

Canonical models and filtrations in three-valued propositional modal logic

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In this paper we present a new and simpler axiomatization for the three-valued modal propositional logic defined by P. Ostermann in [9]. We define the canonical model and the filtration for this logic, and we prove the three-valued versions of the Truth and Filtration Lemmas, well-known results in the propositional modal logic. We obtain a new proof for the completeness of the smallest three-valued normal logic **3-K**. Also, **3-K** has the finite model property.

These results can be easily extended to three-valued versions of other systems of propositional modal logic.

Keywords: modal logic, three-valued Lukasiewicz logic, canonical model, filtration.

1 Introduction

P. Ostermann defined in [9] a system of propositional modal logic based on the n -valued Lukasiewicz logic. He proved the completeness of this logic following the method used for modal logic by S. Kripke in [8].

For the three-valued case we present a very simple and natural axiomatization. Our proof of the completeness theorem is achieved combining the method used to demonstrate the completeness of three-valued Lukasiewicz logic ([5]) with a technique that has been standard in modal logic for many years- the canonical model/filtration technique (see [6, 2, 7]).

2 Syntax and Semantics

In this section we give syntactical and semantical definitions for our logic. We use Lukasiewicz's three-valued logic as defined in [1] and [4]. The basic reference for modal logic is [6](see also [2, 3, 7]).

2.1 Three-valued modal logics

The *language* of the three-valued propositional modal logic is determined by:

- (i) a countable set AF of *atomic formulas*, denoted by v_0, v_1, \dots , etc.

- (ii) the propositional connectives \neg , \rightarrow ,
- (iii) the modal operator \Box .

The set $Fmla$ of *formulas* is defined inductively as follows:

- (i) $AF \subseteq Fmla$,
- (ii) if $p \in Fmla$ and $q \in Fmla$, then $p \rightarrow q$ and $\neg p \in Fmla$,
- (iii) if $p \in Fmla$, then $\Box p \in Fmla$.

We use the standard abbreviations \vee , \wedge , \leftrightarrow , \oplus and \odot from the three-valued Lukasiewicz logic:

$$\begin{aligned} p \vee q &:= ((p \rightarrow q) \rightarrow q), \\ p \wedge q &:= \neg(\neg p \vee \neg q), \\ p \leftrightarrow q &:= (p \rightarrow q) \wedge (q \rightarrow p), \\ p \oplus q &:= \neg p \rightarrow q, \\ p \odot q &:= \neg(\neg p \oplus \neg q). \end{aligned}$$

We also define:

$$\begin{aligned} \sim p &:= p \rightarrow \neg p, \\ \diamond p &:= \neg \Box \neg p. \end{aligned}$$

By a *three-valued modal logic* over the given language we understand any set \mathcal{L} of formulas that satisfies the following conditions:

- (i) \mathcal{L} contains every tautology of the three-valued Lukasiewicz logic,
- (ii) \mathcal{L} is closed under *modus ponens*,
- (iii) \mathcal{L} is closed under *uniform substitution*.

The members of a three-valued modal logic \mathcal{L} are called its *theorems*. We write $\vdash_{\mathcal{L}} p$ to mean that p is a theorem of \mathcal{L} .

An example of a three-valued modal logic is $Fmla$.

Let \mathcal{L} be a three-valued modal logic and $\Sigma \cup \{p\} \subseteq Fmla$. We say that p is *\mathcal{L} -deducible from Σ* and we write $\Sigma \vdash_{\mathcal{L}} p$ if there exist $p_0, \dots, p_{n-1} \in \Sigma$ such that

$$\vdash_{\mathcal{L}} (p_0 \rightarrow (p_1 \rightarrow (\dots \rightarrow (p_{n-1} \rightarrow p) \dots))).$$

(in the case $n = 0$, this means that $\vdash_{\mathcal{L}} p$).

A set Σ of formulas is *\mathcal{L} -consistent* if there is no formula p such that $\Sigma \vdash_{\mathcal{L}} p$ and $\Sigma \vdash_{\mathcal{L}} \neg p$; if not, Σ is *\mathcal{L} -inconsistent*. A \mathcal{L} -consistent set Σ is *maximal \mathcal{L} -consistent* if $p \in \Sigma$ for any formula p such that $\Sigma \cup \{p\}$ is \mathcal{L} -consistent.

In the sequel we shall prove some syntactical properties.

