FMI, CS, Master I
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Techniques of Combinatorial Optimization
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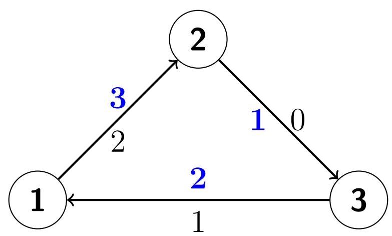
Seminar 5

(S5.1) Give an example where feasible circulations do not exist.

Proof. We consider the following example



Thus,

(i) $D = (V, A)$, where $V = \{1, 2, 3\}, A = \{(1, 2), (2, 3), (3, 1)\}$,

(ii) $c((1, 2)) = 3, c((2, 3)) = 1, c((3, 1)) = 2$ and

(iii) $d((1, 2)) = 2, d((2, 3)) = 0, d((3, 1)) = 1$.

Let $f : A \to \mathbb{R}$ be feasible w.r.t. d, c. Then $in_f(2) = f((1, 2)) \geq d((1, 2)) = 2$, while $out_f(2) = f((2, 3)) \leq c((2, 3)) = 1$. Thus, $in_f(2) \neq out_f(2)$, so f cannot be a circulation. □

(S5.2) Prove that the incidence matrix M of a directed graph $D = (V, A)$ is totally unimodular.

Proof. Let B be a square submatrix of M of order t. We prove by induction on t that $\det(B)$ is -1, 0 or 1. The case $t = 1$ is trivial. Let $t > 1$. We have the following cases:

(i) B has a column with only zeros. Then obviously $\det(B) = 0$.

(ii) B has a column with exactly one nonzero, which is ± 1. Expand the determinant by this column and use the induction hypothesis to conclude that $\det(B) \in \{-1, 0, 1\}$.

(iii) Each column of B contains two nonzeros, one of them 1 and the other -1. Then the sum of all columns of B is $\mathbf{0}$, hence the columns of B are linearly dependent. As a consequence, $\det(B) = 0$.

□

(S5.3) Let $N = (D, s, t)$ be a unit capacity network, $k \geq 1$ and $P_1, \ldots, P_k$ be k arc-disjoint s-t paths in D. Then for all $k \geq 1$,

$$f := \chi^{P_1} + \ldots + \chi^{P_k}$$

is an s-t $\{0, 1\}$-flow f with value$(f) = k$.

Proof. For $k = 1$, we have that $f := \chi^{P_1}$ is an s-t flow of value 1, by (S4.4) and the fact that $0 \leq \chi^{P_1} \leq 1$.

Let $k \geq 2$. Then f satisfies the flow conservation law at every $v \neq s, t$, by (S4.4) and (S4.2), and value$(f) = k$. It remains to prove that f takes values in $\{0, 1\}$. For any $a \in A$, we have one of the two cases:

(i) $a \notin P_1 \cup \ldots \cup P_k$, so $f(a) = 0$.

(ii) there exists a unique $i = 1, \ldots, k$ such that $a \in P_i$, so $f(a) = 1$.

Thus, $f : A \to \{0, 1\}$ is a flow. □

(S5.4) Let $D = (V, A)$ be a digraph. A subset $B \subseteq A$ is said to be an s-t *disconnecting arc set* if B intersects each s-t path. Prove that each s-t cut is an s-t disconnecting arc set.

Proof. Let $\delta^{out}(U)$ be an s-t cut, where $s \in U$ and $t \notin U$. Let $P = sv_1 \ldots v_k t$ be an s-t path and denote $v_0 := s$ and $v_{k+1} := t$. We have two cases:

(i) there exists $i = 1, \ldots, k$ such that $v_i \notin U$ and $v_{i-1} \in U$. Then $(v_{i-1}, v_i) \in \delta^{out}(U)$.

(ii) $v_i \in U$ for all $i = 1, \ldots, k$. Then $(v_k, t) \in \delta^{out}(U)$.

□