oplus e^- = 1$. This is a contradiction with $MV(A)$ being a quasi-perfect MV-algebra. □

Proposition 5.6 For a BL-algebra A the following are equivalent:
(i) A is perfect;
(ii) A is quasi-perfect and directly indecomposable.

Proof: (i)$\Rightarrow$(ii) Since A is perfect it follows that A is local, hence, by Proposition 4.4, A is directly indecomposable. We have also from Proposition 4.9 that A is quasi-local. Because $B(A) = \{0, 1\}$, the condition $(*)$ from the definition of quasi-perfect BL-algebras must be verified only for $e = 1$. Let $a \in A$ such that $a^n = 0$ for some $n \in \omega$, hence $ord(a) < \infty$. Because A is a perfect BL-algebra, we obtain that $ord(a^-) = \infty$. Hence, $(a^-)^m \neq 0$ for all $m \in \omega$.
(ii)$\Rightarrow$(i) Applying Proposition 4.9, it follows that A is local. Since $B(A) = \{0, 1\}$, in the condition $(*)$ we have $e = 1$. We get that, for any $a \in A$, $a^n = 0$ for some $n \in \omega$ implies $(a^-)^m \neq 0$ for all $m \in \omega$. This is equivalent to $ord(a) < \infty$ implies $ord(a^-) = \infty$. That is, A is perfect. □

A proper filter F of A is called *quasi-perfect* if it is quasi-primary and satisfies,
$(**)$ for all $a \in A$, $u \in A$ such that $u \vee u^- \in F$ and $u^- \notin F$,
$(a^n \odot u)^- \in F$ for some $n \in \omega$ implies $((a^-)^m \odot u)^- \notin F$ for all $m \in \omega$.

Proposition 5.7 Let F be a filter of A. The following are equivalent:
(i) A/F is a quasi-perfect BL-algebra;
(ii) F is a quasi-perfect filter of A.

Proof: It follows easily by Proposition 4.11 and the fact that for any $u \in A$, $u/F \in B(A/F) - \{0\}$ iff $u \vee u^- \in F$ and $u^- \notin F$. □

Proposition 5.8 Let A be a nontrivial BL-algebra A. The following are equivalent:
(i) each Stone ultrafilter of $L(A)$ is a perfect filter of A;
(ii) A is quasi-perfect.

(i)$\Rightarrow$ (ii) From Proposition 5.2 we obtain that each Stone ultrafilter of $L(A)$ is a primary filter of A. Applying Proposition 4.14, it results that A is quasi-local. Let $a \in A$ and $e \in B(A) - \{0\}$ such that $a^n \odot e = 0$ for some $n \in \omega$, hence $e \leq (a^n)^-$. Since $e \in B(A) - \{0\}$, there exist a prime filter P of $B(A)$ such that $e \in P$. Hence, $(a^n)^- \in < P >$. Suppose now that $(a^-)^m \odot e = 0$ for some $m \in \omega$. It follows that $((a^-)^m)^- \in < P >$. Hence, $(a^n)^- \in < P >$ for some $n \in \omega$ and $((a^-)^m)^- \in < P >$ for some $m \in \omega$, i.e. $< P >$ is not perfect. But $< P >$ is a Stone ultrafilter of $L(A)$. We have got a contradiction with (i).
(ii)$\Rightarrow$ (i) Let $< P >$ be a Stone ultrafilter of $L(A)$, where P is a prime filter of $B(A)$. Because A is quasi-local, from Proposition 4.14 we obtain that $< P >$ is primary. Suppose that $< P >$ is not perfect. Then there are $a \in A, m, n \in \omega$ such that $(a^n)^- \in < P >$ and $((a^-)^m)^- \in < P >$. So, there are $e_1, e_2 \in P$ with $e_1 \leq (a^n)^-$ and $e_2 \leq ((a^-)^m)^-$. Taking $e = e_1 \wedge e_2$ we obtain $e \in P$, hence $e \in B(A) - \{0\}$ and $e \leq (a^n)^-$, $e \leq ((a^-)^m)^-$. That is, $e \in B(A) - \{0\}$ such that $a^n \odot e = 0$ and $(a^-)^m \odot e = 0$, which contradicts the fact that A is quasi-perfect. $\Box$

From Propositions 5.3 and 5.8 we obtain

Theorem 5.9 A nontrivial BL-algebra A is a weak Boolean product of perfect BL-algebras iff A is quasi-perfect.

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