We remind some tautologies of the three-valued Lukasiewicz logic, which will be used in our proofs ([1, 4]):

- (t1) $p \rightarrow (q \rightarrow p)$,
- (t2) $(p \rightarrow q) \rightarrow ((q \rightarrow r) \rightarrow (p \rightarrow r))$,
- (t3) $p \rightarrow p$,
- (t4) $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$,
- (t5) $p \leftrightarrow \neg \neg p$,
- (t6) $\neg p \rightarrow (p \rightarrow q)$,
- (t7) $(p \rightarrow (p \rightarrow (q \rightarrow r))) \rightarrow ((p \rightarrow (p \rightarrow q)) \rightarrow (p \rightarrow (p \rightarrow r)))$,
- (t8) $\sim \sim p \rightarrow p$,
- (t9) $(p \rightarrow \sim p) \rightarrow \sim p$,

- (t10) $(p \wedge (q \wedge r)) \leftrightarrow ((p \wedge q) \wedge r),$
- (t11) $(p \wedge q) \leftrightarrow (q \wedge p),$
- (t12) $(p \wedge q) \rightarrow p,$
- (t13) $p \rightarrow (q \rightarrow (p \wedge q)),$
- (t14) $(p \vee (q \vee r)) \leftrightarrow ((p \vee q) \vee r),$
- (t15) $(p \vee q) \leftrightarrow (q \vee p),$
- (t16) $p \rightarrow (p \vee q),$
- (t17) $(p \odot (q \odot r)) \leftrightarrow ((p \odot q) \odot r),$
- (t18) $(p \odot q) \leftrightarrow (q \odot p),$
- (t19) $p \odot q \rightarrow p,$
- (t20) $p \rightarrow (q \rightarrow p \odot q),$
- (t21) $(p \rightarrow (q \rightarrow r)) \leftrightarrow (p \odot q \rightarrow r),$
- (t22) $(p \rightarrow q) \rightarrow (p \odot r \rightarrow q \odot r),$
- (t23) $(p \oplus (q \oplus r)) \leftrightarrow ((p \oplus q) \oplus r),$
- (t24) $(p \oplus q) \leftrightarrow (q \oplus p),$
- (t25) $p \rightarrow p \oplus q,$
- (t26) $(p \vee q) \rightarrow (p \oplus q).$

Let \mathcal{L} be a three-valued modal logic.

Remark 2.1 By (t21) and (t17), $\vdash_{\mathcal{L}} p$ iff there exist $p_0, \dots, p_{n-1} \in \Sigma$ such that $\vdash_{\mathcal{L}} p_0 \odot p_1 \odot \dots \odot p_{n-1} \rightarrow p.$

Proposition 2.2 Let $\Sigma \subseteq Fmla$ and $p, q \in Fmla.$

- (i) $\vdash_{\mathcal{L}} p$ iff $\emptyset \vdash_{\mathcal{L}} p.$
- (ii) If $\vdash_{\mathcal{L}} p,$ then $\Sigma \vdash_{\mathcal{L}} p.$
- (iii) If \mathcal{L}' is another three-valued modal logic such that $\mathcal{L} \subseteq \mathcal{L}'$, then $\Sigma \vdash_{\mathcal{L}} p$ implies $\Sigma \vdash_{\mathcal{L}'} p.$
- (iv) If $\Sigma \vdash_{\mathcal{L}} p,$ then $\Gamma \vdash_{\mathcal{L}} p$ for any superset Γ of $\Sigma.$
- (v) If $\Sigma \vdash_{\mathcal{L}} p,$ then there exists a finite subset Γ of Σ such that $\Gamma \vdash_{\mathcal{L}} p.$
- (vi) If $p \in \Sigma,$ then $\Sigma \vdash_{\mathcal{L}} p.$
- (vii) If $\Sigma \vdash_{\mathcal{L}} p$ and $\Sigma \vdash_{\mathcal{L}} p \rightarrow q,$ then $\Sigma \vdash_{\mathcal{L}} q.$
- (viii) $\Sigma \vdash_{\mathcal{L}} p$ iff there exists a finite sequence $q_0, \dots, q_m = p$ such that for any $i \leq m,$ we have one of the following possibilities:
 - (a) $q_i \in \mathcal{L} \cup \Sigma,$
 - (b) there exist $j, k < i$ such that $q_k = q_j \rightarrow q_i.$

Proof: The proof is immediate by the definitions. See, for example, [1], Lemma 1.7, pag. 465-466 and [6], Exercises 2.2, pag. 18. \square

A sequence $q_0, \dots, q_m = p$ as in Proposition 2.2.(viii) is called a *formal proof* of p from $\Sigma.$

A Deduction Theorem is also true.

Theorem 2.3 Let $\Sigma \subseteq Fmla$ and $p, q \in Fmla.$ Then $\Sigma \cup \{p\} \vdash_{\mathcal{L}} q$ iff $\Sigma \vdash_{\mathcal{L}} p \rightarrow (p \rightarrow q).$

Proof: Similar to [1], Lemma 1.7.(d), pag. 466. \square

We give now some properties of \mathcal{L} -consistent and maximal \mathcal{L} -consistent sets of formulas.

Proposition 2.4 Let $\Sigma \subseteq Fmla$ and $p \in Fmla$.

