FMI, CS, Master I
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Techniques of Combinatorial Optimization
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Seminar 4

(S4.1) Let $N = (D, c, s, t)$ be a flow network with the property that all capacities are even natural numbers. Prove that the value of the maximum flow is even.

Proof. Since all capacities are even, the capacity of every cut is even, hence the minimum capacity of a cut is even. Apply the Max-Flow Min-Cut theorem to conclude that the maximum value of a flow is even. □

(S4.2) Let $D = (V, A)$ be a digraph. Assume that $v \in V$ and $f_1, \ldots, f_n : A \to \mathbb{R}$ are mappings satisfying the flow conservation law at v. Then any linear combination of $f_1, \ldots, f_n$ satisfies the flow conservation law at v.

Proof. Let $\lambda_i \in \mathbb{R}, i = 1, \ldots, n$ and $f := \sum_{i=1}^{n} \lambda_i f_i$. Then for all $v \in V$,

$$in_f(v) = \sum_{a \in \delta^{in}(v)} f(a) = \sum_{a \in \delta^{in}(v)} \sum_{i=1}^{n} \lambda_i f_i(a) = \sum_{i=1}^{n} \lambda_i \left(\sum_{a \in \delta^{in}(v)} f_i(a) \right) = \sum_{i=1}^{n} \lambda_i in_{f_i}(v)$$

and similarly

$$out_f(v) = \sum_{i=1}^{n} \lambda_i out_{f_i}(v).$$

Apply now the fact that $in_{f_i}(v) = out_{f_i}(v)$ for all $i = 1, \ldots, n$ to conclude that $in_f(v) = out_f(v)$. □

(S4.3) Prove Proposition 3.4.2..

Proof. Apply the Flow Decomposition Theorem 3.4.1. Then there exist $K, L \in \mathbb{Z}_+$, positive numbers $w_1, \ldots, w_K, \mu_1, \ldots, \mu_L$, s-t paths $P_1, \ldots, P_K$ and circuits $C_1, \ldots, C_L$ such that

$$f = \sum_{i=1}^{K} w_i \chi^{P_i} + \sum_{j=1}^{L} \mu_j \chi^{C_j} \quad \text{and} \quad \text{value}(f) = \sum_{i=1}^{K} w_i.$$

Furthermore, the w_i's, μ_j's are positive integers. Since f is a $\{0,1\}$-flow, we must have $w_i = \mu_j = 1$ for all i, j. Thus,

$$f = \sum_{i=1}^{K} \chi^{P_i} + \sum_{j=1}^{L} \chi^{C_j} \quad \text{and} \quad \text{value}(f) = K.$$

It remains to show that the family $\mathcal{F} = \{P_1, \ldots, P_K, C_1, \ldots, C_L\}$ is arc-disjoint. If $Q_1, Q_2 \in \mathcal{F}$ have an arc a in common, then $f(a) \geq \chi^{Q_1}(a) + \chi^{Q_2}(a) = 2$, which contradicts the fact that f is a $\{0,1\}$-flow. □

(S4.4) For any s-t path P in D, prove that χ^P satisfies the flow conservation law at every $v \neq s, t$ and that $\text{value}(\chi^P) = 1$.

Proof. If $P = st$, then $\chi^P(a) = 0$ for all $a \neq (s,t)$, hence $in_{\chi^P}(v) = out_{\chi^P}(v) = 0$ for all $v \neq s, t$. Assume that $P = sv_1 \ldots v_k t$ with $k \geq 1$. Let us denote $v_0 := s, v_{k+1} := t$. Then $\chi^P((s,v_1)) = \chi^P((v_1,v_2)) = \ldots = \chi^P((v_{k-1},v_k)) = \chi^P((v_k,t)) = 1$ and $\chi^P(a) = 0$ for all the other arcs a. For an arbitrary $v \neq s, t$ we have two cases:

(i) $v \notin P$. Then $in_{\chi^P}(v) = out_{\chi^P}(v) = 0$.

(ii) $v = v_i, i = 1, \ldots, k$. Then

$$out_{\chi^P}(v_i) = \sum_{a \in \delta^{in}(v_i)} \chi^P(a) = \chi^P((v_{i-1}, v_i)) + 0 = 1,$$
$$out_{\chi^P}(v_i) = \sum_{a \in \delta^{out}(v_i)} \chi^P(a) = \chi^P((v_i, v_{i+1})) + 0 = 1.$$

Finally,

$$\text{value}(\chi^P) = out_{\chi^P}(s) - in_{\chi^P}(s) = \chi^P((s, v_1)) - 0 = 1.$$

□