- (i) Σ is \mathcal{L} -inconsistent iff $\Sigma \vdash_{\mathcal{L}} r$ for any formula r .
- (ii) $\Sigma \cup \{p\}$ is \mathcal{L} -inconsistent iff $\Sigma \vdash_{\mathcal{L}} \sim p$.
- (iii) $\Sigma \cup \{\sim p\}$ is \mathcal{L} -inconsistent iff $\Sigma \vdash_{\mathcal{L}} p$.
- (iv) Σ is \mathcal{L} -consistent iff every finite subset of Σ is \mathcal{L} -consistent.
- (v) If Σ is \mathcal{L} -consistent, then for any formula p , at least one of $\Sigma \cup \{p\}$ and $\Sigma \cup \{\sim p\}$ is \mathcal{L} -consistent.
- (vi) If $\Sigma \subseteq \mathcal{L}$, then Σ is \mathcal{L} -consistent iff $\mathcal{L} \neq Fmla$.

Proof: (i) (\Rightarrow) There exists $p \in Fmla$ such that $\Sigma \vdash_{\mathcal{L}} p$ and $\Sigma \vdash_{\mathcal{L}} \neg p$. Apply (t6) and Proposition 2.2.(vii) to get $\Sigma \vdash_{\mathcal{L}} r$.

(\Leftarrow) Obviously.

(ii) (\Rightarrow) Suppose that $\Sigma \cup \{p\}$ is \mathcal{L} -inconsistent. Then $\Sigma \cup \{p\} \vdash_{\mathcal{L}} \neg p$, by (i). Applying Deduction Theorem we obtain $\Sigma \vdash_{\mathcal{L}} p \rightarrow \sim p$. Using (t9) and Proposition 2.2.(ii) and (vii), it follows that $\Sigma \vdash_{\mathcal{L}} \sim p$.

(\Leftarrow) By $\Sigma \vdash_{\mathcal{L}} \sim p$ and Proposition 2.2.(vi) and (iv) we get $\Sigma \cup \{p\} \vdash_{\mathcal{L}} p$ and $\Sigma \cup \{p\} \vdash_{\mathcal{L}} \sim p$. Using the definition of $\sim p$ it follows that $\Sigma \cup \{p\} \vdash_{\mathcal{L}} p$ and $\Sigma \cup \{p\} \vdash_{\mathcal{L}} \neg p$, hence $\Sigma \cup \{p\}$ is \mathcal{L} -inconsistent.

(iii) (\Rightarrow) By the fact that $\Sigma \cup \{\sim p\}$ is \mathcal{L} -inconsistent, using (ii) we obtain $\Sigma \vdash_{\mathcal{L}} \sim \sim p$. Applying (t8), it follows that $\Sigma \vdash_{\mathcal{L}} p$.

(\Leftarrow) Similar to (ii).

(iv) (\Rightarrow) Obviously.

(\Leftarrow) Suppose that Σ is \mathcal{L} -inconsistent and let p be any formula. Then, $\Sigma \vdash_{\mathcal{L}} \neg(p \rightarrow p)$. Applying Proposition 2.2.(v), there exists a finite subset Γ of Σ such that $\Gamma \vdash_{\mathcal{L}} \neg(p \rightarrow p)$. But, by (t3), $\Gamma \vdash_{\mathcal{L}} (p \rightarrow p)$. We get Γ is \mathcal{L} -inconsistent, so we contradict the hypothesis.

(v) Suppose that $\Sigma \cup \{p\}$ and $\Sigma \cup \{\sim p\}$ are \mathcal{L} -inconsistent. By (ii) and (iii) we have $\Sigma \vdash_{\mathcal{L}} p$ and $\Sigma \vdash_{\mathcal{L}} \sim p$, hence Σ is \mathcal{L} -inconsistent.

(vi) (\Rightarrow) Obviously.

(\Leftarrow) Suppose that Σ is \mathcal{L} -inconsistent, hence $\Sigma \vdash_{\mathcal{L}} p$ for any formula p . Because $\Sigma \subseteq \mathcal{L}$, we get $\mathcal{L} \vdash_{\mathcal{L}} p$ for any formula $p \in Fmla$. Using the fact that \mathcal{L} is closed under modus ponens it follows that $\mathcal{L} = Fmla$. \square

Proposition 2.5 Let Σ be a maximal \mathcal{L} -consistent set and $p, q \in Fmla$.

- (i) $\Sigma \vdash_{\mathcal{L}} p$ implies $p \in \Sigma$.
- (ii) $\mathcal{L} \subseteq \Sigma$.
- (iii) If $\Sigma \subseteq \Gamma$ and Γ is \mathcal{L} -consistent, then $\Sigma = \Gamma$.
- (iv) $p \in \Sigma$ iff $\sim p \notin \Sigma$.
- (v) $p \vee q \in \Sigma$ iff $(p \in \Sigma$ or $q \in \Sigma)$.
- (vi) $p \wedge q \in \Sigma$ iff $(p \in \Sigma$ and $q \in \Sigma)$.

- (vii) $p \odot q \in \Sigma$ iff $(p \in \Sigma \text{ and } q \in \Sigma)$.
- (viii) If $p \in \Sigma$ or $q \in \Sigma$, then $p \oplus q \in \Sigma$.
- (ix) If $(p \rightarrow q) \in \Sigma$, then $p \in \Sigma$ implies $q \in \Sigma$.
- (x) If $(p \leftrightarrow q) \in \Sigma$, then $p \in \Sigma$ iff $q \in \Sigma$.

Proof: (i) Suppose that $p \notin \Sigma$. It follows that $\Sigma \cup \{p\}$ is \mathcal{L} -inconsistent, hence, using Lemma 2.4.(ii), $\Sigma \vdash_{\mathcal{L}} \sim p$. By $\Sigma \vdash_{\mathcal{L}} p$ and $\Sigma \vdash_{\mathcal{L}} \sim p$ we get $\Sigma \vdash_{\mathcal{L}} \perp$, so Σ is \mathcal{L} -inconsistent.

(ii) Immediately using (i).

(iii) Let $p \in \Gamma$. Then $\Sigma \cup \{p\} \subseteq \Gamma$ and Γ is \mathcal{L} -consistent. Hence, $\Sigma \cup \{p\}$ is \mathcal{L} -consistent. It follows that $p \in \Sigma$.

(iv) (\Leftarrow) Let $p \in \Sigma$ and suppose that $\sim p \in \Sigma$. It follows that Σ is \mathcal{L} -inconsistent.

(\Rightarrow) Suppose that $\sim p \notin \Sigma$. Then $\Sigma \cup \{\sim p\}$ is \mathcal{L} -inconsistent and, by Proposition 2.4.(vi), it follows that $\Sigma \cup \{p\}$ is \mathcal{L} -consistent. Hence, $p \in \Sigma$.

(v) (\Rightarrow) Suppose that $p \vee q \in \Sigma$ and $p \notin \Sigma$. By (iv) it follows that $\sim p \in \Sigma$. By $p \vee q \in \Sigma$, (t26) and the definition of \oplus we get $\neg p \rightarrow q \in \Sigma$. Using (t2), $p \rightarrow \neg p \in \Sigma$ and $\neg p \rightarrow q \in \Sigma$ we obtain $p \rightarrow q \in \Sigma$. So, we have $p \vee q = (p \rightarrow q) \rightarrow q \in \Sigma$ and $p \rightarrow q \in \Sigma$. Hence, $q \in \Sigma$.

(\Leftarrow) It follows immediately by (t16).

(vi) (\Rightarrow) Apply (t12)

(\Leftarrow) Apply (t13).

(vii) (\Rightarrow) Apply (t19)

(\Leftarrow) Apply (t20).

(viii) Apply (t25).

(ix) Apply (i) and Proposition 2.2.(vii).

(x) Apply (vi) and (i). \square

Lemma 2.6 (Lindebaum's Lemma) Every \mathcal{L} -consistent set of formulas is contained in a maximal \mathcal{L} -consistent set.

Proof: Similar to [1], Chapter 9, Proposition 1.9, pag. 467. \square

Let $W_{\mathcal{L}}$ be the set of all maximal \mathcal{L} -consistent sets of formulas.

Remark 2.7 By Proposition 2.4.(vi) and Lindebaum's Lemma we get

$$W_{\mathcal{L}} \neq \emptyset \text{ iff } \mathcal{L} \text{ is } \mathcal{L}\text{-consistent iff } \mathcal{L} \neq Fmla.$$

Proposition 2.8 Let \mathcal{L} be an \mathcal{L} -consistent three-valued modal logic.

(i) $\{p \in Fmla \mid \Gamma \vdash_{\mathcal{L}} p\} = \bigcap \{\Sigma \in W_{\mathcal{L}} \mid \Gamma \subseteq \Sigma\}$, i. e. $\Gamma \vdash_{\mathcal{L}} p$ iff p belongs to every maximal \mathcal{L} -consistent set that includes Γ .

(ii) $\mathcal{L} = \bigcap \{\Sigma \in W_{\mathcal{L}}\}$, i. e. $\vdash_{\mathcal{L}} p$ iff p belongs to every maximal \mathcal{L} -consistent set.

Proof: (i) (\Rightarrow) Suppose that $\Gamma \vdash_{\mathcal{L}} p$ and let $\Sigma \in W_{\mathcal{L}}$ be such that $\Gamma \subseteq \Sigma$. Then $\Sigma \vdash_{\mathcal{L}} p$ and, by Proposition 2.5.(i), $p \in \Sigma$.

(\Leftarrow) Suppose that p belongs to every maximal \mathcal{L} -consistent set that includes Γ and that we don't have $\Gamma \vdash_{\mathcal{L}} p$. Then, by Proposition 2.4.(iii), $\Gamma \cup \{\sim p\}$ is

\mathcal{L} -consistent. Applying Lindebaum's Lemma we obtain a maximal \mathcal{L} -consistent set Θ such that $\Gamma \cup \{\sim p\} \subseteq \Theta$. Hence $\Gamma \subseteq \Theta$, therefore $p \in \Gamma$. It follows that $\sim p, p \in \Theta$, which contradicts Proposition 2.5.(iv).

(ii) It follows by (i). \square

2.2 Three-valued normal logics

By a *three-valued normal logic* we understand a three-valued modal logic \mathcal{L} that also satisfies the following conditions:

(i) \mathcal{L} contains every formula of the form

(K) $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$,

(A1) $\Box(p \oplus p) \leftrightarrow (\Box p \oplus \Box p)$,

(ii) \mathcal{L} is closed under *necessitation*; that is, if $p \in \mathcal{L}$, then also $\Box p \in \mathcal{L}$.

Proposition 2.9 Let \mathcal{L} be a three-valued normal logic. For any formulas p, q ,

(i) If $\vdash_{\mathcal{L}} p \rightarrow q$, then $\vdash_{\mathcal{L}} \Box p \rightarrow \Box q$.

(ii) If $\vdash_{\mathcal{L}} p \leftrightarrow q$, then $\vdash_{\mathcal{L}} \Box p \leftrightarrow \Box q$.

(iii) $\vdash_{\mathcal{L}} \Diamond \neg p \leftrightarrow \neg \Box p$.

(iv) $\sim \Diamond p \leftrightarrow \Box \sim p$.

Proof: (i) Apply (K) and the closure of \mathcal{L} under necessitation.

(ii) By the definition of \leftrightarrow , $\vdash_{\mathcal{L}} p \leftrightarrow q$ is equivalent to $\vdash_{\mathcal{L}} (p \rightarrow q) \wedge (q \rightarrow p)$. By (t12) we get $\vdash_{\mathcal{L}} p \rightarrow q$ and $\vdash_{\mathcal{L}} q \rightarrow p$. Using (i) twice we obtain $\vdash_{\mathcal{L}} \Box p \rightarrow \Box q$ and $\vdash_{\mathcal{L}} \Box q \rightarrow \Box p$. Now, (t13) gives us $\vdash_{\mathcal{L}} (\Box p \rightarrow \Box q) \wedge (\Box q \rightarrow \Box p)$, hence $\vdash_{\mathcal{L}} \Box p \leftrightarrow \Box q$.

(iii) By (t5) and (i) we have $\vdash_{\mathcal{L}} \Box p \leftrightarrow \Box \neg \neg p$. By (t4) we get $\vdash_{\mathcal{L}} \neg \Box \neg \neg p \leftrightarrow \neg \Box p$, hence $\vdash_{\mathcal{L}} \Diamond \neg p \leftrightarrow \neg \Box p$.

(iv) By the definition of \oplus and (t5) we obtain $\vdash_{\mathcal{L}} \Box(\neg p \oplus \neg p) \leftrightarrow \Box \sim p$ and $\vdash_{\mathcal{L}} (\Box \neg p \oplus \Box \neg p) \leftrightarrow \sim \neg \Box \neg p$. Using (A1) it follows that $\vdash_{\mathcal{L}} \Box \sim p \leftrightarrow \sim \neg \Box \neg p$, hence $\vdash_{\mathcal{L}} \Box \sim p \leftrightarrow \sim \Diamond p$. \square

Proposition 2.10 Let \mathcal{L} be a three-valued normal logic. For any $\Sigma \subseteq Fmla$ and any formula p ,

$\Sigma \vdash_{\mathcal{L}} p$ implies $\{\Box q \mid q \in \Sigma\} \vdash_{\mathcal{L}} \Box p$.

Proof: There exist $p_0, \dots, p_{n-1} \in \Sigma$ such that $\vdash_{\mathcal{L}} (p_0 \rightarrow (p_1 \rightarrow (\dots \rightarrow (p_{n-1} \rightarrow p) \dots)))$. Applying (K) and the fact that \mathcal{L} is closed under modus ponens, we get $\vdash_{\mathcal{L}} (\Box p_0 \rightarrow (\Box p_1 \rightarrow (\dots \rightarrow (\Box p_{n-1} \rightarrow \Box p) \dots)))$. Hence, $\{\Box q \mid q \in \Sigma\} \vdash_{\mathcal{L}} \Box p$. \square

It is clear that the intersection of any collection of three-valued normal logics is also a three-valued normal logic. In particular, the intersection of all three-valued normal logics is the *smallest* three-valued normal logic, denoted **3-K**.

2.3 Semantics

The semantics of our system is defined as in [9]. A *frame* is a pair $\mathcal{F} = \langle W, R \rangle$, where W is a non-empty set, and R is a binary relation on W . A *model* based on a frame $\mathcal{F} = \langle W, R \rangle$ is a triple $\mathcal{M} = \langle W, R, V \rangle$, where $V : Fmla \times W \rightarrow L_3 = \{0, \frac{1}{2}, 1\}$ is a function called *valuation* that have the properties:

- (i) $V(\neg p, s) = 1 - V(p, s)$,
- (ii) $V(p \rightarrow q, s) = \min\{1, 1 - V(p, s) + V(q, s)\}$,
- (iii) $V(\Box p, s) = \min\{V(p, t) \mid sRt\}$,

for all $p, q \in Fmla, s \in W$.

Let p be a formula. We define the next notions:

- p is *true at point* s in the model \mathcal{M} and we write $\mathcal{M}, s \models p$ if $V(p, s) = 1$,
- p is *true* in model \mathcal{M} and we write $\mathcal{M} \models p$ if $\mathcal{M}, s \models p$ for any $s \in W$,
- p is *valid* in a frame \mathcal{F} and we write $\mathcal{F} \models p$ if $\mathcal{M} \models p$ for any model \mathcal{M} based on \mathcal{F} .

If \mathcal{C} is a class of models (frames), then p is *true* (respectively *valid*) in \mathcal{C} , $\mathcal{C} \models p$, if p is true (respectively valid) in all members of \mathcal{C} .

Let \mathcal{C} be a class of frames, or of models and \mathcal{L} be a three-valued modal logic.

We say that \mathcal{L} is *sound* with respect to \mathcal{C} if for all formulas p ,

if $\vdash_{\mathcal{L}} p$, then $\mathcal{C} \models p$.

\mathcal{L} is *complete* with respect to \mathcal{C} if for any p ,

if $\mathcal{C} \models p$, then $\vdash_{\mathcal{L}} p$.

\mathcal{L} is *determined* by \mathcal{C} if \mathcal{L} is both sound and complete with respect to \mathcal{C} .

3 Canonical models and completeness

Let \mathcal{L} be an \mathcal{L} -consistent three-valued normal logic.

The *canonical model* of \mathcal{L} is the model $\mathcal{M}_{\mathcal{L}} = \langle W_{\mathcal{L}}, R_{\mathcal{L}}, V_{\mathcal{L}} \rangle$, where:

- (i) $W_{\mathcal{L}}$ is the set of all maximal \mathcal{L} -consistent sets of formulas,
- (ii) for all $s, t \in W_{\mathcal{L}}$,
 $sR_{\mathcal{L}}t$ iff $\{p \in Fmla \mid \Box p \in s\} \subseteq t$,
- (iii) for every atomic formula p and every $s \in W_{\mathcal{L}}$,

$$V_{\mathcal{L}}(p, s) = \begin{cases} 1, & \text{if } p \in s \\ 0, & \text{if } \neg p \in s \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

The *canonical frame* for \mathcal{L} is $\mathcal{F}_{\mathcal{L}} = \langle W_{\mathcal{L}}, R_{\mathcal{L}} \rangle$.

Proposition 3.1 For any $s \in W_{\mathcal{L}}$, and any $p \in Fmla$,

$\Box p \in s$ iff for all $t \in W_{\mathcal{L}}$, $sR_{\mathcal{L}}t$ implies $p \in t$.

Proof: (\Rightarrow) By the definition of $R_{\mathcal{L}}$.

(\Leftarrow) Suppose that for all $t \in W_{\mathcal{L}}$,

$sR_{\mathcal{L}}t$ implies $p \in t$,

i. e.

$\{q \in Fmla \mid \Box q \in s\} \subseteq t$ implies $p \in t$.

Let $\Gamma = \{q \in Fmla \mid \Box q \in s\}$. We get $p \in \bigcap \{t \in W_{\mathcal{L}} \mid \Gamma \subseteq t\}$, hence, using Proposition 2.8.(i), $\Gamma \vdash_{\mathcal{L}} p$. Applying Proposition 2.10 it follows that $\{\Box q \mid q \in \Gamma\} \vdash_{\mathcal{L}} \Box p$, so $s \vdash_{\mathcal{L}} \Box p$. By the fact that s is a maximal \mathcal{L} -consistent set and Proposition 2.5.(i), we obtain $\Box p \in s$. \square

Corollary 3.2 For any $s \in W_{\mathcal{L}}$, and any $p \in Fmla$,
 $\Box p \notin s$ iff there exists $t \in W_{\mathcal{L}}$ such that $sR_{\mathcal{L}}t$ and $p \notin t$.

Theorem 3.3 For every formula p and every $s \in W_{\mathcal{L}}$,

$$V_{\mathcal{L}}(p, s) = \begin{cases} 1, & \text{if } p \in s \\ 0, & \text{if } \neg p \in s \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

Proof: We prove by induction on p that for any formula p :

(i) $p \in s$ iff $V_{\mathcal{L}}(p, s) = 1$,

(ii) $\neg p \in s$ iff $V_{\mathcal{L}}(p, s) = 0$.

The intended result is then obvious.

The cases $p \in AF$, p is $\neg q$ and p is $q \rightarrow r$ are similar to [1], Chapter 9, Proposition 1.9 and [5].

Consider now that p is $\Box q$, where q satisfies the inductive hypothesis. Then

(i) By Proposition 3.1, we have that

$p \in s$ iff $\Box q \in s$ iff for all $t \in W_{\mathcal{L}}$, $sR_{\mathcal{L}}t$ implies $q \in t$.

By the inductive hypothesis for q , it follows that

$p \in s$ iff for all $t \in W_{\mathcal{L}}$, $sR_{\mathcal{L}}t$ implies $V_{\mathcal{L}}(q, t) = 1$
iff $\min\{V_{\mathcal{L}}(q, t) \mid sR_{\mathcal{L}}t\} = 1$ iff $V_{\mathcal{L}}(\Box q, s) = 1$.

Hence, $p \in s$ iff $V_{\mathcal{L}}(p, s) = 1$.

(ii) By Proposition 2.5.(iv), we have that

$\neg p \in s$ iff $\neg \Box q \in s$ iff $\sim \neg \Box q \notin s$.

By Proposition 2.9.(iii), Proposition 2.5.(i) and (x), we get

$\sim \neg \Box q \notin s$ iff $\sim \Diamond \neg q \notin s$.

Apply now Proposition 2.9.(iv) to get

$\sim \Diamond \neg q \notin s$ iff $\Box \sim \neg q \notin s$.

By Corollary 3.2, Proposition 2.5.(iv), and the inductive hypothesis we have

$\Box \sim \neg q \notin s$ iff there exist $t \in W_{\mathcal{L}}$ such that $sR_{\mathcal{L}}t$ and $\sim \neg q \notin t$ iff there exists $t \in W_{\mathcal{L}}$ such that $sR_{\mathcal{L}}t$ and $\neg q \in t$ iff there exists $t \in W_{\mathcal{L}}$ such that $sR_{\mathcal{L}}t$ and $V_{\mathcal{L}}(q, t) = 0$ iff $V_{\mathcal{L}}(\Box q, s) = 0$.

Hence, $\neg p \in s$ iff $V_{\mathcal{L}}(p, s) = 0$. \square

By this theorem we get a three-valued version of the Truth Lemma, a well-known result in the classical modal logic.

Proposition 3.4 (the Truth Lemma) Let p be a formula. Then, for all $s \in W_{\mathcal{L}}$,

$$\mathcal{M}_{\mathcal{L}}, s \models p \text{ iff } p \in s.$$

Corollary 3.5 $\mathcal{M}_{\mathcal{L}}$ determines \mathcal{L} , i. e. for all formulas p ,

$$\mathcal{M}_{\mathcal{L}} \models p \text{ iff } \vdash_{\mathcal{L}} p.$$

Proof: Applying the Truth Lemma and Proposition 2.8.(ii) we get $\mathcal{M}_{\mathcal{L}} \models p$ iff for all $s \in W_{\mathcal{L}}$, $\mathcal{M}_{\mathcal{L}}, s \models p$ iff for all $s \in W_{\mathcal{L}}$, $p \in s$ iff $p \in \bigcap \{s \mid s \in W_{\mathcal{L}}\}$ iff $p \in \mathcal{L}$ iff $\vdash_{\mathcal{L}} p$. \square

Thus we obtain the completeness theorem for **3-K**.

Theorem 3.6 **3-K** is determined by the class of all frames.

Proof: (Soundness) We prove easily that for any frame \mathcal{F} , $\mathcal{L}_{\mathcal{F}} = \{p \in Fmla \mid \mathcal{F} \models p\}$ is a three-valued normal logic, so **3-K** $\subseteq \mathcal{L}_{\mathcal{F}}$. Hence, $\vdash_{\mathbf{3-K}} p$ implies $\mathcal{F} \models p$ for any frame \mathcal{F} .

(Completeness) If we don't have $\vdash_{\mathbf{3-K}} p$, then, by Corollary 3.5, p is false in $\mathcal{M}_{\mathbf{3-K}}$, and so is not valid in the frame $\mathcal{F}_{\mathbf{3-K}}$. \square

4 Filtrations and Decidability

In this section we extend the filtration technique from the classical modal logic [6] to the three-valued modal logic.

Fix a model $\mathcal{M} = \langle W, R, V \rangle$ and a set $\Sigma \subseteq Fmla$ that is closed under subformulas. For every $s \in W$, let

$$\Sigma_s^1 = \{p \in \Sigma \mid V(p, s) = 1\}, \Sigma_s^{\frac{1}{2}} = \{p \in \Sigma \mid V(p, s) = \frac{1}{2}\}, \Sigma_s^0 = \{p \in \Sigma \mid V(p, s) = 0\}, \text{ and } \Sigma_s = (\{1\} \times \Sigma_s^1) \cup (\{\frac{1}{2}\} \times \Sigma_s^{\frac{1}{2}}) \cup (\{0\} \times \Sigma_s^0).$$

We define the relation \sim_{Σ} on W by :

$$s \sim_{\Sigma} t \text{ iff } V(p, s) = V(p, t) \text{ for all } p \in \Sigma.$$

Then it is clear that \sim_{Σ} is an equivalence relation and that for any $s, t \in W$,

$$s \sim_{\Sigma} t \text{ iff } \Sigma_s = \Sigma_t.$$

For every $s \in W$, let $[s]$ be its \sim_{Σ} -equivalence class and W^{Σ} the set of all such equivalence classes.

Lemma 4.1 If Σ is finite, then W^{Σ} is finite and has at most $3 \cdot 2^n$ elements, where n is the number of elements of Σ .

Proof: Since $[s] = [t]$ iff $s \sim_{\Sigma} t$ iff $\Sigma_s = \Sigma_t$, putting

$$f([s]) = \Sigma_s$$

gives a well-defined one-to-one mapping

$$f : W^{\Sigma} \rightarrow (\{1\} \times \mathcal{P}(\Sigma)) \cup (\{\frac{1}{2}\} \times \mathcal{P}(\Sigma)) \cup (\{0\} \times \mathcal{P}(\Sigma)).$$

Hence, if $|\Sigma| = n$, then $|W^{\Sigma}| \leq |(\{1\} \times \mathcal{P}(\Sigma)) \cup (\{\frac{1}{2}\} \times \mathcal{P}(\Sigma)) \cup (\{0\} \times \mathcal{P}(\Sigma))| = 3 \cdot 2^n$.

\square

A binary relation R' on W^Σ is called a Σ -filtration of R if it satisfies:

(F1) if sRt , then $[s]R'[t]$,

(F2) if $[s]R'[t]$, then for all $p \in Fmla$,

$$\Box p \in \Sigma \text{ implies } V(\Box p, s) \leq V(p, t).$$

Now let $AF^\Sigma = AF \cap \Sigma$ be the set of atomic formulas that belong to Σ , and define $V_{AF}^\Sigma : AF^\Sigma \times W^\Sigma \rightarrow L_3$ by $V_{AF}^\Sigma(p, [s]) = V(p, s)$. It follows immediately that V_{AF}^Σ is well-defined.

A model $\mathcal{M}' = \langle W^\Sigma, R', V^\Sigma \rangle$ in which R' is a Σ -filtration of R , and $V^\Sigma : AF \times W^\Sigma \rightarrow L_3$ is an arbitrary extension of V_{AF}^Σ is called a Σ -filtration of \mathcal{M} .

Theorem 4.2 (Filtration Lemma) If $p \in \Sigma$, then for any $s \in W$,

$$V(p, s) = V^\Sigma(p, [s]).$$

Proof: By induction on the formula p .

- $p \in AF$.

Then, by the hypothesis, $p \in AF^\Sigma$. Hence, $V^\Sigma(p, [s]) = V_{AF}^\Sigma(p, [s]) = V(p, s)$.

- p is $\neg q$.

Since Σ is closed under subformulas, by $\neg q \in \Sigma$ we get $q \in \Sigma$. So, we can apply the induction hypothesis for q to obtain $V(q, s) = V^\Sigma(q, [s])$. It follows that $V(p, s) = 1 - V(q, s) = 1 - V^\Sigma(q, [s]) = V^\Sigma(p, [s])$.

- p is $q \rightarrow r$.

By the fact that Σ is closed under subformulas, we similarly obtain that $V(q, s) = V^\Sigma(q, [s])$ and $V(r, s) = V^\Sigma(r, [s])$. Hence,

$$V(p, s) = \min\{1, 1 - V(q, s) + V(r, s)\} = \min\{1, 1 - V^\Sigma(q, [s]) + V^\Sigma(r, [s])\} = V^\Sigma(p, [s]).$$

- p is $\Box q$.

By the fact that Σ is closed under subformulas and $\Box q \in \Sigma$ we get $q \in \Sigma$, hence, by the induction hypothesis, $V(q, s) = V^\Sigma(q, [s])$.

We prove that $V(p, s) \leq V^\Sigma(q, [s])$. Let $t \in W$ be such that $[s]R'[t]$. Since $\Box q \in \Sigma$, we have by (F2) that $V(\Box q, s) \leq V(q, t)$. Hence,

$$V^\Sigma(p, [s]) = \min\{V^\Sigma(q, [t]) \mid [s]R'[t]\} = \min\{V(q, t) \mid [s]R'[t]\} \geq V(\Box q, s) = V(p, s).$$

The fact that $V(p, s) \geq V^\Sigma(q, [s])$ follows immediately by (F1). \square

Theorem 4.3 **3-K** is determined by the class of all finite frames. Moreover, if a formula p has n subformulas, then

$$\vdash_{\mathbf{3-K}} p \text{ iff } p \text{ is valid in all frames having at most } 3 \cdot 2^n \text{ elements.}$$

Proof: Similar to [6], Theorem 4.6, pag. 34. \square

Using this theorem we obtain

Corollary 4.4 **3-K** has the finite model property. Hence, **3-K** is decidable.

Acknowledgements

I would like to express my gratitude to Anna Gomolińska. Her valuable comments helped me to improve the final version of the paper.

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